Surfaces and the representation theory of finite groups

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1 Introduction

Let Σ be a closed connected orientable surface and let G be a finite group. Mednykh's formula states that

$$\frac{|\mathrm{Hom}(\pi_1(\Sigma), G)|}{|G|} = \frac{1}{|G|^{\chi(\Sigma)}} \sum_V (\dim V)^{\chi(\Sigma)} \tag{1}$$

where the sum on the RHS runs over all complex irreducible representations of G and $\chi(\Sigma) = 2 - 2g$ is the Euler characteristic of Σ (here g is the genus).

The goal of this paper is to outline a proof of Mednykh's formula using ideas from topological quantum field theory. In particular, we will interpret both sides of Mednykh's formula as the partition function of Σ associated to a suitable 2-dimensional topological quantum field theory, namely (untwisted) Dijkgraaf-Witten theory with gauge group G.

2 Generalities

An n-dimensional topological quantum field theory (TQFT) is, for our purposes, a symmetric monoidal functor

$$Z : \mathrm{nCob} \to \mathrm{FinVect}. \tag{2}$$

Here the source category nCob is the symmetric monoidal category whose objects are compact orientable $(n-1)$-manifolds and whose morphisms $X \to Y$ are (diffeomorphism classes of) compact orientable n-manifolds with boundary $\overline{X} \sqcup Y$, with monoidal product given by disjoint union. The target category is the symmetric monoidal category of finite-dimensional complex vector spaces, with monoidal product given by the tensor product.

This definition was introduced by Atiyah and is intended to capture using category theory the locality of the Feynman integral.

3 Classical gauge theory with finite gauge group

Dijkgraaf-Witten theory with gauge group a finite group G is a toy model of Chern-Simons theory, in which G is replaced by a Lie group. It has a single field, which is a principal G-bundle on some manifold M. Since G is finite, a principal G-bundle on M is precisely a G-cover, which is in turn precisely a groupoid homomorphism

$$\Pi_1(M) \to BG \tag{3}$$

from the fundamental groupoid of M to the groupoid BG with one object whose automorphism group is G. The category of principal G-bundles on M is precisely the functor category $\Pi_1(M) \Rightarrow BG$, and classical Dijkgraaf-Witten theory assigns this functor category to M. This category, which is really a groupoid, should be thought of as the "moduli stack" $A_G(M)$ of principal G-bundles on M, since we keep track of automorphisms of bundles.

If M is connected, we may instead talk about group homomorphisms $\pi_1(M) \to G$. The morphisms between group homomorphisms are given by conjugation in G.

Example Let $M = S^1$. A principal G-bundle on M may then be identified with a homomorphism $\mathbb{Z} \to G$, hence with an element of G. In this case $\Pi_1(S^1) \Rightarrow BG$ is the groupoid $\frac{G}{G}$ of elements of G up to conjugation. That is, it is the groupoid whose objects are the elements of G and where the morphisms $g \to h$ are given by elements $a \in G$ such that $aga^{-1} = h$.

Since we only consider compact manifolds M, the fundamental group of each connected component of M is finitely generated, hence $\Pi_1(M) \to BG$ is an essentially finite groupoid (a groupoid equivalent to a groupoid with finitely many morphisms).

4 The classical functor

The assignment $M \mapsto A_G(M)$ gives a contravariant functor

$$A_G : \mathrm{Man} \to \mathrm{Gpd} \tag{4}$$

from the category of manifolds to the category of groupoids. It is a composition of two functors, namely the covariant functor $\Pi_1(-) : \mathrm{Man} \to \mathrm{Gpd}$ and the contravariant functor $(-) \Rightarrow BG : \mathrm{Gpd} \to \mathrm{Gpd}$.

