Theory X and Geometric Representation Theory II

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Last time we were discussing the cobordism hypothesis. If O is an object in a symmetric monoidal (∞, n)-category $(C, \otimes)$, there is a list of conditions called k-dualizability, $k \leq n$, which allows us to define

$$Z(M^k) = \int_{M^k} O \tag{1}$$

where Z is a putative extended topological field theory with $Z(\text{pt}) = O$. Here M^k is an n-framed k-manifold, or equivalently k-manifold together with a trivialization of the tangent bundle of $M^k \times \mathbb{R}^{n-k}$.

If we want to define the TFT on manifolds with less structure, such as an orientation or a spin structure, we do the following. The cobordism hypothesis equips the space of n-dualizable objects in C with an action of $O(n)$ (by change of framing), and if $G \to O(n)$ is a topological group mapping to $O(n)$ (for example $SO(n)$ or $\text{Spin}(n)$), then allowing our TFT to be defined on manifolds with G-structure is equivalent to equipping the object O with the data of a homotopy fixed point for the induced action of G.

Today's goal is to define a 2d TFT $X[T^2 \times S^2]$ (dimensional reduction of $N = 2$ SYM on T^2, or of $N = 4$ SYM on S^2). This turns out to be a topological version of 2d Yang-Mills.

1 Dijkgraaf-Witten theory

First let's discuss the theory for a finite group G. There is a resulting 2d TFT Z_G describing G-gauge theory (Dijkgraaf-Witten theory). The fields are

$$F(M) = \text{Bun}_G(M) \tag{2}$$

where $\text{Bun}_G(M)$ denotes the groupoid of principal G-bundles on M; when M is connected, this is the action groupoid of G acting by conjugation on homomorphisms $\pi_1(M) \to G$ for a choice of basepoint. To a point we'd like to assign a category

$$Z(\text{pt}) = \text{Vect}(F(\mathbb{R}^2)) = \text{Rep}(G). \tag{3}$$

Equivalently, we can think about the group algebra $\mathbb{C}[G]$, as a stand-in for its category of modules. The fact that this defines a 2d TFT is equivalent to saying that $\mathbb{C}[G]$ is finite-dimensional semisimple, or that $\text{Rep}(G)$ is semisimple with finitely many simple objects (this requires that we are working over $\mathbb{C}$).

But Z_G is an oriented field theory: this requires the extra data of $SO(2)$-fixed point structure, which in this case turns out to be the extra data of a trace on $\mathbb{C}[G]$ making it a Frobenius algebra.

To a circle we assign the vector space

$$Z(S^1) = \text{Fun}(F(S^1)) = \mathbb{C}\left[\frac{G}{G}\right] \tag{4}$$

of functions on isomorphism classes of principal G-bundles on S^1, or equivalently of class functions.

To a connected surface Σ we assign the number

$$Z(\Sigma) = |F(\Sigma)| = \frac{|\operatorname{Hom}(\pi_1(\Sigma), G)|}{|G|} \tag{5}$$

of isomorphism classes of principal G-bundles on Σ, but weighted by one over the automorphisms (the groupoid cardinality). More generally there is a push-pull formula describing what we assign to a surface with boundary.

We can think of the assignment to the circle as follows. In the context of 1d oriented TFT, we can assign a vector space V to a point with some orientation and another vector space V^* to a point with the opposite orientation. There are two half-circles we can write down corresponding to an evaluation map

$$\text{ev} : V \otimes V^* \to 1 \tag{6}$$

and a coevaluation map

$$\text{coev} : 1 \to V \otimes V^* \tag{7}$$

realizing V as the dual / adjoint of V^*. In particular, V is finite-dimensional, and the circle gets assigned the composition of the coevaluation and the evaluation, which is $\dim V$.

In our case, suppose Z is a 2d oriented TFT with $Z(\text{pt}) = C = \text{Mod}(A)$. Then we assign a pair of points the category of (A, A)-bimodules. The coevaluation map is the inclusion of the identity bimodule ${}_AA_A$, and the evaluation map is the following trace operation

$$\text{tr}(M) = HH_0(M) = M \otimes_{A \otimes A^{op}} A \tag{8}$$

on bimodules; this is an underived form of Hochschild homology. Hence the circle gets assigned

$$Z(S^1) = \text{tr}(\text{id}_A) = HH_0(A) = A \otimes_{A \otimes A^{op}} A. \tag{9}$$

This should be thought of as the dimension, in an appropriate sense, of $\text{Mod}(A)$. Explicitly, it is the quotient $A/[A, A]$ where $[A, A]$ is the subspace spanned by commutators; the map $A \to A/[A, A]$ is the universal trace on A. This recovers our computation about class functions earlier.

However, there is another interesting framing to look at. With the blackboard framing coming from its inclusion into an annulus, $Z(S^1)$ has a natural multiplication coming from the pair of pants (with a suitable framing) which makes $Z(S^1)$ a commutative algebra.

