274 Curves on Surfaces, Lecture 22

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Fall 2012

24 Additive categorification of surface cluster algebras (Christof)

Surface cluster algebras can be categorified (additively) as follows. We consider certain triangulated 2-Calabi-Yau $\mathbb{C}$-linear categories C with a cluster tilting object $T = T_1 \oplus ... \oplus T_n$. Triangulated categories generalize derived categories: their main feature is the existence of a self-equivalence $\Sigma : C \to C$ and a collection of distinguished triangles

$$X \to Y \to Z \to \Sigma X \tag{1}$$

satisfying various axioms and generalizing exact sequences. One of these is that for every morphism $X \xrightarrow{f} Y$ there exists a distinguished triangle

$$X \xrightarrow{f} Y \to Z \to \Sigma X. \tag{2}$$

2-Calabi-Yau means that there is a natural isomorphism

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(Y, \Sigma^2 X)^* \tag{3}$$

where * denotes the linear dual. In particular,

$$\mathrm{Hom}(X, \Sigma Y) \cong \mathrm{Hom}(Y, \Sigma X)^*. \tag{4}$$

Cluster tilting means that for every X we have

$$\mathrm{Hom}(T, \Sigma X) = 0 \Leftrightarrow X \text{ is direct summand of } T. \tag{5}$$

In particular, there is a functor

$$F : C \ni X \mapsto \mathrm{Hom}(T, \Sigma X) \in \mathrm{End}(T)^{op}\text{-Mod} \tag{6}$$

whose kernel is given by morphisms which factor through T.

The above conditions imply that every Hom-space is finite-dimensional. In particular, $\mathrm{End}(T)^{op}$ is finite-dimensional. It can be written as $\mathbb{C}[Q]/I$ where $\mathbb{C}[Q]$ is the path algebra of a quiver Q and I is an admissible ideal. This quiver Q is canonically determined by the algebra.

The summands $T = T_1 \oplus ... \oplus T_n$ can be exchanged; for each i there exists $T_i' \neq T_i$ such that $T/T_i \oplus T_i'$ is again a cluster tilting object. The corresponding quiver Q changes according to quiver mutation.

There is a map called the cluster character sending an object Z to a certain sum

$$C^T(Z) = \underline{x}^{g(Z)} \sum_{\underline{\ell}} \chi(\mathrm{Gr}_{\underline{\ell}}^{Q^{op}}(F(Z)))\underline{y}^{\underline{\ell}} \tag{7}$$

over Euler characteristics of quiver Grassmannians which evaluates to a Laurent polynomial in $\mathbb{Z}[x_1^{\pm}, ...x_n^{\pm}]$. Here $g(Z)$ is defined as follows: if

$$\Sigma^{-1}Z \to T_Z^{\underline{b}} \to T_Z^{\underline{a}} \to Z \tag{8}$$

is a distinguished triangle, where $\underline{a} = (a_1, a_2, ...)$ and

$$T_Z^{\underline{a}} = T_1^{a_1} \oplus T_2^{a_2} \oplus ... \tag{9}$$

then

$$g(Z) = \underline{a} - \underline{b}. \tag{10}$$

Furthermore,

$$y_k = \prod_{i=1}^{n} x_i^{|Q(i,k)|-|Q(k,i)|} \tag{11}$$

where $|Q(i,k)|$ is the number of edges from i to k.

Ideally $C^T(Z)$ is contained in the upper cluster algebra associated to Q. The indecomposable rigid objects (those satisfying $\mathrm{Hom}(Z, \Sigma Z) = 0$) give cluster variables.

Quiver Grassmannians are defined as follows. If M is a module over a quiver algebra $\mathbb{C}[Q]$ and $\underline{\ell}$ is a dimension vector, then $\mathrm{Gr}_{\underline{\ell}}^{Q}(M)$ is a projective variety parameterizing submodules of M with dimension vector $\underline{\ell}$. Any projective variety appears as some quiver Grassmannian, so they can be come arbitrarily complicated.

We can recognize the images of rigid objects in $\mathrm{End}(T)^{op}$-Mod as follows. For M a module, take a minimal projective presentation

$$P_n \xrightarrow{\pi} P_0 \to M \to 0 \tag{12}$$

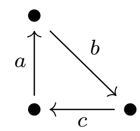
and consider the cokernel

$$\mathrm{Hom}(P_0, M) \xrightarrow{\mathrm{Hom}(\pi, M)} \mathrm{Hom}(P_n, M) \to \mathcal{E}(M). \tag{13}$$

Then $M = F(Z)$ with Z rigid if and only if $\mathcal{E}(M) = 0$. In particular, we need $\mathrm{Ext}^1(M, M) = 0$.

To find categories and objects T of them satisfying the above conditions, we can start from a quiver with potential. If Q is a quiver, a potential W is a linear combination of cycles in Q. From this data one can construct a Ginzburg dg-algebra, and from this dg-algebra one can construct the required category and object. This object T satisfies $\mathrm{End}(T)^{op} = \mathbb{C}[Q]/\langle \partial W \rangle$ where ∂W is the ideal generated by all cyclic derivatives of the potential W. Actually one needs to take a suitable completion of this in general.

Exercise 24.1. *Consider the quiver*



(14)

with potential $W = cba$. In this case the completed and uncompleted algebras are the same and $\langle \partial W \rangle = \{ba, cb, ac\}$.