

# 274 Curves on Surfaces, Lecture 5

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## 5 Ideal polygons

Previously we discussed three models of the hyperbolic plane: the Poincaré disk, the upper half-plane, and the hyperboloid. We did not describe geodesics in the hyperboloid model, but they are given by intersections of planes through the origin.



Figure 1: A plane giving a geodesic on the hyperboloid.

This is precisely the same way we obtain geodesics on the sphere.

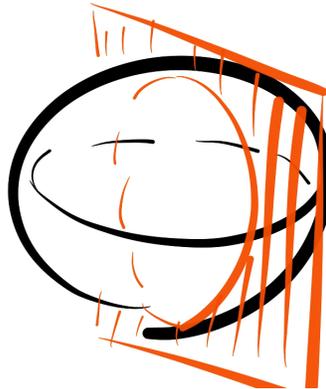


Figure 2: A plane giving a geodesic on the sphere.

To relate the models, in the hyperboloid model we can look at the hyperboloid from the tip of the other hyperboloid, and we get the disk model. (For an observer at the origin, the corresponding model is the Klein disk model, which is not conformal. Its geodesics are straight chords. This is analogous to how stereographic projection is conformal on the sphere, but projection from a random point is not.)

$\mathbb{H}^2$  has an *ideal boundary*. For example, in the disk model the boundary of the disk is this boundary. It is often convenient to complete  $\mathbb{H}^2$  to include this boundary, obtaining  $\overline{\mathbb{H}^2}$ . The metric does not extend, but we still get a topological space.  $\overline{\mathbb{H}^2}$  is obtained from  $\mathbb{H}$  by adding *ideal points*, or equivalence classes of geodesic rays. Most pairs of geodesics get exponentially far away from each other, but sometimes two geodesic rays will approach each other; intuitively they are approaching the same point of the boundary. (We do not get this behavior in the Euclidean plane, where two geodesic rays can be parallel.)

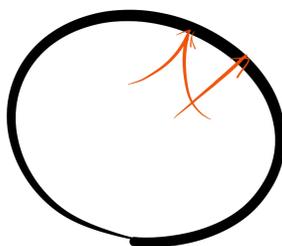


Figure 3: Some equivalent and inequivalent geodesic rays.

A *triangle* in  $\mathbb{H}^2$  can be constructed from three geodesics. If  $\alpha, \beta, \gamma$  are the corresponding angles of the triangle, the claim was that the area of the triangle is  $\pi - (\alpha + \beta + \gamma)$ . This is a corollary of the Gauss-Bonnet theorem, one intuitive statement of which is that the total turning of a curve bounding a disk  $D$  in a Riemannian manifold is

$$2\pi - \int_D K dA \tag{1}$$

where  $K$  is the curvature.

**Exercise 5.1.** *Measure the curvature of some surface. (A leaf? An orange peel? Can you find a leaf with positive curvature?)*

The Gauss-Bonnet theorem can be used to compute the area of a sphere. Considering a great circle on a sphere of radius 1, we compute that  $2\pi$  is half the surface area of such a sphere, hence the total surface area is  $4\pi$ . When applied to a

geodesic triangle, the total turning of a geodesic triangle with angles  $\alpha, \beta, \gamma$  is given by  $(\pi - \alpha) + (\pi - \beta) + (\pi - \gamma)$ , which is  $2\pi$  plus the area of the triangle.

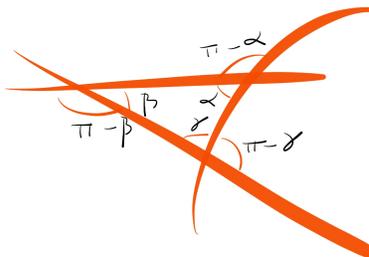


Figure 4: The turning angles of a hyperbolic triangle.

In particular, this area is positive, so  $\alpha + \beta + \gamma < \pi$ . In addition, we conclude that the area of a geodesic triangle is always strictly less than  $\pi$ . As the three vertices approach the ideal boundary, the corresponding angles become very small, so the area approaches  $\pi$ . In the limiting case, the three vertices are on the ideal boundary, and we get an *ideal triangle*. Since geodesics meet the boundary at right angles, the angles in an ideal triangle are all equal to 0, so an ideal triangle has area  $\pi$ .