Since we want to build a TQFT, we would like to extend A_G to cobordisms. This is done as follows. If M is an n-dimensional compact oriented manifold with boundary $\bar{X} \sqcup Y$ (so a cobordism $X \to Y$), then M, X, Y together define a cospan

$$\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & M & \end{tikzcd} \tag{5}$$

of manifolds. Applying $\Pi_1(-)$ gives a cospan

$$
\begin{tikzcd}
\Pi_1(X) \arrow[dr] & & \Pi_1(Y) \arrow[dl] \\
& \Pi_1(M) &
\end{tikzcd}
\tag{6}
$$

of groupoids, and applying $(-) \Rightarrow BG$ gives a span

$$
\begin{tikzcd}
& A_G(M) \arrow[dl] \arrow[dr] & \\
A_G(X) & & A_G(Y)
\end{tikzcd}
\tag{7}
$$

of groupoids. There is a category Span(FinGpd) whose morphisms are (isomorphism classes of) spans of essentially groupoids, where composition of spans is given by taking pullbacks as follows:

$$
\begin{tikzcd}
& & Y_1 \times_{X_2} Y_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
& Y_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & Y_2 \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
X_1 & & X_2 & & X_3
\end{tikzcd}
. \tag{8}
$$

The assignment $M \mapsto A_G(M)$ then extends, by the groupoid Seifert-van Kampen theorem, to a (symmetric monoidal) functor

$$A_G : \mathrm{nCob} \to \mathrm{Span}(\mathrm{FinGpd}) \tag{9}$$

which we might call classical (untwisted) Dijkgraaf-Witten theory.

5 Linearization and quantization

Quantum (untwisted) Dijkgraaf-Witten theory Z_G is obtained from the classical theory A_G by applying a linearization functor $\mathbb{C}^{(-)} : \mathrm{Span}(\mathrm{FinGpd}) \to \mathrm{FinVect}$. This functor is heuristically given by Feynman integrals over "moduli stacks" of principal G-bundles and rigorously defined using a push-pull construction as follows.

If X is an essentially finite groupoid, let $\mathbb{C}^X$ denote the space of complex-valued functions on the objects of X such that if there exists a morphism $p : x \to y$ in X, then $f(x) = f(y)$. Equivalently, $\mathbb{C}^X$ denotes the space of complex-valued functions on the isomorphism classes $\pi_0(X)$ of objects of X. A functor $F : X \to Y$ induces two linear maps between these spaces of functions. One of them is the pullback $F^* : \mathbb{C}^Y \to \mathbb{C}^X$ given by

$$\left(F^*(f)\right)(x) = f(F(x)). \tag{10}$$

The other is the pushforward $F_* : \mathbb{C}^X \to \mathbb{C}^Y$ given by

$$(F_*(f))\,(y) = \sum_{x \in \pi_0(X) : F(x) \cong y} \frac{|\mathrm{Aut}(y)|}{|\mathrm{Aut}(x)|} f(x). \tag{11}$$

The pushforward is adjoint to the pullback in the following sense. $\mathbb{C}^X$ admits a distinguished linear functional

$$\int_X : \mathbb{C}^X \ni f \mapsto \sum_{x \in \pi_0(X)} \frac{f(x)}{|\mathrm{Aut}(x)|} \in \mathbb{C} \tag{12}$$

which should be thought of as integration over X. The vector space $\mathbb{C}^X$ is also a commutative algebra under pointwise multiplication, and the integral of the identity recovers the groupoid cardinality of X. These two structures combine to give an inner product

$$\langle f, g \rangle_X = \int_X \overline{f(x)} g(x)\, dx = \sum_{x \in \pi_0(X)} \frac{\overline{f(x)} g(x)}{|\mathrm{Aut}(x)|} \tag{13}$$

on $\mathbb{C}^X$, and the pushforward is adjoint to the pullback with respect to this inner product. It should be thought of as integration along fibers.

