Qualifying Exam Syllabus and Transcript

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Major Topic: Lie Algebras (Algebra)

- **Basic definitions:** Lie algebras, ideals, subalgebras, the center, representations.
- **Structure theory of Lie algebras:** nilpotent Lie algebras, solvable Lie algebras, semisimple Lie algebras, Engel's theorem, Lie's theorem, Cartan's criterion, the Levi decomposition.
- **Structure theory of semisimple Lie algebras:** Cartan subalgebras, root systems, Dynkin diagrams, Weyl groups, classification of simple complex Lie algebras.
- **Representation theory:** Universal enveloping algebras, the Poincaré-Birkhoff-Witt theorem, highest weight modules, Verma modules, classification of finite-dimensional simple modules over semisimple Lie algebras, the BGG resolution, the Weyl character formula.
- **Examples:** representation theory of $\mathfrak{sl}_2, \mathfrak{sl}_3$, Weyl character formula for $\mathfrak{sl}_n$.

Reference: Kirillov's *An Introduction to Lie Groups and Lie Algebras*, Chapters 2-7, 8.1-8.5.

Major Topic: Algebraic Topology (Geometry)

- **The fundamental group:** Covering spaces, Seifert-van Kampen, Eilenberg-MacLane spaces.
- **Homology and cohomology:** Singular homology and cohomology, Mayer-Vietoris, cup products, Poincaré duality, universal coefficients, the Künneth formula.
- **The homotopy groups:** Commutativity of the higher homotopy groups, the long exact sequence of a fibration, Whitehead's theorem, the Hurewicz theorem.

- **Examples:** Fundamental groups of graphs $\Rightarrow$ subgroups of free groups are free, (co)homology of surfaces, $\pi_k(S^n)$ for $k \leq n$, the Hopf fibration and $\pi_3(S^2)$.

Reference: Hatcher's *Algebraic Topology*, Chapters 0-3.A, 4.1-4.2

Minor Topic: Homological Algebra (Algebra)

- **Basic category theory:** categories, functors, natural transformations, abelian categories, limits and colimits, adjoint functors, adjointness and exactness.
- **Chain complexes:** homology, mapping cones, mapping cylinders, chain homotopy, the long exact sequence of a short exact sequence of chain complexes.
- **Derived functors:** projective resolutions, injective resolutions, left derived functors, right derived functors, Ext and Tor.
- **Examples:** Ext and Tor for abelian groups, $H^n(G, -) \cong \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, -)$ via the bar resolution, $H^n(\mathfrak{g}, -) \cong \mathrm{Ext}^n_{U(\mathfrak{g})}(\mathbb{C}, -)$ via the Chevalley-Eilenberg resolution, Whitehead's lemma.

Reference: Weibel's *An Introduction to Homological Algebra*, Appendix A, Chapters 1-3, 6, 7.

Note: this transcript is from memory and is probably far from the exact words that were said during the exam. At several points I am unsure which examiner asked which question.

Lie Theory

Nadler: Can you define $\mathfrak{sl}_2(\mathbb{C})$ and describe its finite-dimensional irreducible representations?

Me: $\mathfrak{sl}_2(\mathbb{C})$ is the Lie algebra of traceless 2×2 complex matrices. If V is the defining 2-dimensional representation, then the symmetric powers $S^n(V)$, with dimension $n+1$, are a complete list of the finite-dimensional irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$. Should I go into more detail?

Nadler: That's fine. Where is the adjoint representation in this classification?

Me: The obvious guess is that it's the 3-dimensional irreducible $S^2(V)$. The adjoint representation must be irreducible because if it were reducible then $\mathfrak{sl}_2$ would have a proper ideal of dimension either 1 or 2, and in either case that would imply that $\mathfrak{sl}_2$ is solvable, which we know is not true because it's equal to its own commutator subalgebra. Since the adjoint representation is 3-dimensional we're done.

Nadler: Can you write down an explicit isomorphism between the adjoint representation and $S^2(V)$?

