Factorization Algebras Associated to the (2, 0) Theory IV

Kevin Costello
Notes by Qiaochu Yuan

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Last time we saw that 5d $N = 2$ SYM has a twist that looks like

$$\mathrm{Maps}(\mathbb{C}^{2|2}_{\bar\partial} \times \mathbb{R}_{dR}, BG) \tag{1}$$

which has a further A-twist

$$\mathrm{Maps}(\mathbb{C}^{2|1} \times \mathbb{R}, BG)_{dR} \tag{2}$$

where $Z(X)$, for a complex surface X, is the cohomology of instanton moduli space $\mathrm{Bun}_G(X)$, a B-twist

$$\mathrm{Maps}(\mathbb{R}^5_{dR}, BG) \tag{3}$$

which is a 5d analogue of Chern-Simons, and a half-twist

$$\mathrm{Maps}(\mathbb{C}^{1|1}_{\bar\partial} \times \mathbb{R}^3_{dR}, BG). \tag{4}$$

Today we'll discuss how the 6d $(2, 0)$ theory is expected to reduce to these. This theory has a twist which is topological (in a somewhat weak sense) in 4 directions and holomorphic in 2 via some linear algebra involving its SUSY. To do this you start with some supersymmetry Q and you look at the translations in the image of $[Q, -]$.

Let's consider this theory on $\mathbb{R}^4 \times \mathbb{C}^\times$, then reduce on the circle in $\mathbb{C}^\times$. A SUSY argument shows that the result is topological in 5 directions.

Conjecture 0.1. *This is the A-twist of 5d $N = 2$ SYM.*

It follows that if X is a complex surface then $X_{\mathfrak{g}}(X \times S^1)$ is the cohomology $H^\bullet(\mathrm{Bun}_G(X))$.

Now let's reduce on a topological circle, so work on $\mathbb{R}^3 \times S^1 \times \mathbb{C}$. The resulting 5d theory is mixed topological and holomorphic.

Conjecture 0.2. *This is the half-twist of 5d $N = 2$ SYM.*

Note that reducing in topological directions is very nice because there's no need to worry about sizes.

Now let's work on a $\mathbb{C}$ fibration over $\mathbb{R}^3 \times S^1$ with monodromy given by some vector field v. This should give us a deformation of the half-twist involving $v\frac{\partial}{\partial\varepsilon}$. Witten considered the case where $v = z\frac{\partial}{\partial z}$ and says that this gives Khovanov homology. There is a further supercharge $z\frac{\partial}{\partial\varepsilon}$. When $z \neq 0$ the resulting theory is topological, in a way where we count things. When $z = 0$ this is a defect, and Chern-Simons lives on the $\mathbb{R}^3$.

This has been verified in the sense that we can write down Wilson line operators supported at $z = 0$ whose expectation values are Chern-Simons knot invariants. This is joint work with John Francis. In fact we can write down an E_3 algebra whose E_2 category of representations is representations of the quantum group, at least in perturbation theory (writing $q = e^\hbar$ and working in formal power series over $\hbar$).

Here is a guess for what a poor man's version of the 6d $(2, 0)$ theory itself looks like after the twist and in perturbation theory. In the abelian case, fields should be

$$\Omega^\bullet(\mathbb{R}^4) \otimes \Omega^{1,\bullet}(\mathbb{C}) \tag{5}$$

or equivalently locally constant maps from $\mathbb{R}^4$ to holomorphic 1-forms on $\mathbb{C}$. This is a degenerate field theory: the Poisson structure is degenerate. Writing

$$(\Omega^\bullet(\mathbb{R}^4) \otimes \Omega^{1,\bullet}(\mathbb{C}))^\vee = \Omega_c^\vee(\mathbb{R}^4) \otimes \Omega_c^{0,\bullet}(\mathbb{C})[5] \tag{6}$$

we will take two elements α_1, α_2 of this dual and send them to

$$\pi(\alpha_1, \alpha_2) = \int \alpha_1 \partial \alpha_2. \tag{7}$$

In particular, replace $\mathbb{R}^4$ by a compact complex surface X. Then we find the free boson valued in $H^\bullet(X)$. The corresponding vertex algebra is the one considered by Grojnowski and Nakajima structured on the cohomology of the Hilbert schemes

$$\bigoplus H^\bullet(\mathrm{Hilb}^n(X)). \tag{8}$$

Now reducing on S^1 gives

$$X_{\mathbb{R}}(X \times S^1) = \bigoplus H^\bullet(\mathrm{Hilb}^n(X)) \tag{9}$$

and reducing on an elliptic curve E_q gives

$$X_{\mathbb{R}}(X \times E_q) = \bigoplus q^n \chi(\mathrm{Hilb}^n(X)) \tag{10}$$

where the LHS is the Vafa-Witten twist of 4d $N = 4$ and q is the modular parameter. This is the elliptic genus of X, which is the expected answer.

