3 More about mapping class groups

Some background reading:


2. *Papers on Group Theory and Topology*, Dehn (introduction of Dehn-Thurston coordinates). Alex will be talking about this paper.


Let $S$ be a surface with $\chi(S) < 0$ and $x$ a marked point. The Birman exact sequence is a short exact sequence

$$1 \rightarrow \pi_1(S,x) \rightarrow \text{MCG}(S,x) \rightarrow \text{MCG}(S) \rightarrow 1.$$  \hfill (1)

It can be iterated; for example, we can write down a short exact sequence

$$1 \rightarrow \pi_1(S \setminus 5 \text{ pts}) \rightarrow \text{MCG}(S,6 \text{ pts}) \rightarrow \text{MCG}(S,5 \text{ pts}) \rightarrow 1.$$  \hfill (2)

The map from $\pi_1(S,x)$ is the *point-dragging map* or *push map*. Given a curve $\gamma \in \pi_1(S,x)$, we want to send it to an element of $\text{MCG}(S,x)$ which is trivial in $\text{MCG}(S)$, hence it needs to be isotopic to the identity. It suffices to describe this isotopy. This isotopy will drag a neighborhood of the marked point $x$ along $\gamma$ and will be trivial outside a neighborhood of $\gamma$.

![Figure 1: A marked point being pushed along a closed curve.](image)
Why does this describe the entire kernel of the map $\text{MCG}(S, x) \to \text{MCG}(S)$? The general picture is as follows. For $X$ any smooth manifold and $x \in X$ a marked point, there is a fibration

$$\text{Diff}^+(X, x) \hookrightarrow \text{Diff}^+(X) \twoheadrightarrow X$$

(3)

where the map $\text{Diff}^+(X) \to X$ sends a diffeomorphism to the image of $x$. (A fibration behaves like a fiber bundle. The crucial property is a lifting property: in particular, any path in $X$ lifts to a path in $\text{Diff}^+(X)$.) This fibration induces a long exact sequence in homotopy

$$\cdots \pi_1(\text{Diff}^+(X)) \to \pi_1(X) \to \pi_0(\text{Diff}^+(X, x)) \to \pi_0(\text{Diff}^+(X)) \to \pi_0(X).$$

(4)

But $\pi_0(\text{Diff}^+(X)) = \text{MCG}(X)$ and $\pi_0(\text{Diff}^+(X, x)) = \text{MCG}^+(X, x)$, and $\pi_0(X)$ is a point when $X$ is connected. The next term in the long exact sequence is a map $\pi_1(X) \to \pi_0(\text{Diff}^+(X, x))$.

**Theorem 3.1.** (*Hamstrom*) Let $S$ be a surface with $\chi(S) < 0$. Then $\pi_1(\text{Diff}^+(X))$ is trivial. In fact, the connected component of the identity in $\text{Diff}^+(X)$ is contractible.

This is an aspect of hyperbolic geometry. The same is true for higher-dimensional hyperbolic manifolds; this is an aspect of Mostow rigidity. (But Mostow rigidity is false for hyperbolic surfaces.)

What happens when $S = T^2$? We claimed that the map $\text{MCG}(T^2, x) \to \text{MCG}(T^2)$ is an isomorphism. The long exact sequence ends

$$\cdots \pi_1(\text{Diff}^+(T^2)) \to \pi_1(T^2) \to \text{MC}(T^2, x) \to \text{MCG}(T^2) \to 1$$

(5)

so the map $\pi_1(T^2) \to \text{MCG}(T^2, x)$ needs to be trivial. There is a map $T^2 \to \text{Diff}_0(T^2)$ given by $T^2$ acting on itself by translation, and it is a difficult theorem that this is a homotopy equivalence. (This can be proven by removing a point, which makes the Euler characteristic $-1$, and applying the big theorem above.) Consequently

$$\pi_1(\text{Diff}^+(T^2)) = \pi_1(\text{Diff}^0(T^2)) \cong \pi_1(T^2).$$

(6)

Similarly, $T^2$ admits an action by affine linear maps, and this is a homotopy equivalence to $\text{Diff}(T^2)$.

In summary, the end of the long exact sequence looks like
$\pi_1(\text{Diff}^+(T^2, x)) \rightarrow \pi_1(\text{Diff}^+(T^2)) \rightarrow \pi_1(T^2) \rightarrow \text{MCG}(T^2, x) \rightarrow \text{MCG}(T^2) \rightarrow 1$

where $\cong$ denotes an isomorphism.

More generally, if $G$ is a connected Lie group, we get a map $G \rightarrow \text{Diff}_0(G)$ coming from the action of $G$ on itself by translation, and we also get a map in the other direction coming from evaluation. This is not a homotopy equivalence in general. When $G = \text{SU}(2)$ we know that $\text{SU}(2) \cong S^3$, and $\text{Diff}^+(S^3)$ is homotopy equivalent to $\text{SO}(4)$ (the Smale conjecture, proved by Hatcher).