Me: First observe that the adjoint representation fits into a short exact sequence

$$0 \to \mathfrak{sl}(V) \to V \otimes V^* \xrightarrow{\text{tr}} \mathbb{C} \to 0 \tag{1}$$

since by definition $\mathfrak{sl}(V)$ consists of the 2×2 matrices of trace zero. On the other hand, V is a self-dual representation, so we can replace the middle representation with $V \otimes V$. This is because $\Lambda^2(V)$ is the trivial representation, hence the action of $\mathfrak{sl}(V)$ preserves a nondegenerate skew-symmetric bilinear form $\omega : V \otimes V \to \mathbb{C}$. Explicitly, if e_1, e_2 is a basis for V then this form can be chosen so that $\omega(e_1, e_2) = 1$, so the corresponding isomorphism

$$V \ni v \mapsto \omega(v, -) \in V^* \tag{2}$$

sends e_1 to e_2^* and e_2 to $-e_1^*$. Using this isomorphism I claim that the short exact sequence above is isomorphic to the short exact sequence

$$0 \to S^2(V) \to V \otimes V \to \Lambda^2(V) \to 0. \tag{3}$$

Olsson (?): I don't think we need to be quite so explicit. More abstractly, what can we say about maps from the tensor square $V^{\otimes 2}$ of the defining representation to the trivial representation?

Me: We always have $V^{\otimes 2} \cong S^2(V) \oplus \Lambda^2(V)$. In this case $\Lambda^2(V) \cong \mathbb{C}$ is the trivial representation and $S^2(V)$ is irreducible, so

$$\text{Hom}(S^2(V) \oplus \mathbb{C}, \mathbb{C}) \cong \text{Hom}(S^2(V), \mathbb{C}) \cong \text{Hom}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C} \tag{4}$$

since there are no nonzero maps from a nontrivial irreducible representation to a trivial one. Hence any two nonzero morphisms $V^{\otimes 2} \to \mathbb{C}$ are related by multiplication by a scalar, and the kernel of any such morphism must be isomorphic to $S^2(V)$.

Nadler: How would you decompose the tensor square of the adjoint representation as a direct sum of irreducibles, by hook or by crook?

Me: The shortest method is probably to use the character of the adjoint representation, which I'll write as $z^2 + 1 + z^{-2}$, and —

Nadler: What kind of mathematical object is that expression?

Me: There's a restriction map from representations of $\mathfrak{sl}_2$ to representations of its Cartan subalgebra $\mathfrak{h}$, and I prefer to think of characters as elements of the representation ring of the Cartan subalgebra. Equivalently, characters describe weight decompositions.

Continuing, the tensor square of the adjoint representation has character $(z^2+1+z^{-2})^2$, which I just need to write as a sum of characters of irreducible representations. This looks like

$$(z^2 + 1 + z^{-2})^2 = (z^4 + z^2 + 1 + z^{-2} + z^{-4}) + (z^2 + 1 + z^{-2}) + 1 \tag{5}$$

hence we deduce an isomorphism

$$\mathfrak{sl}(V)^{\otimes 2} \cong S^4(V) \oplus S^2(V) \oplus \mathbb{C}. \tag{6}$$

Nadler: Can you tell me what the BGG resolution of the trivial representation of $\mathfrak{sl}_3$ looks like?

Me: First I might as well state the BGG resolution in general. Let $\mathfrak{g}$ be a semisimple Lie algebra, W its Weyl group, $\ell(w)$ the length function on the Weyl group, Δ its root system, Δ^+ a choice of positive roots, and

$$\rho = \sum_i \omega_i = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \tag{7}$$

its Weyl vector. Define the dot action of W weights λ as $w \cdot \lambda = w(\lambda + \rho) - \rho$. For λ a dominant (integral) weight, the simple module $L(\lambda)$ has a resolution of the form

$$\cdots \to \bigoplus_{\ell(w)=2} M(w \cdot \lambda) \to \bigoplus_{\ell(w)=1} M(w \cdot \lambda) \to M(\lambda) \to L(\lambda) \to 0 \tag{8}$$

where $M(\lambda)$ is the Verma module of highest weight λ.

Nadler: Is this resolution infinite?

Me: No, it ends at the longest element of W.

Nadler: How many longest elements are there?

Me: Exactly one.

Nadler: Can you tell me what $M(0)$ looks like explicitly?

Me: First I might as well describe Verma modules in general. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition of $\mathfrak{g}$, where $\mathfrak{h}$ is the Cartan subalgebra, $\mathfrak{n}^-$ is the sum of the negative root spaces, and —

Nadler: I notice we've gone this entire time talking about Lie algebras without writing down any matrices!

