Theory X and Geometric Representation Theory III

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If Z is an n-dimensional TFT, we should think of $Z(S^{n-1})$ (where S^{n-1} has a particular Hochschild cohomology framing) as the space of local operators. One reason is that in any n-manifold M we can cut out a small ball at a point; the result turns the manifold into a cobordism with source S^{n-1}, and so the resulting linear functional on $Z(S^{n-1})$ can be evaluated on an element of $Z(S^{n-1})$ to produce a number, which we interpret as (more or less) an expectation value

$$\langle O_x \rangle = \int_{F(M)} O_x(\varphi) e^{iS(\varphi)} \, D\varphi. \tag{1}$$

A reason to call these operators is that they can be inserted into any cobordism $N_1 \to N_2$, which induces a linear map $Z(N_1) \to Z(N_2)$, to produce a cobordism $N_1 \sqcup S^{n-1} \to N_2$, or a family of linear maps

$$Z(N_1) \otimes Z(S^{n-1}) \to Z(N_2) \tag{2}$$

depending linearly on $Z(S^{n-1})$.

Moreover, if P is a codimension-2 manifold, then inserting S^{n-1} into an identity cobordism shows that $Z(S^{n-1})$ naturally acts by endomorphisms on $\mathrm{id}_{Z(P)}$, and so forth. In fact the entire theory is acted on by $Z(S^{n-1})$ in this way.

1 E_n-algebras

To justify this statement we want to think of $Z(S^{n-1})$ as an algebra. In fact it is an E_n-algebra. When $n = 2$ this says that $Z(S^1)$, where S^1 is thought of as an annulus, has an operation for every pairs of pants, or equivalently for every pair of disks cut out of a disk. In general this says that $Z(S^{n-1})$ has an operation for every pair of n-disks cut out of an n-disk. The configuration space of such pairs itself has the homotopy type of S^{n-1}, so if we were working in chain complexes, then passing to homology we get a map

$$H_\bullet(S^{n-1}) \otimes Z(S^{n-1}) \to Z(S^{n-1}). \tag{3}$$

So this gives us two operations, one coming from $H_0(S^{n-1})$ in degree 0, and one coming from $H^{n-1}(S^{n-1})$ in (cohomological) degree $1 - n$. These operations satisfy relations telling us that $Z(S^{n-1})$ is a graded commutative algebra ($n \geq 2$) with a Poisson bracket of degree $1 - n$. When $n = 2$ this is a Gerstenhaber algebra.

In general $Z(S^{n-1})$ has an action of a certain topological operad E_n whose homology is a variant of the Poisson operad P_n. Kontsevich formality asserts that chains on E_n is quasi-isomorphic to P_n.

Q: is this for a framed TFT?

A: right. For an oriented TFT we have more structure because we can rotate the sphere; we get n-disk algebras, and in particular when $n = 2$ we get BV algebras.

In particular, inserting local operators turns $Z(N^{n-1})$ into a module over $Z(S^{n-1})$ as an E_n-algebra.

As before, let $M_Z = \operatorname{Spec} Z(S^{n-1})$ be the moduli of vacua. Then the entire field theory lives over M_Z in a suitable sense. But what is this thing? Here is a silly definition.

Definition The category of affine E_n-schemes is the opposite of the category of E_n-algebras.

A better but more suspicious thing to say is that, by formality, $Z(S^{n-1})$ is in particular a commutative differential graded algebra, hence M_Z is a (derived) affine scheme. The Poisson bracket equips M_Z with the structure of a (derived) Poisson scheme. We will really only keep track of the parity of the degree; in particular, when n is odd we'll think of M_Z as a Poisson scheme. The claim is that this explains the appearance of Poisson brackets in Costello's talk yesterday.

Q: can we localize by inverting an element of $Z(S^{n-1})$?

A: these things are mildly noncommutative so we should be careful. We should think harder and use factorization homology.