Pushforward and pullback allow us to linearize a span of groupoids

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd} \tag{14}$$

by associating to it the composition

$$q_* \circ p^* : \mathbb{C}^X \to \mathbb{C}^Y \tag{15}$$

which explicitly takes the form

$$((q_* \circ p^*)(f))\,(y) = \sum_{z \in \pi_0(Z) : q(z) \cong y} \frac{|\mathrm{Aut}(y)|}{|\mathrm{Aut}(z)|} f(p(z)). \tag{16}$$

This defines a functor $\mathbb{C}^{(-)} : \mathrm{Span}(\mathrm{FinGpd}) \to \mathrm{FinVect}$ which, when composed with A_G, gives untwisted Dijkgraaf-Witten theory

$$Z_G = \mathbb{C}^{A_G} : \mathrm{nCob} \to \mathrm{FinVect}. \tag{17}$$

Example Suppose M is a compact oriented n-manifold without boundary. Then $Z_G(M) : \mathbb{C} \to \mathbb{C}$ is multiplication by the groupoid cardinality

$$\sum_{x \in \pi_0(A_G(M))} \frac{1}{|\mathrm{Aut}(x)|} \tag{18}$$

of $A_G(M)$. It is a general fact about groupoid cardinality, which can be proven using the orbit-stabilizer theorem, that when M is connected this sum is equal to

$$\frac{|\mathrm{Hom}(\pi_1(M), G)|}{|G|}. \tag{19}$$

6 The 2-dimensional theory

We now restrict our attention to the case $n = 2$. In this case 2-dimensional TQFTs have a known classification as follows. If Z is such a TQFT, then $Z(S^1)$ is a commutative Frobenius algebra (an algebra A equipped with a linear functional λ such that the bilinear form $\lambda(ab)$ is nondegenerate), and conversely given any commutative Frobenius algebra A there is a unique (up to equivalence) TQFT such that $Z(S^1) \cong A$. The structure maps of A and the axioms they satisfy come from a description of 2Cob in terms of generators and relations; for example, the multiplication in A comes from the pair of pants and the linear functional comes from the cup.

The commutative Frobenius algebra associated to untwisted Dijkgraaf-Witten theory can be explicitly described. We saw above that $A_G(S^1)$ is the groupoid of elements of G up to conjugation. It follows that $Z_G(S^1)$ can be identified with the space of class functions on G. The linear functional on $Z_G(S^1)$ is induced by the cap, which, after applying the classical functor $A_G(-)$, gives the span of groupoids

$$\begin{tikzcd} & BG \arrow[dl] \arrow[dr] & \\ \frac{G}{G} & & 1 \end{tikzcd} \tag{20}$$

where $\frac{G}{G}$, as above, denotes the groupoid of elements of G up to conjugation; BG, as above, denotes the groupoid with one object and automorphism group G; and 1 denotes the trivial groupoid. The map $BG \to \frac{G}{G}$ is the inclusion of the identity into $\frac{G}{G}$. Linearizing, we obtain the linear functional

$$Z_G(S^1) \ni f \mapsto \frac{f(1)}{|G|} \in Z_G(\emptyset) \cong \mathbb{C} \tag{21}$$

where 1 denotes the identity element of G.

The multiplication on $Z_G(S^1)$ is induced by the pair of pants, which, after applying the classical functor $A_G(-)$, gives the span of groupoids

$$\begin{tikzcd} & \frac{G \times G}{G} \arrow[dl] \arrow[dr] & \\ \frac{G}{G} \times \frac{G}{G} & & \frac{G}{G} \end{tikzcd} \tag{22}$$

where $\frac{G\times G}{G}$, which is Z_G applied to the pair of pants, denotes the groupoid of pairs of elements of $G \times G$ up to simultaneous conjugation, the map to $\frac{G}{G} \times \frac{G}{G}$ is induced by the identity $G\times G \to G\times G$, and the map $\frac{G}{G} \times \frac{G}{G} \to \frac{G}{G}$ is induced by multiplication. Linearizing, we obtain a multiplication map