Me: Okay, let's specialize to $\mathfrak{g} = \mathfrak{sl}_3$. There is a direct sum decomposition

$$\mathfrak{sl}_3 \cong \left\{ \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \right\} \oplus \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\} \tag{9}$$

of $\mathfrak{sl}_3$ into strictly lower triangular and upper triangular matrices. Let $\mathfrak{b}$ denote the upper triangular matrices. The Verma module $M(0)$ is the induced representation

$$M(0) \cong U(\mathfrak{sl}_3) \otimes_{U(\mathfrak{b})} \mathbb{C} \tag{10}$$

where $\mathbb{C}$ is the trivial 1-dimensional representation of $U(\mathfrak{b})$.

Olsson: What's the action of $U(\mathfrak{sl}_3)$ here?

Me: It's induced from left multiplication.

Continuing, by the PBW theorem, $U(\mathfrak{sl}_3)$ is a free $U(\mathfrak{b})$-module on monomials in three variables corresponding to the three entries of a strictly lower triangular matrix, call them e_{21}, e_{31}, e_{32}. So as a vector space, and in fact as a weight module, we can write $M(0) \cong \mathbb{C}[e_{21}, e_{31}, e_{32}]$.

Nadler: What do the maps in the BGG resolution look like? Or, how would you go about constructing something like the BGG resolution?

Me: $L(\lambda)$ is the unique simple quotient of $M(\lambda)$, hence we want to look for any proper submodules of $M(\lambda)$ at all. One idea is that we want to look for weight vectors in $M(\lambda)$ which generate proper submodules. If we've found such a weight vector of weight μ (note: I didn't say this, but we also need the weight vector to be singular, or annihilated by $\mathfrak{n}^+$) then we get an induced map $M(\mu) \to M(\lambda)$, and the maps in the BGG resolution are maps like this.

Nadler: Can you explain where these maps come from in terms of an adjunction?

Me: Yes, it's the tensor-hom adjunction

$$\mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_\mu, V) \cong \mathrm{Hom}_{U(\mathfrak{b})}(C_\mu, V) \tag{11}$$

where C_μ is the 1-dimensional $U(\mathfrak{b})$-module generated by a vector of weight μ.

Nadler: What are the irreducible Verma modules of $\mathfrak{sl}_2$?

Me: Let H, X, Y be the standard basis of $\mathfrak{sl}_2$. The Verma module $M(\lambda)$ is generated by a vector v_λ satisfying $Hv_\lambda = \lambda v_\lambda$ and $Xv_\lambda = 0$ and has basis $v_\lambda, Yv_\lambda, Y^2 v_\lambda, \ldots$. The first thing we should do is probably write down the action of X, so commuting —

Nadler: Commuting operators is near and dear to my heart, but perhaps we should work more abstractly. You already know that some of these Verma modules aren't irreducible. What about the others? Is there a more indirect argument you could use?

Me: I know that $M(\lambda)$ has a finite-dimensional quotient $L(\lambda)$ whenever $\lambda \in \mathbb{Z}_{\geq 0}$ and that these are precisely the finite-dimensional irreducible representations of $\mathfrak{sl}_2$. For all other values of λ, if $M(\lambda)$ were reducible then it would have a proper submodule containing some weight vector of weight μ. But then this submodule contains all lower weights, so the quotient is finite-dimensional and this can't happen.

Nadler: Can you describe a geometric object on which $\mathfrak{sl}_2$ acts?

Me: SL_2 acts on $\mathbb{P}^1$ in the usual way, and differentiating this action gives an action of $\mathfrak{sl}_2$ by vector fields on $\mathbb{P}^1$ (more precisely on, say, smooth functions on $\mathbb{P}^1$).

Algebraic Topology

Nadler: What is your favorite space?

Me: Can I pick a family of spaces? Let's go with the configuration space $C_n(\mathbb{C})$ of n distinct unordered points in $\mathbb{C}$, which is an Eilenberg-MacLane space $K(B_n, 1)$ for the braid groups B_n.

Nadler: Can you define an Eilenberg-MacLane space?

Me: A connected topological space X is a $K(G, 1)$ if $\pi_1(X) \cong G$ and the higher homotopy vanishes.

Olsson: In what sense is an Eilenberg-MacLane space unique?

Me: Up to weak homotopy equivalence, or up to homotopy equivalence if we restrict our attention to CW complexes by Whitehead's theorem. But I'm not entirely sure how to prove this.