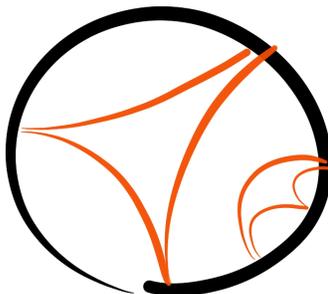


Figure 5: Some ideal triangles.

Ideal triangles turn out to be simpler and easier to understand than ordinary triangles. For example:

**Proposition 5.2.** *Any two ideal triangles are related by an orientation-preserving isometry.*

*Proof.* We will work in the upper half-plane model. In this model the ideal boundary is  $\mathbb{RP}^1$  acted on by the isometry group  $\mathrm{PSL}_2(\mathbb{R})$ . We know that  $\mathrm{PGL}_2(\mathbb{R})$  acts triply transitively. A short way to see this is to try to send any three points  $a, b, c$  to  $0, 1, \infty$ . This gives, uniquely,

$$x \mapsto \frac{(x-a)(b-c)}{(x-c)(b-a)}. \quad (2)$$

By choosing an appropriate ordering of  $a, b, c$  we can arrange for this isometry to lie in  $\mathrm{PSL}_2(\mathbb{R})$ .  $\square$

It follows that the *moduli space of ideal triangles* (the space of ideal triangles up to orientation-preserving isometry) is a point. What about *ideal quadrilaterals*? All of the interior angles are still 0, so by Gauss-Bonnet any ideal quadrilateral has area  $2\pi$ ; alternately, an ideal quadrilateral is the union of two ideal triangles. Now not all ideal quadrilaterals are equivalent. By the previous proof, we can send three of the vertices to  $0, 1, \infty$ , and the fourth point cannot be moved because the pointwise stabilizer of  $0, 1, \infty$  is the identity.

This gives an invariant of ideal quadrilaterals: take three of the points  $a, b, c$  to  $0, 1, \infty$  and look at where the fourth point  $d$  goes. This is

$$d \mapsto \frac{(d-a)(b-c)}{(d-c)(b-a)} \quad (3)$$

or the *cross-ratio* of  $a, b, c, d$ . This can be used to describe the moduli space of ideal quadrilaterals. In general, we expect the moduli space of ideal  $n$ -gons to have dimension  $n - 3$ . We can think of the  $n$  as the number of parameters describing vertices and the 3 as the dimension of  $\mathrm{PSL}_2(\mathbb{R})$ .

There is something funny going on here. If we think of an ideal quadrilateral as being composed of two ideal triangles, we can slide the ideal triangles against each other by moving the opposite vertex, and this does not change the triangles themselves up to isometry. Ordinary Euclidean triangles do not have this property.

We would like to make ideal geometry look more like ordinary geometry. One way to do this would be to associate a number to an edge of an ideal polygon, which we want to think of as its length. It has infinite length in the hyperbolic metric, so we will have to be clever. First, what do circles (points equidistant to a given point) look like in the hyperbolic plane?

**Proposition 5.3.** *Circles in the disk model or half-plane model are still ordinary circles.*

*Proof.* In the disk model, take the center of the circle to be the origin. Isometries include rotational symmetries, so the circle must be an ordinary Euclidean circle.

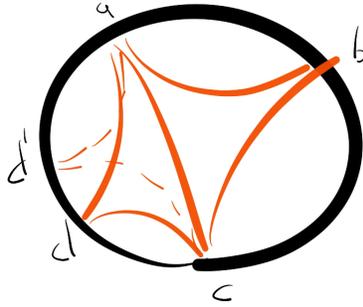


Figure 6: Sliding triangles in an ideal quadrilateral.

Conformal maps preserve Euclidean circles, so the same is true for any center, and the same is true for the half-plane.  $\square$

However, the Euclidean center is usually not the center. As a circle gets closer to the boundary, its center also gets closer to the boundary.

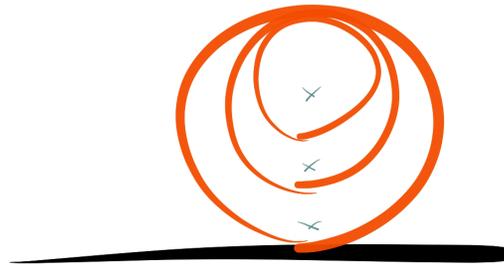


Figure 7: Centers getting closer to the boundary.

As we send the circle to the boundary and send the radius to  $\infty$  so that the circle continues to pass through a given point, we obtain a *horocycle*. In the disk and half-plane models these look like Euclidean circles tangent to the boundary (and in the latter case this includes circles of infinite radius). The corresponding construction in the Euclidean plane gives a line, but horocycles are not geodesics in the hyperbolic plane. They can be thought of as circles centered at an ideal point.