$$(f_1 * f_2)(g) = \sum_{h_1 h_2 = g} \frac{|\mathrm{Aut}(g)|}{|\mathrm{Aut}(h_1, h_2)|} f_1(h_1) f_2(h_2) \tag{23}$$

where $f_1, f_2 \in Z_G(S^1)$, $g \in G$, the sum runs over all simultaneous conjugacy classes of pairs of elements $h_1, h_2 \in G$ whose product is g, $\mathrm{Aut}(g)$ denotes the centralizer of g (its automorphism group in $\frac{G}{G}$), and $\mathrm{Aut}(h_1, h_2)$ denotes the intersection of the centralizers of h_1 and h_2 (the automorphism group of (h_1, h_2) in $\frac{G\times G}{G}$).

An easier way to understand the above multiplication is as follows. There is a natural linear isomorphism from $Z_G(S^1)$, the space of class functions, to the center of the group algebra $\mathbb{C}[G]$. This isomorphism takes the form

$$Z_G(S^1) \ni f \mapsto \sum_{g\in G} f(g) g \in \mathbb{C}[G]. \tag{24}$$

The linear functional on $Z_G(S^1)$ then becomes the restriction of the linear functional on $\mathbb{C}[G]$ assigning $\frac{1}{|G|}$ to the identity and 0 to other elements to the center. Moreover, this isomorphism respects multiplication. To see this, let c_1, c_2 be two conjugacy classes and let f_1, f_2 be their indicator functions in $Z_G(S^1)$. Then we compute that

$$\left(\sum_{g\in G} f_1(g) g\right)\left(\sum_{g\in G} f_2(g) g\right) = \sum_{h_1 \in c_1, h_2 \in c_2} h_1 h_2. \tag{25}$$

Writing this in the form $\sum_{g\in G} f(g) g$, we have that

$$f(g) = |\{h_1 \in c_1, h_2 \in c_2 : h_1 h_2 = g\}|. \tag{26}$$

The set $\{h_1 \in c_1, h_2 \in c_2 : h_1 h_2 = g\}$ admits an action of $\mathrm{Aut}(g)$ by simultaneous conjugation. The orbit containing a particular pair (h_1, h_2) has stabilizer $\mathrm{Aut}(h_1, h_2)$, and it follows from the orbit-stabilizer theorem that

$$f(g) = (f_1 \otimes f_2)(g) \tag{27}$$

as desired.

To prove Mednkyh's formula we will also need to determine what linear map is assigned to the span

$$\begin{tikzcd} & \frac{G\times G}{G} \arrow[dl] \arrow[dr] & \\ \frac{G}{G} & & \frac{G}{G}\times\frac{G}{G} \end{tikzcd} \tag{28}$$

coming from the transpose of the pair of pants. We compute that this map is

$$(\Delta(f))(h_1, h_2) = \sum_{(h_1', h_2'): h_i' \sim h_i} \frac{|\mathrm{Aut}(h_1)||\mathrm{Aut}(h_2)|}{|\mathrm{Aut}(h_1', h_2')|} f(h_1' h_2') \tag{29}$$

where the sum runs over all simultaneous conjugacy classes of pairs of elements (h_1', h_2') such that h_1' is conjugate to h_1 and h_2' is conjugate to h_2.

7 Mednykh's formula

We are now ready to prove Mednykh's formula. First we will prove a more general result. Let $Z : \mathrm{2Cob} \to \mathrm{FinVect}$ be a 2-dimensional TQFT. Suppose that $Z(S^1)$ is semisimple. Then as an algebra it is a finite product of copies of $\mathbb{C}$. If $e_1, ...e_n$ denote its primitive idempotents, then $Z(S^1)$ is determined up to isomorphism as a commutative Frobenius algebra by the values $\lambda(e_1), ...\lambda(e_n)$, which are necessarily all nonzero (this is necessary and sufficient for $\lambda(ab)$ to be nondegenerate). In particular, these values determine $Z(\Sigma)$ for all closed connected orientable surfaces Σ.