Nadler: What if you use the universal property?

Me: We can use the fact that in, say, the homotopy category of CW complexes, $K(G, 1)$ represents the functor sending a space X to the principal G-bundles on X, which are G-covers since G is discrete here, and then $K(G, 1)$ is unique in the homotopy category of CW complexes by the Yoneda lemma.

Nadler: Why is $C_n(\mathbb{C})$ a $K(B_n, 1)$?

Me: I worked this out at some point. The idea is to work by induction on n, writing down a sequence of fibrations relating the spaces for different n, and then using the long exact sequence in homotopy of these fibrations to show that all of the higher homotopy vanishes. Here I'm defining B_n to be $\pi_1(C_n(\mathbb{C}))$. I can't remember how to write down the fibrations though.

Wodzicki: Why don't you just identify the universal cover?

Me: I can write down what the universal cover should be in terms of paths but I don't know how I would go about showing that it's contractible.

Nadler: To write down fibrations it looks like you would need an ordering.

Me: Yes, that's what I need. Let $F_n(\mathbb{C})$ denote the configuration space of n distinct ordered points in $\mathbb{C}$; then we have an S_n-cover $F_n(\mathbb{C}) \to C_n(\mathbb{C})$ inducing an isomorphism on higher homotopy, so to prove that $C_n(\mathbb{C})$ is a $K(B_n, 1)$ it suffices to prove that $F_n(\mathbb{C})$ is a $K(P_n, 1)$ where P_n is the pure braid group, the kernel of the natural homomorphism $B_n \to S_n$.

The fibrations I want are now obtained by forgetting the first point. This gives fibrations

$$\mathbb{C} \setminus \{1, 2, ...n - 1\} \to F_n(\mathbb{C}) \to F_{n-1}(\mathbb{C}) \tag{12}$$

with fiber homotopy equivalent to a wedge of $n-1$ circles, which is a $K(F_{n-1}, 1)$. The long exact sequence in homotopy and induction now shows that each $F_n(\mathbb{C})$ has vanishing higher homotopy and that the pure braid groups are iterated extensions of free groups.

Olsson (?): What is your favorite compact manifold?

Me: Can I pick a family of manifolds? Let's go with $\mathbb{CP}^n$.

Nadler: Compute its cohomology.

Me: First we recall that $\mathbb{CP}^n$ admits a cell structure with one cell in each even dimension from 0 to $2n$. It follows that the differentials in the cellular chain complex are trivial, hence the homology of $\mathbb{CP}^n$ is free abelian on cells and the cohomology is dual to that. This gives

$$H^k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, 2, ...2n \\ 0 & \text{otherwise} \end{cases}. \tag{13}$$

Nadler: What about the ring structure?

Me: We can use the fact that on a compact oriented manifold, the cup product is Poincaré dual to transverse intersection of oriented submanifolds. The cohomology $H^{2i}(\mathbb{CP}^n)$ can be taken to be generated by the Poincaré dual of the fundamental class of an inclusion $\mathbb{CP}^{n-i} \to \mathbb{CP}^n$ of a subspace of codimension i, and generically a copy of $\mathbb{CP}^{n-i}$ and a copy of $\mathbb{CP}^{n-j}$ intersect in a copy of $\mathbb{CP}^{n-i-j}$. It follows that

$$H^\bullet(\mathbb{CP}^n) \cong \mathbb{Z}[c]/c^{n+1}. \tag{14}$$

Nadler: What happens to the cohomology when we remove a point?

Me: The obvious guess is that we remove the top cell, so the top homology and cohomology, but nothing else happens. More formally we can use Mayer-Vietoris. Let $X = \mathbb{CP}^n$ and write $X = Y \cup B$ where B is a small ball around the point we removed and Y is the closure of the complement; in particular Y is homotopy equivalent to X minus a point. Now $X \cap B$ is a sphere S^{2n-1}. Let me see if I remember which way the homological vs. the cohomological Mayer-Vietoris sequences go...

Nadler: That's fine. What if we glue together two points?

Me: I think we can use Mayer-Vietoris again.

(note: here I waffle for a bit and eventually realize that my idea requires me to use Mayer-Vietoris twice, which I don't like)

Wodzicki: Why don't you simply observe that gluing together two points is equivalent to attaching them by a path, which is in turn equivalent to taking the wedge sum with a circle?