One property of Euclidean circles is that a geodesic through the center is perpendicular to the boundary. This is still true for hyperbolic circles. It is also true that a horocycle is perpendicular to any geodesic approaching its ideal point.

We now know how to give meaning to a circle tangent to the boundary.

**Exercise 5.4.** *What is the meaning of a circle intersecting the boundary at an arbitrary angle?*

General hint: find the right model and put the interesting point in a convenient place.

We now want to describe *decorated* ideal polygons, which are given by an ideal polygon together with a choice of horocycle around each vertex. This gives us a way to measure length: we can take the length of an edge of a decorated ideal polygon to be the oriented distance between the horocycles around the corresponding vertices (so it is negative if the horocycles overlap).

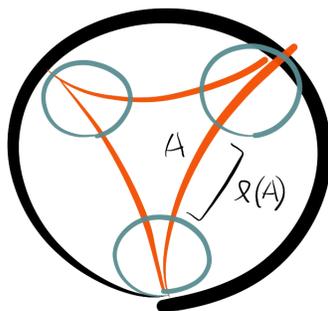


Figure 8: A decorated triangle and a length.

**Lemma 5.5.** *Any three choices  $\ell(A), \ell(B), \ell(C) \in \mathbb{R}$  of lengths is realizable by a unique decorated ideal triangle (up to isometry).*

*Proof.* We first consider the case where all three lengths are equal to 0. This means that the horocycles are all tangent to each other as well as to the ideal boundary. The corresponding configuration of four circles is unique up to fractional linear transformations.

Alternately, we can work with an ideal triangle in the upper half-plane with one of the vertices at infinity. One of the horocycles is then a line parallel to the  $x$ -axis, and the other two horocycles are uniquely determined by this.



Figure 9: Four tangent circles.

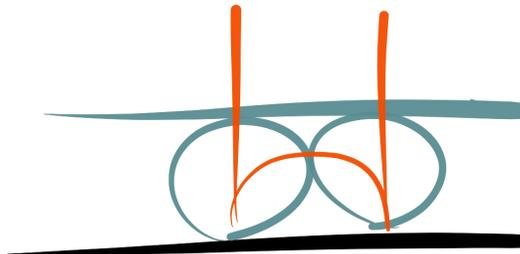


Figure 10: Three tangent horocycles in the upper half-plane.

If from here we move one of the horocycles by some distance along one geodesic, we move it by the same distance along the other geodesic. If we move all three horocycles by distances  $a, b, c$ , we get distances  $a + b, b + c, c + a$ , and this linear transformation is invertible.  $\square$

We can also decorate ideal quadrilaterals and assign lengths to their edges. This is equivalent to gluing decorated ideal triangles with matching lengths, and now there is no sliding provided that all lengths are fixed. This leads to a parameterization of decorated ideal polygons.

**Theorem 5.6.** *Given a decorated ideal polygon and a triangulation, the lengths between horocycles of edges determine the polygon (up to isometry).*

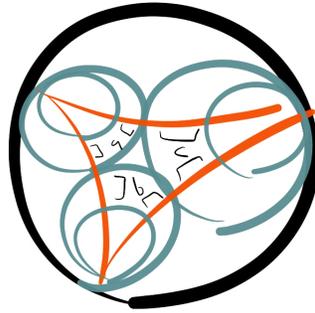


Figure 11: The general case.

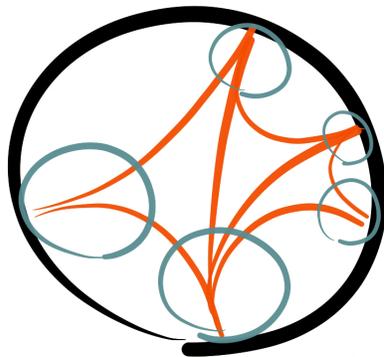


Figure 12: A decorated and triangulated pentagon.

This gives us some number of parameters describing a decorated ideal  $n$ -gon. First there are  $n$  edges to consider. A triangulation adds  $n - 3$  edges, so we get  $2n - 3$  parameters. The moduli space of ideal  $n$ -gons has dimension  $n - 3$ , and the difference between these are the  $n$  parameters describing the horocycles.

Next time we will discuss how these parameters change under change of triangulations.