Here is a guess for the nonabelian case. Choose a principal $\mathfrak{sl}_2$-triple $E, F, H \in \mathfrak{g}$. The corresponding weight decomposition gives subalgebras $\mathfrak{n}_-, \mathfrak{b}_-, \mathfrak{n}_+, \mathfrak{b}_+$ such that $E \in \mathfrak{n}_+, F \in \mathfrak{n}_-$. Fields on $\mathbb{R}^4 \times \mathbb{C}$ are locally constant maps from $\mathbb{R}^4$ to the space of G-opers on $\mathbb{C}$, or more explicitly the space of pairs of N_--bundles and a holomorphic connection on a corresponding G-bundle of the form

$$E\, dz + B_-\text{-connection}. \tag{11}$$

We can describe a dgla describing formal deformations of opers, namely

$$\Omega^{0,\bullet}(\mathbb{C}) \otimes \mathfrak{n}_- \xrightarrow{\partial + [E,-]\, dz} \Omega^{1,\bullet}(\mathbb{C}) \otimes \mathfrak{b}_-. \tag{12}$$

The 4d field theory has fields

$$\Omega^\bullet(\mathbb{R}^4) \otimes \text{the above} \tag{13}$$

where the first term is in degree -1 and the second is in degree 0. A slick way to describe the Poisson structure is to use the fact that $\mathrm{Opers}(\Sigma) \to \mathrm{Loc}_G(\Sigma)$ is Lagrangian, and to use a theorem (joint work with Nick Rozenblyum) that Lagrangian things inside symplectic things have a canonical odd Poisson structure. This Poisson structure transgresses to include $\mathbb{R}^4$. Concretely, the dual of the dgla above is

$$\Omega_c^{0,\bullet}(\mathbb{C}) \otimes \mathfrak{b}_+ \xrightarrow{\partial + [E,-]\,dz} \Omega_c^{1,\bullet}(\mathbb{C}) \otimes \mathfrak{n}_+ \tag{14}$$

and the Poisson structure has a constant coefficient part

$$\int \alpha \wedge d\alpha, \alpha \in \Omega_c^{0,\bullet}(\mathbb{C}) \otimes \mathfrak{b}_+ \tag{15}$$

and a linear part which measures the failure of the above to be the cotangent bundle of opers. This is very similar to giving the dgla a Lie bialgebra structure.

As a check, let's look on $\mathbb{R}^2 \times D^2 \times \mathbb{C}$ and reduce along the boundary circle of D^2. We should get a boundary condition for the mixed twist

$$\mathrm{Maps}(\mathbb{R}^3_{dR} \times \mathbb{C}^{1|1}_{\bar{\partial}}, BG) \tag{16}$$

of 5d SYM. We can think of this as an infinite-dimensional Rozansky-Witten theory

$$\mathrm{Maps}(\mathbb{R}^3_{dR}, \mathrm{Higgs}_G(\mathbb{C})) \tag{17}$$

(this is just a tensor-hom adjunction). Gaiotto-Witten show that the boundary condition comes from the Hitchin section $\mathrm{HS}(\mathbb{C})$ in $\mathrm{Higgs}_G(\mathbb{C})$. It follows that local operators in the 6d theory can be identified with functions on

$$\mathrm{Maps}(\mathbb{R}^4_{dR}, \mathrm{HS}(\mathbb{C})). \tag{18}$$

Earlier we said that local operators would be functions on

$$\mathrm{Maps}(\mathbb{R}^4_{dR}, \mathrm{Opers}(\mathbb{C})). \tag{19}$$

There is an ugly fact: $\mathrm{HS}(\mathbb{C}) \cong \mathrm{Opers}(\mathbb{C})$. This doesn't seem to be canonical and depends on extra data. In particular it doesn't work if we replace $\mathbb{C}$ with a Riemann surface.

Why did we get the correct Poisson structure? Theory X on $\mathbb{R}^3 \times S^1 \times \mathbb{C}$ should S^1-equivariantly reduce to the 5d theory on $\mathbb{R}^3 \times \mathbb{C}^{1|1}$. This turns on the $\varepsilon \frac{\partial}{\partial z}$ differential which gets us to a theory on $\mathbb{R}^5_{dR}$.

Working on $\mathbb{R}^2 \times D^2 \times \mathbb{C}$ and reducing S^1-equivariantly, we get a boundary condition for the 5d theory

$$\mathrm{Maps}(\mathbb{R}^3_{dR}, \mathrm{Loc}_G(\mathbb{C}^2)) \tag{20}$$

and the boundary condition is $\mathrm{Opers}(\mathbb{C}) \to \mathrm{Loc}_G(\mathbb{C})$. The compatibility with the Poisson structure in the 5d theory is a sanity check on our Poisson structure in 6d.

Q: it seems like at least some of the topological theories we've written down can be written down in any dimension, or at least any even dimension or something. But I guess the point is that without the relationships to non-topological theories coming from SUSY we don't know anything about the values of these theories.

A: yes. Theory X lives in 6d for a reason.

Beem-Rastelli computed the 6d theory on $\mathbb{R}^4 \times \mathbb{C}$ working equivariantly on $\mathbb{R}^4$. They found a vertex algebra on $\mathbb{C}$ which is a W-algebra obtained by quantizing Opers($\mathbb{C}$).

Suppose we take our model for the 6d theory on $\mathbb{R}^4 \times \mathbb{C}$ and naively reduce to 5d along a topological S^1 by taking the Hochschild homology of local operators. In the abelian case (to simplify notation), we find

$$\Omega^\bullet(\mathbb{R}^3) \otimes \left(\varepsilon\Omega^{1,\bullet}(\mathbb{C}) \to \Omega^{1,\bullet}(\mathbb{C})\right) \tag{21}$$

where ε is a class in $H^1(S^1)$. On the other hand, the 5d theory itself gives

$$\Omega^\bullet(\mathbb{R}^3) \otimes \left(\Omega^{0,\bullet}(\mathbb{C}) \to \Omega^{1,\bullet}(\mathbb{C})\right). \tag{22}$$

So there is a map on the level of fields given by taking the differential ∂. So there is a map from Hochschild homology of the 6d operators to the 5d operators preserving all of the expected structures, including SUSY. This map is not quite an isomorphism (in the abelian case or otherwise) because we miss the kernel of ∂. This is a problem because this kernel contains what we need for Chern-Simons.