Recall that last time we skewered a torus (quotiented it by the central element $-I$ in $\text{MCG}(T^2) \cong \text{SL}_2(\mathbb{Z})$) to obtain a double cover $T^2 \rightarrow S^2$ branched at 4 points. The claim was that this showed

$$\text{MCG}(S^2, 1 \text{ pt}, 3 \text{ pts}) \cong \text{PSL}_2(\mathbb{Z}).$$

(The 1 point is the identity in $T^2$ regarded as a group and the 3 points are the non-identity points of order 2.)

What is the mapping class group of $S^2$ fixing four points pointwise? This is the congruence subgroup $\Gamma(2)$, which consists of the image of the kernel of the map $\text{SL}_2(\mathbb{Z}) \rightarrow \text{SL}_2(\mathbb{Z}/2\mathbb{Z})$ in $\text{PSL}_2(\mathbb{Z})$. It is in fact the free group $\mathbb{Z} \ast \mathbb{Z}$ on two generators.

The relationship to the braid group $B_3$ comes from the map

$$(D^2, 3 \text{ pts}) \rightarrow (S^2, 3 \text{ pts}, 1 \text{ pt})$$

given by identifying the boundary to a point (which becomes the fourth marked point).

Figure 2: A 3-punctured disc getting its boundary identified to form a 4-punctured sphere.
The mapping class group $B_3$ of $(D^2, 3 \text{ pts})$ (fixing the boundary pointwise) has a center generated by \textit{Dehn twist} along a boundary curve. As a braid it is given by the \textit{full twist}. The image of Dehn twist in $\text{MCG}(S^2, 3 \text{ pts}, 1 \text{ pt})$ is trivial (we can untwist). Thus we obtain an exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow \text{MCG}(D^2, 3 \text{ pts}, \partial D^2) \rightarrow \text{MCG}(S^2, 3 \text{ pts}, 1 \text{ pt}) \rightarrow 1$$

showing that $B_3$ is a central extension of $\text{PSL}_2(\mathbb{Z})$.

Recall that before we were permuting curves on the thrice-punctured disc and, looking at Dehn-Thurston coordinates, we saw the Fibonacci numbers appear. This can now be explained as follows. The element of the mapping class group we were applying was a braid in $B^3$ whose image in $\text{PSL}_2(\mathbb{Z})$ is given by the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

\textbf{Exercise 3.2.} \textit{Verify this.}

Hint: look at how the braid group generators lift to the torus. They can be thought of as Dehn twists.

Dehn twists in general look like the following: if $C$ is a simple closed curve on $S$, the Dehn twist $T_C \in \text{MCG}(S)$ rotates an annular neighborhood $[0,1] \times C$ of $C$ as follows: $\{t\} \times C$ is rotated by $2\pi t$. 

\textbf{Figure 3:} A full twist and a half twist.

\textbf{Figure 4:} Some hints.
Question from the audience: is this the same as the push map?
Answer: no. The push map gives a trivial element of the mapping class group. However, there is a relationship. Let $\gamma$ is a simple closed curve and $C_1, C_2$ curves which bound an annular neighborhood of $\gamma$.

**Exercise 3.3.** $\text{Push}(\gamma) = T_{C_1} \circ T_{C_2}^{-1}$.

**Theorem 3.4.** *(Lickorish, …)* Let $S$ be a closed surface. Then $\text{MCG}^+(S)$ is generated by Dehn twists.

Dehn twists cannot generate the mapping class group of a surface with marked points because they cannot permute the marked points. With marked points, the Dehn twists instead generate the pure mapping class group (the subgroup fixing the marked points pointwise).

The basic invariant of an element $M \in \text{SL}_2(\mathbb{Z})$ up to conjugacy is its trace (this determines its characteristic polynomial). If $\text{tr}(M) = 2$ then
Figure 7: Dehn twists and the push map.

\[ M = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \]  
(11)

for some \( x \), and similarly if \( \text{tr}(M) = -2 \) then

\[ M = \begin{bmatrix} -1 & x \\ 0 & -1 \end{bmatrix}. \]  
(12)

These are the **parabolic elements**, and they look like Dehn twists when acting on the torus.

If \( |\text{tr}(M)| > 2 \) then \( M \) has 2 distinct real eigenvalues, and iterating \( M \) we obtain exponential growth. (In the particular case above, \( M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \) and the eigenvalues are \( \phi^2, \varphi^2 \) where \( \phi, \varphi \) are the golden ratios.) These are the **hyperbolic elements**.

If \( |\text{tr}(M)| < 2 \) then \( M \) is in fact torsion. These are the **elliptic** or **periodic** elements. The two basic possibilities are

\[ M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \]  
(13)

and variants.

**Exercise 3.5.** Which braids do these correspond to?