274 Curves on Surfaces, Lecture 16

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17 Quantum skeins (Muller)

The skein relation can be modified by adding an extra parameter q and keeping track of crossings. This gives a noncommutative algebra.

A *link diagram* in a marked surface Σ (with no punctures) is a collection of curves (beginning and ending at marked points) with normal transverse crossings together with a choice of which one passes over the other at a crossing. We will define a generalization of the skein algebra by considering $\mathbb{Z}[q^{\pm 1/2}]$-linear combinations of link diagrams (considered up to isotopy) modulo the following relations:

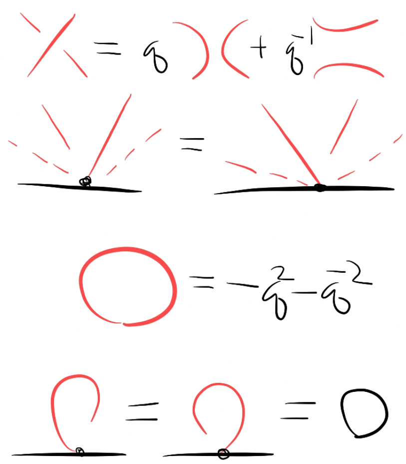

Figure 1: Quantum skein relations.

A modified version of Reidemeister I holds, as well as Reidemeister II, a marked version of Reidemeister II, and Reidemeister III. These allow us to think of diagrams almost topologically except that we do not have Reidemeister I.

To get a topological interpretation, we should instead interpret these diagrams as describing framed links (ribbons), which are links with a distinguished normal direction at each point. Given a framed link L in $\Sigma \times [0, 1]$, we can project it down to a link diagram in Σ, and the above relations guarantee that we get a well-defined element $\langle L \rangle$ of the skein algebra.

Exercise 17.1. *Use RII and RIII to show that you can remove two loops in a different way from modified RI.*

Given two link diagrams L_1, L_2, we can choose representatives L_1', L_2' up to isotopy so that their union has normal transverse crossings (where everything in L_1' passes

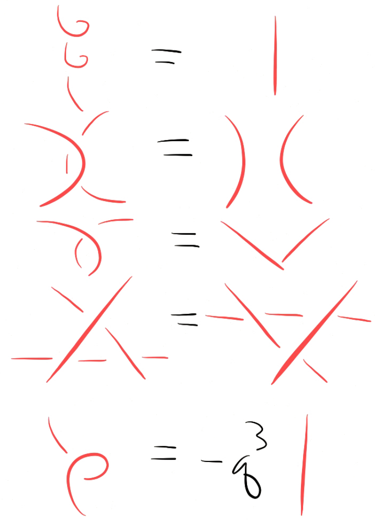

Figure 2: Reidemeister moves.

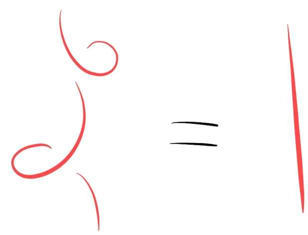

Figure 3: An exercise.

over everything in L_2'). Call this operation $L_1' \odot L_2'$. This is well-defined after passing to the skein algebra and defines the multiplication

$$\langle L_1' \rangle \langle L_2' \rangle = \langle L_1' \odot L_2' \rangle \tag{1}$$

in the skein algebra $\mathrm{Sk}_q(\Sigma)$. The unit is the class of the empty diagram.

The skein algebra has an involution sending $q^{1/2}$ to $q^{-1/2}$ and sending the class of a link to the class of the mirror image (the link obtained by reversing all crossings). $\mathrm{Sk}_q(\Sigma)$ is multigraded by non-negative combinations of marked points: the multigrading of a diagram is given by the number of endpoints at each marked point.

Consider $\mathrm{Sk}_q(\mathbb{R}^2)$. Given a link diagram, we can use the skein relation to get a $\mathbb{Z}[q^{\pm 1/2}]$-linear combination of products of contractible loops, which can then be set to $-q^2 - q^{-2}$. This associates to any link diagram a Laurent polynomial (times the empty diagram). In particular, the skein algebra is just $\mathbb{Z}[q^{\pm 1/2}]$ as an algebra. This gives an invariant of framed links in $\mathbb{R}^3$ called the Kauffman bracket.

The Kauffman bracket can be normalized to give an invariant of links (one satisfying Reidemeister I) called the Jones polynomial. This is done by defining

$$J(L) = (-q^3)^{-w(L)} \langle L \rangle \tag{2}$$

where $w(L)$ is the writhe of L. This is computed by choosing an orientation of each component of L and computing a signed sum of crossings, some of which are chosen to be positive and some of which are chosen to be negative. (There may be a small change of variables here.)

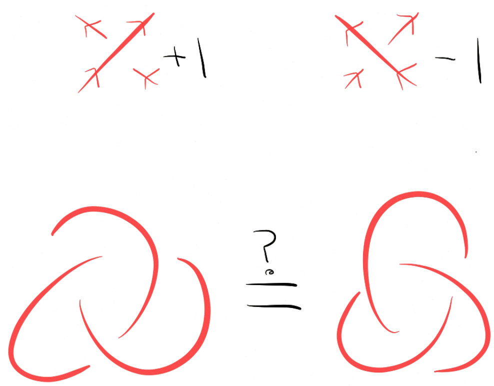

Figure 4: Computing the writhe and trefoil knots.

Exercise 17.2. *Compute the Kauffman bracket and the Jones polynomial of the trefoil knot. Conclude that the trefoil knot is not isotopic to its mirror image.*

Jones introduced the Jones polynomial by studying certain algebras very far from knot theory. Kauffman observed that it could be computed using the Kauffman bracket, and Turaev observed that the Kauffman bracket made sense for arbitrary surfaces.