Theorem 7.1. *Let Σ be a closed connected orientable surface of genus g. Then*

$$Z(\Sigma) = \sum_{i=1}^{n} \lambda(e_i)^{1-g}. \tag{30}$$

Proof. We first need to compute the map $Z(S^1) \to Z(S^1) \otimes Z(S^1)$ assigned to the transpose of the pair of pants. This map takes the form

$$e_i \mapsto \sum_{j,k} c_i^{jk} e_j \otimes e_k \tag{31}$$

for some coefficients c_i^{jk}. It has the property that composing with either $\mathrm{id} \otimes \lambda$ or with $\lambda \otimes \mathrm{id}$ gives the identity; this gives

$$\sum_{j,k} c_i^{jk} e_j \otimes \lambda(e_k) = \sum_{j,k} c_i^{jk} \lambda(e_j) \otimes e_k = e_i. \tag{32}$$

Since the $\lambda(e_i)$ are all nonzero, it follows that c_i^{jk} is 0 if either $j \neq i$ or $k \neq i$, and $c_i^{ii} = \frac{1}{\lambda(e_i)}$. Hence the map assigned to the transpose of the pair of pants is

$$e_i \mapsto \frac{e_i \otimes e_i}{\lambda(e_i)}. \tag{33}$$

It follows that the map $\theta : Z(S^1) \to Z(S^1)$ assigned to the composite of the transpose of the pair of pants and the pair of pants is

$$e_i \mapsto \frac{e_i}{\lambda(e_i)}. \tag{34}$$

We now observe that the closed orientable surface of genus g can be obtained as the composite of g copies of the cobordism $S^1 \to S^1$ given by the composite of the transpose of the pair of pants and the pair of pants, together with a cup and a cap. The corresponding composition of linear maps is given by

$$1 \mapsto \sum_i e_i \mapsto \sum_i \frac{e_i}{\lambda(e_i)^g} \mapsto \sum_i \lambda(e_i)^{1-g} \tag{35}$$

as desired. $\square$

To prove Mednykh's formula it now suffices to compute the numbers $\lambda(e_i)$ for $Z_G(S^1)$. We will use the identification with the center of the group algebra $\mathbb{C}[G]$. The primitive idempotents here are given by

$$e_V = \frac{\dim V}{|G|} \sum_{g \in G} \chi_V(g) g \tag{36}$$

where V is an irreducible complex representation of G, and we have

$$\lambda(e_V) = \left(\frac{\dim V}{|G|} \right)^2 \tag{37}$$

so it follows by the above theorem that

$$Z_G(\Sigma) = \sum_V \left(\frac{\dim V}{|G|} \right)^{2-2g}. \tag{38}$$

On the other hand, we observed earlier that

$$Z_G(\Sigma) = \frac{|\text{Hom}(\pi_1(\Sigma), G)|}{|G|}. \tag{39}$$

Mednykh's formula follows.

Example Let $g = 0$. Then Mednykh's formula gives

$$\frac{1}{|G|} = \sum_V \left(\frac{\dim V}{|G|} \right)^2 \tag{40}$$

which is just the familiar statement that the sum of the squares of the dimensions of the irreducible representations of G is the size of G.

Example Let $g = 1$. Then Mednykh's formula gives

$$\frac{|\text{Hom}(\mathbb{Z}^2, G)|}{|G|} = \sum_V 1. \tag{41}$$

Here $|\mathrm{Hom}(\mathbb{Z}^2, G)|$ is the number of pairs of commuting elements of G. The LHS of the above sum can also be written

$$\sum_{g \in G} \frac{|\mathrm{Aut}(g)|}{|G|} \tag{42}$$

where as before $\mathrm{Aut}(g)$ denotes the centralizer of g, and by Burnside's lemma this is precisely the number of conjugacy classes of G. Hence Mednykh's formula in this case recovers the familiar statement that the number of irreducible representations of G is the number of conjugacy classes of G (the Peter-Weyl theorem).