Hitchin Systems in Supersymmetric Field Theory I

Andrew Neitzke
Notes by Qiaochu Yuan

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Fix a compact Lie group G, although we'll be a little sloppy about compact forms of the same Lie algebra $\mathfrak{g}$. Also fix a compact Riemann surface C. From this data we can write down a famous space $M_G[C]$, namely Hitchin's integrable systems. It is a complex integrable system: it is a torus fibration over some base B (with base and fiber the same dimension) and it is hyperkähler, so has a family of holomorphic symplectic structures labeled by a parameter $\xi \in \mathbb{CP}^1$.

The main point of today's talk is that $M_G[C]$ is part of a much bigger structure, namely a supersymmetric QFT. It ends up being the 3d QFT given by Theory X (for $\mathfrak{g}$) compactified on $C \times S^1$. We can use this to see new things about $M_G[C]$; for example we get distinguished local coordinates $\chi_\gamma(\xi)$.

The base B has a singular part B_{disc} and a regular part $B_{reg} = B \setminus B_{disc}$. There is a local system of lattices $\Gamma \to B_{reg}$ given by the homology of the fibers, and the distinguished local coordinates around a point of M indexed by $\gamma \in \Gamma$. Among other things, this lets us give a more explicit construction of the hyperkähler structure.

As before, fix a Lie algebra $\mathfrak{g}$ of ADE type. It's believed that there is a 6-dimensional interacting QFT $X_\mathfrak{g}$. (It's also believed that there are no interacting QFTs in higher dimensions.) Some things that are believed about this theory:

1. It has $N = (2, 0)$ supersymmetry. For starters, it should be invariant under the Poincaré group $\mathrm{Iso}(\mathbb{R}^{5,1})$, but this symmetry admits a super extension (e.g. from a Lie algebra symmetry to a super Lie algebra symmetry).

2. It has no parameters (e.g. coupling constants). In particular it is scale invariant, and in fact conformally invariant (so in fact invariant under a superconformal Lie algebra).

3. It does not have a Lagrangian description (or else there would of course be coupling constants). Hence it can't be described in terms of path integrals over a space of fields. All known constructions use string theory.

4. It has local operators ϕ_F^i. Here $1 \le i \le 5$ is an integer and F is an invariant polynomial on $\mathfrak{g}$. The 5 here is $11-6$. The 11 here comes from M-theory, which is an 11-dimensional theory, and the 5 comes from the defining representation of an extra $\mathrm{SO}(5)$ action coming from looking at a copy of $\mathbb{R}^6$ inside a bigger $\mathbb{R}^{11}$.

5. It has surface operators labeled by irreducible representations of $\mathfrak{g}$ together with a vector in $\mathbb{R}^5$ (the same defining representation of $\mathrm{SO}(5)$).

6. It has 4-dimensional operators labeled by (roughly) nilpotent orbits in the complexification $\mathfrak{g}_\mathbb{C}$ together with a 2-plane in $\mathbb{R}^5$.

So, roughly, the Euclidean version of $X_\mathfrak{g}$ assigns a number to a compact Riemannian 6-manifold. However, we can also insert local operators at points, surface operators on surfaces, and 4-dimensional operators on 4-manifolds in such a 6-manifold, and this affects the resulting number. $X_\mathfrak{g}$ should also assign vector spaces to 5-manifolds, something like categories to 4-manifolds, etc., and we can also insert operators here. Very little is known about how to calculate any of this.

However, we can compactify. In d dimensions, starting from a QFT T, instead of working on $\mathbb{R}^{d-1,1}$, we work on $Y \times \mathbb{R}^{d-k-1,1}$ where Y is a compact Riemannian k-manifold such that the local physics is the same (whatever that means). There is some choice about how to modify the theory to do this (e.g. adding terms to the Lagrangian involving the curvature of Y, for a theory with a Lagrangian description). The resulting theory $T[Y]$ has less symmetry and hence will usually have less supersymmetry, if T had some supersymmetry.

In particular we can try to compactify $X_{\mathfrak{g}}$ on a circle S^1 of length R. If we consider $T[S^1]$ at energy scales much less than $\frac{1}{R}$ (whatever that means), the physics becomes that of 5-dimensional supersymmetric Yang-Mills theory with gauge group G. This theory has a coupling constant

$$g_{YM}^2 \sim R. \tag{1}$$

Question from the audience: why did physicists miss 5d SUSY Yang-Mills when they thought that there weren't interacting QFTs above dimension 4?

Answer: 5d SUSY Yang-Mills is well-defined as an effective field theory, below a certain energy scale (as above). It is not obviously well-defined at arbitrarily high energies, which is a desirable property of an actual as opposed to an effective field theory. Effective field theories are relatively easy to write down: we can just write down some Lagrangian.

5d SUSY Yang-Mills has an honest Lagrangian description. Its fields are

1. a G-bundle with connection P,
2. sections Φ^i of $\text{Ad}(P), i = 1, \ldots, 5$,
3. fermions.

The Lagrangian density is

$$\mathcal{L} = \frac{1}{8\pi^2 R} \text{tr} \left(F_D \wedge *F_D + R^2 \sum_{i=1}^{5} D\Phi^i \wedge *D\Phi^i + \text{fermions} \right) \tag{2}$$

where $D = d + A$ is the covariant derivative attached to the connection and F_D is the curvature. This theory is not free in the sense that $F_D = dA + A \wedge A$ is not linear in A.

The observables of $X_{\mathfrak{g}}[S^1]$ can be mapped to observables of 5d SYM. For example, the local operators $\phi_F^i(\vec{x}, x^6), \vec{x} \in \mathbb{R}^{4,1}, x^6 \in R$ become operators $F(\Phi^i(\vec{x}))$, and the surface operators labeled by a representation R become Wilson line operators given by a trace of holonomy in the bundle with connection associated to R.

We haven't really described what it means to say that $X_{\mathfrak{g}}[S^1]$ reduces to 5d SYM at low energies much less than $\frac{1}{R}$. The idea is that if we compare the Fourier transforms of the high-energy (ultraviolet) and low-energy (infrared) operators in the $\vec{x}$ directions, then their low-wavelength Fourier modes agree up to corrections of order $\|\vec{k}\|R$ ($\vec{k}$ the Fourier variable).

Where does the Hitchin space come into this story?

We shouldn't think of passing to the low-energy / IR limit as a poor man's substitute for the high-energy / UV theory. Consider the following analogy: suppose you want to study a bowl of water. One thing you could do is write down the QCD Lagrangian to describe every particle in it, and then do some massive computations to describe the bowl. This is bad. What you want to do instead is to figure out what happens at the scales you care about, e.g. water waves. In other words what you want is an IR description. Seiberg-Witten theory can be thought of as the IR version of Donaldson theory in this regard, and that's why it's easier to work with.

Question from the audience: what about operators in odd dimensions?
Answer: the operators in even dimensions were easier to discover in some sense because they protect some supersymmetry. We don't know any in odd dimensions.

Question from the audience: why is it believed that there aren't any interacting theories in higher dimensions?
Answer: this is related to Nahm's theorem, which classifies the possible superconformal Lie algebras that could be involved.

Question: M-theory is 11-dimensional. Why doesn't that count as an interacting QFT?
Answer: it is not a QFT! It is not defined on manifolds with a fixed metric, but is a gravitational theory, so the metric varies.

Question: why can't we go beyond ADE?
Answer: the construction we have involves ADE singularities.