It is also possible to associate skein modules to 3-manifolds, and the skein module of a knot complement is a powerful knot invariant in the category of $\mathrm{Sk}_q(S^1 \times S^1)$-modules. It might be reasonable to expect that this is a complete knot invariant.

The above discussion does not involve marked points. As it turns out, marked points give us a lot of additional power.

Lemma 17.3. *Let x be a simple arc in Σ and let y be any link diagram. Then $\langle x\rangle\langle y\rangle$ is a linear combination of diagrams which intersect x one fewer times than y.*

This is specific to arcs.

Corollary 17.4. *There exists n such that $\langle x\rangle^n\langle y\rangle$ does not intersect x.*

Now assume that we have enough arcs to obtain a triangulation. Repeatedly multiplying by these arcs, we can obtain a sum of link diagrams supported in triangles of the triangulation, which are homotopic to the identity. This is a geometric manifestation of the Laurent phenomenon.

Let x_i, x_j be arcs of a triangulation. Then there exists $a \in \{-2, -1, 0, 1, 2\}$ such that

$$x_i x_j = q^a x_j x_i. \tag{3}$$

In other words, x_i, x_j quasi-commute. Thus a triangulation consists of a collection of quasi-commuting elements.

Given quasi-commuting elements $y_1, y_2, ...y_n$, let

$$\boxed{y_1 y_2 ... y_n} = q^a y_1 y_2 ... y_n \tag{4}$$

where a is chosen such that the result is fixed by the involution.

Lemma 17.5. *Given a triangulation $x_1, ...x_n$ of Σ and a link diagram Y, there exist $\alpha_1, ...\alpha_n \geq 0$ such that*

$$\boxed{x_1^{\alpha_1} ... x_n^{\alpha_n}}\langle Y\rangle = \text{polynomial in the } x_i. \tag{5}$$

If we could divide, this would imply that $\langle Y\rangle$ is a (quasi-commuting) Laurent polynomial in the x_i. This requires a little algebra. We need to know that for any triangulation, the corresponding arcs $x_1, ...x_n$ generate an Ore set inside $\mathrm{Sk}_q(\Sigma)$ without zero divisors. This implies the following.

Proposition 17.6. *$Sk_q(\Sigma)$ embeds into its localization at the x_i above.*

But by the above results this localization is generated by $x_i^{\pm 1}$. It is a quantum torus (a ring of Laurent polynomials in quasi-commuting variables).

This suggests that it would be reasonable to find a quantum version of the classical result that

$$A(\Sigma) \subseteq \mathrm{Sk}_1(\Sigma) \subseteq U(\Sigma) \tag{6}$$

where $A(\Sigma)$ is the cluster algebra associated to Σ and $U(\Sigma)$ is the upper cluster algebra.

This requires a notion of quantum cluster algebra (see Berenstein and Zelevinsky), which is defined by an exchange matrix B satisfying the same conditions as in an ordinary cluster algebra, but also by a compatibility matrix Λ, a skew-symmetric integral $N \times N$ matrix (where N is the number of variables) such that ΛB is positive on the diagonal. Rather than a ring of Laurent polynomials we will work in a quantum torus generated by $x_1, ...x_n$ such that

$$x_i x_j = q^{\Lambda_{ij}} x_j x_i. \tag{7}$$

There is a quantum exchange relation

$$x_i' = \boxed{x_i^{-1} \prod_{B_{ji}>0} x_j^{B_{ji}}} + \boxed{x_i^{-1} \prod_{B_{ji}<0} x_j^{-B_{ij}}} \tag{8}$$

and all of the standard results continue to hold. We have cluster variables obtained by arbitrary sequences of mutations giving an algebra $A_q(\Sigma)$ inside a quantum torus (where we do not invert frozen variables), and we also have an upper cluster algebra $U_q(\Sigma)$ given by the intersection of the quantum tori associated to each cluster.

Theorem 17.7. *For triangulable Σ, there are canonical inclusions $A_q(\Sigma) \subseteq Sk_q(\Sigma) \subseteq U_q(\Sigma)$.*

The first inclusion is never an equality when we have nontrivial loops. The second inclusion is never an equality because $U_q(\Sigma)$ contains inverses to boundary arcs. This can be fixed by considering localizations $A_q^\circ(\Sigma), \mathrm{Sk}_q^\circ(\Sigma)$ at the boundary arcs. This gives another sequence of inclusions

$$A_q^\circ(\Sigma) \subseteq \mathrm{Sk}_q^\circ(\Sigma) \subseteq U_q(\Sigma) \tag{9}$$

and these are equalities if the number of marked points is at least 2.

Exercise 17.8. *Write an arc in a hexagon as a linear combination of monomials with respect to a triangulation.*

Exercise 17.9. *Show that a certain set of 5 elements generates $Sk_q(S^1 \times I)$ (with one marked point in each boundary component).*

Exercise 17.10. *Check that a certain triangulation with interior marked points has degenerate exchange matrix. (This implies that we do not have a quantization in the sense above.)*

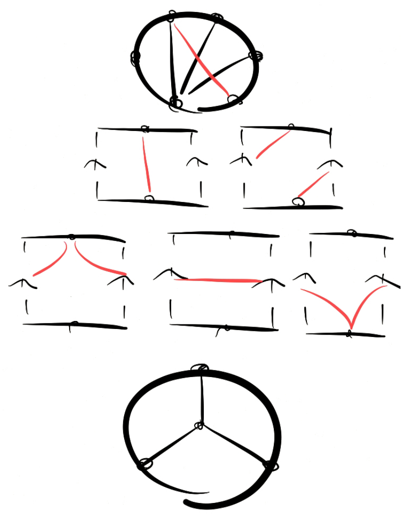

Figure 5: Exercises.