Another way to think about this S^1 is to think of the trivial cobordism between two copies of the identity functor $\text{Mod}(A) \to \text{Mod}(A)$, then cut out a small hole. This gives a natural action of $Z(S^1)$ by endomorphisms of $\text{id}_{\text{Mod}(A)}$ which is in fact an equivalence; that is,

$$Z(S^1) = \mathrm{End}(\mathrm{id}_{\mathrm{Mod}(A)}) = \mathrm{End}({}_A A_A) = HH^0(A) \tag{10}$$

where HH^0 is underived Hochschild cohomology, or the center of A. Hence we also recover the center of the group algebra, which is a dual description of class functions.

What is the significance of endomorphisms of the identity? It is a ring over which everything is linear. In this case, one basis we can choose for the space of class functions on G is a normalized version of the central idempotents projecting to each irreducible representation of G.

Definition The *moduli of vacua* of Z is $\mathrm{Spec}\, Z(S^1)$.

This is a finite set of points that can be identified with the irreps of G. Hence our field theory breaks up as a field theory over this space (this can also be described as realizing Z_G as a relative field theory). In particular, $Z(\mathrm{pt}) = \mathrm{Rep}(G)$ breaks up as a direct sum of copies of Vect labeled by the irreps of G. Similarly, the invariant attached to a surface breaks up as a sum

$$Z(\Sigma) = \sum_{V \in \hat{G}} Z(\Sigma)_V \tag{11}$$

over the irreps of G. Hence we can refine $Z(\Sigma)$ from a number to a function on the moduli of vacua.

2 Yang-Mills theory

Now let G be a compact connected Lie group. We would like to assign $Z(\mathrm{pt}) = \mathrm{Rep}(G)$, the unitary representations of G. This is a perfectly fine 1-dualizable category. It won't be 2-dualizable, e.g. because it has infinitely many simple objects or because $Z(S^1 \times S^1) = \dim Z(S^1)$ should be the dimension of the space of class functions, which is infinite. So this theory won't make sense on arbitrary 2-manifolds.

But it can make sense as a relative theory. Class functions $Z(S^1)$, as a vector space, can be identified with $\mathbb{C}[T]^W$, or W-invariant functions on a maximal torus. But $Z(S^1)$ also has the sructure of a commutative ring: as a commutative ring, it is more or less a direct sum

$$Z(S^1) = \bigoplus_{V \in \hat{G}} \mathbb{C} \tag{12}$$

of a copy of $\mathbb{C}$ for each irrep of G. This corresponds to a convolution structure on $\mathbb{C}[T]^W$. So the moduli of vacua $M = \mathrm{Spec}\, Z(S^1)$ can be identified with the infinite set of irreducible representations of G, and we can try to break up our field theory over this infinite set.

In particular, $Z(\mathrm{pt}) = \mathrm{Rep}(G)$ looks like vector bundles over M, $Z(S^1)$ looks like functions on M, and $Z(S^1 \times S^1)$ looks like the function with constant value 1 on M. More generally, $Z(\Sigma)$ is a well-defined function on M, and then we can ask whether the sum of this function over M exists.

So this theory is a 2d TFT in a weaker sense than before: it is defined relative to the moduli of vacua M. This is the kind of structure we'll see from Theory X.

$Z(\Sigma)$ is attempting to be the volume of the moduli space of flat principal G-bundles with connection on Σ, or equivalently of stable $\mathbb{G}_{\mathbb{C}}$-bundles. The infinitudes above come from the fact that this doesn't always make sense.

For G complex, $X[T^2 \times S^2]$ is a version of this theory (Yang-Mills-Higgs theory) which assigns to $Z(\text{pt})$ the category of algebraic representations of G and which assigns to $Z(S^1)$ a version of class functions

$$\mathbb{C}[G]^G \cong \mathbb{C}[T]^W \cong \bigoplus_{\Lambda^+} \mathbb{C} \tag{13}$$

and $Z(\Sigma)$ is attempting to be the volume of the Hitchin moduli space $\text{Hitch}_G(\Sigma)$. This won't always make sense, but we can write it as a sum of contributions that make sense.

This presentation of Yang-Mills-Higgs theory predicts some extra symmetries which we will explain. A fancy name for this category is the spherical Hecke category, and it will appear when discussing geometric Langlands on S^2. There is a subtle $\text{SL}_2(\mathbb{Z})$ action involving switching G and its Langlands dual; this comes from some subtleties involving extra data we've neglected to discuss.

Example Let $G = \text{SL}_n(\mathbb{C})$. Then class functions can be identified with functions on T/W, which we can write as $\text{Sym}^{n-1}(\mathbb{C}^\times)$, or the space of configurations of $n-1$ particles living on $\mathbb{C}^\times$. The L^2 space of this space admits a natural action of $Z(S^1)$, which breaks up into a bunch of commuting generators, so we have written down a quantum integrable system in some sense. This seems a little silly, but it is possible to realize more complicated quantum integrable systems basically by this construction, e.g. Calogero-Moser.

In general, we will define the moduli of vacua of a TFT to be the spectrum of local operators, which for n dimensions will turn out to be $\text{Spec}\, Z(S^{n-1})$. When $n = 4$ this will be closer to what physicists mean by moduli of vacua.