2 The B-model

Let M be a variety. We want to study the 2d sigma model of maps into M. The fields on a manifold X will be locally constant maps $X \to M$, denoted $[X, M]$. This is a derived algebro-geometric object, and $Z(X)$ will be a linearization of this. We already saw an example of this where M was a stack pt/G and $[X, M]$ was principal G-bundles on X. We will take the linearization so that

$$Z(\mathrm{pt}) = \mathrm{Coh}(M) \tag{4}$$

is the dg category of coherent sheaves on M. The result is always 1-dualizable. It is 2-dualizable iff M is smooth and proper (e.g. projective). It is orientable if M is Calabi-Yau.

Now $F(S^1)$ is the loop space $LM = [S^1, M]$ in some derived sense. What is this? We should think of S^1 combinatorially as two dots and two arrows. The two dots give us two points in M, so we start with $M \times M$. The two arrows equate the two copies of M twice; we can think of this as $M \times_{M \times M} M$, or as the self-intersection of the diagonal $\Delta : M \to M \times M$. The claim is that

$$F(S^1) = [S^1, M] = LM = T_M[-1] \tag{5}$$

is the shifted tangent bundle of M, and the space of functions on it is

$$Z(S^1) = HH_\bullet(\mathrm{Coh}(M)) = O(LM) = \Gamma(O_M \otimes^L_{O_M \otimes O_M} O_M) = \Omega^\bullet(M) = H^{\bullet,\bullet}(M) \tag{6}$$

is differential forms, and this is a version of the Hochschild-Kostant-Rosenberg theorem. With the Hochschild cohomology framing,

$$Z(S^1) = HH^\bullet(\mathrm{Coh}(M)) = \Gamma(\Lambda^\bullet(T)) \tag{7}$$

is polyvector fields, and this is another version of the Hochschild-Kostant-Rosenberg theorem. This is a commutative dga, but it also has a bracket, the Schouten-Nijenhius bracket, which gives it its E_2-algebra structure; this is a version of the Deligne conjecture.

If we have an oriented TFT, so M is Calabi-Yau, then we can evaluate Z on a 2-sphere, and we get

$$Z(S^2) = \mathrm{vol}(M). \tag{8}$$

We also have that $Z(T^2)$ is the dimension of $Z(S^1)$, or

$$Z(T^2) = \chi(\oplus H^{p,q}) \tag{9}$$

This is a consistency check on some claims about reductions of Theory X: $X[S^2 \times T^2 \times \Sigma]$ should be the volume of $\mathrm{Loc}_G(\Sigma)$, and we can get this from reducing $X[\Sigma \times T^2]$ (geometric Langlands B-model on $\mathrm{Hitch}_G(\Sigma)$) on S^2 or from reducing $S^2 \times T^2$ (Yang-Mills-Higgs) on Σ.

Now, as the free loop space, $Z(S^1) = LM = [S^1, M]$ admits an action of S^1. How should we interpret this? Homologically it looks like an action of the group algebra of chains

$$C_\bullet(S^1) \otimes O(LM) \to O(LM). \tag{10}$$

Passing to homology, this gives an odd vector field on LM, or a derivation on differential forms. There is an obvious candidate, which is the de Rham differential. Taking (derived / homotopy) invariants of this vector field will give us

$$Z(S^1)^{S_1} = HC_-(M) = \Gamma(BS^1, Z(S^1)). \tag{11}$$

Here HC_- denotes negative cyclic homology. The third description of taking invariants tells us that we get extra structure, namely an action of cochains $C^\bullet(BS^1) \cong \mathbb{C}[\varepsilon]$ where $|\varepsilon| = 2$. Setting $\varepsilon = 0$ gives $Z(S^1)$, whereas setting $\varepsilon = 1$ gives de Rham cohomology. But setting $\varepsilon = 1$ makes no sense because ε has mass, or more formally because it has a nontrivial grading. We really have a family over the line, where at 0 we get differential forms and at nonzero values we get de Rham cohomology.

From here what we can do is to invert ε. This has the effect of only letting us recover de Rham cohomology with its $\mathbb{Z}_2$-grading. This is periodic cyclic homology $HP(M)$.

Q: in what sense is ε a mass parameter?

A: most simply, it has a nonzero grading. The mass parameters that appear in the other talks can be interpreted in terms of equivariant cohomologies as above, where we were doing S^1-equivariant cohomology, and in particular taking various limits in ε correspond to equivariant localization and so forth.