Nadler: I don't think we're supposed to solve the problem for him!

Me: Oh, of course. That makes everything much simpler.

Nadler: Well, in any case, does $\mathbb{CP}^1$ with two points glued together satisfy Poincaré duality?

Me: The homology and cohomology are both $\mathbb{Z}, \mathbb{Z}, \mathbb{Z}$, but I think capping with the fundamental class doesn't induce an isomorphism. I'm not sure how I would show that, though.

Nadler: What can you say about cup products?

Me: Oh, right. $\mathbb{CP}^1$ with two points glued together is the wedge $S^1 \vee S^2$, so a generator of H^1 has cup square zero, hence the cup product $H^1 \otimes H^1 \to H^2$ is not nondegenerate and Poincaré duality must fail.

Wodzicki: Why does the generator of H^1 have cup square zero?

Me: There's a map $S^1 \vee S^2 \to S^1$ which collapses S^2 to a point, and the generator of H^1 is the pullback of the generator of $H^1(S^1)$ along this map. The cup square of the generator of $H^1(S^1)$ is zero since $H^2(S^2)$ is zero, so by functoriality the conclusion follows.

Homological Algebra

Wodzicki: You have "Chevalley-Eilenberg resolution" written on your syllabus. This is strange; I think you might be mixing up the Chevalley-Eilenberg complex with Cartan-Eilenberg resolutions. In any case, can you define this term for me?

Me: I just mean the following distinguished resolution of the trivial representation k of a Lie algebra $\mathfrak{g}$ by free $U(\mathfrak{g})$-modules:

$$\cdots \to U(\mathfrak{g}) \otimes \Lambda^3(\mathfrak{g}) \to U(\mathfrak{g}) \otimes \Lambda^2(\mathfrak{g}) \to U(\mathfrak{g}) \otimes \mathfrak{g} \to U(\mathfrak{g}) \to k \to 0. \tag{15}$$

Ganor: What are those tensor products over?

Me: Over the ground field k.

Wodzicki: What are the differentials in this resolution?

Me: I don't know them off the top of my head. The first map $U(\mathfrak{g}) \to k$ is the quotient by the ideal generated by $\mathfrak{g}$; said another way, it's the counit map for the Hopf algebra structure on $U(\mathfrak{g})$.

In general, we can define the subsequent maps by describing maps $\Lambda^k(\mathfrak{g}) \to U(\mathfrak{g}) \otimes \Lambda^{k-1}(\mathfrak{g})$ of vector spaces and using the universal property. So the first differential is given by the linear inclusion map $\mathfrak{g} \to U(\mathfrak{g})$, since the resulting map of $U(\mathfrak{g})$-modules surjects onto the kernel of the counit map. The second differential is given by a linear map $\Lambda^2(\mathfrak{g}) \to U(\mathfrak{g}) \otimes \mathfrak{g}$ which I'm guessing is

$$v \wedge w \mapsto v \otimes w - w \otimes v. \tag{16}$$

Wodzicki: That is incorrect.

Nadler: Perhaps it would help to answer the following question first. What is a resolution of the skyscraper sheaf at 0 on $X = \mathbb{C}^2$ as an $\mathcal{O}_X$-module?

Me: You're saying I should write down the Koszul resolution first. That sounds reasonable; I know that the Chevalley-Eilenberg resolution is supposed to be a deformation of the Koszul resolution (note: I didn't say this, but in addition the Chevalley-Eilenberg resolution reduces to the Koszul resolution when $\mathfrak{g}$ is abelian). For V a finite-dimensional vector space the Koszul resolution is a resolution of the trivial module k of $S(V)$ (all elements of V act by zero) by free $S(V)$-modules. It takes the form

$$\cdots \to S(V) \otimes \Lambda^3(V) \to S(V) \otimes \Lambda^2(V) \to S(V) \otimes V \to S(V) \to k \to 0. \tag{17}$$

The maps $S(V) \to k$ and $V \to S(V)$ are the same as before. The map $\Lambda^2(V) \to S(V) \otimes V$ is the map I wrote down above because this generates the kernel of the previous map. But to modify this for the case of Lie algebras we need the map

$$v \wedge w \mapsto v \otimes w - w \otimes v - [v, w]. \tag{18}$$

So that should be the correct next differential in the Chevalley-Eilenberg resolution.