274 Curves on Surfaces, Lecture 10

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11 More about cluster algebras

Last time we discussed conjugate horocycles. This gave a relation \( \lambda(A)\lambda(A') = \lambda(B) \) where \( \lambda(A') \) is a \( \lambda \)-length measured with respect to the conjugate horocycle. On the other hand, we know that \( \ell(h) = \frac{\lambda(B)}{\lambda(A)^2} \), which gives

\[
\lambda(A') = \lambda(A)\ell(h)
\]  

or equivalently taking logarithms,

\[
\ell(A') - \ell(A) = 2 \ln \ell(h).
\]

This can be proven using a scaling argument. The result is clear when \( \ell(h) = 1 \), since then the horocycle is its own conjugate. In general, a suitable scaling multiplies \( \ell(h) \) by \( c \), multiplies \( \lambda(A) \) by \( \frac{1}{\sqrt{c}} \), and multiplies \( \lambda(A') \) by \( \sqrt{c} \), so the conclusion follows.

Last time we also asked for a surface giving rise to the affine Dynkin diagrams as quivers. To get \( \tilde{A}_{k,\ell} \) we can triangulate an annulus.

![Figure 1: A triangulation giving \( \tilde{A}_{2,4} \).](image)

We also asked for a surface giving rise to \( D_4 \) in the orientation where all of the arrows point outward. On the quiver level this can be obtained from the other \( D_4 \) we had by mutating twice.

The corresponding geometric exchange relation for the first mutation is

\[
x_1y = x_4x_3 + x_3
\]

but the actual exchange relation is

\[
x_1x'_1 = x_4 + 1.
\]
As before, this suggests measuring a $\lambda$-length with respect to some conjugate horocycle.

Question from the audience: where is the triality symmetry here?

Answer: it appears to be somewhat hidden and is not readily accessible geometrically. Note that quotienting $D_4$ by triality gives $G_2$, which is exceptional and does not come from a surface at all.

Another example with hidden symmetry is the 4-punctured sphere. With a tetrahedral triangulation, the corresponding quiver is the octahedron with a certain triangulation. This octahedral quiver can be obtained from a triangulation in a second
way, which gives a hidden symmetry (related to Regge symmetry?). More precisely, it can be glued from Type II blocks (see below) in two different ways.

We will now clarify the geometric meaning of what we have been doing.

A tagged simple arc is an arc with one or both ends marked with a notch which does not self-intersect and which does not bound a monogon or a 1-punctured monogon. Notches can only appear at punctures in the interior and should agree at common endpoints if an arc goes from a puncture to itself. Geometrically, a notch indicates that $\lambda$-lengths should be measured with respect to the conjugate horocycle. Two tagged arcs are compatible if they don’t cross and if either

1. the tags agree at common endpoints or
2. the arcs are parallel, one is notched, and one is plain.

![Diagram of compatible and incompatible tagged arcs]

**Figure 6: Compatible and incompatible tagged arcs.**

A *tagged triangulation* on a surface with a fixed set of marked points is a maximal collection of (distinct) compatible tagged arcs between marked points.

**Theorem 11.1.** Any tagged triangulation may be obtained from an ordinary triangulation $T$ by

1. replacing self-folded triangles with parallel arcs and
2. flipping all tags at some vertices.

We can construct quivers from a tagged triangulation. The way to remember how this construction works is to remember the relation $\lambda(A)\lambda(A') = \lambda(B)$ for $A'$ a tagged arc parallel to $A$ and $B$ an arc around them. This suggests that when we replace a self-folded triangle with parallel arcs, we effectively double the corresponding vertex in the quiver.

Conversely, to determine when a quiver can come from a tagged triangulation, we can glue *blocks* together (not to themselves) along vertices in such a way that we cancel edges of opposite orientations. Blocks can only be glued along vertices which have not been previously glued.

Any cluster algebra occurring in this way is mutation-finite. However, we don’t get some interesting examples, such as the exceptional series.

**Exercise 11.2.** Show that it is not possible to obtain $E_6$, $E_7$, $E_8$ by gluing blocks.

Here is a more precise statement of the classification theorem we stated previously.
Figure 7: Removing a self-folded triangle and doubling the corresponding vertex.

Figure 8: Blocks which glue together to form quivers coming from tagged triangulations.

**Theorem 11.3.** Every mutation-finite skew-symmetric cluster algebra is either

1. rank 2,
2. a surface cluster algebra, or
3. $E_6, E_7, E_8, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}, X_6, X_7$.

It would be interesting to find a better proof of this.

**Exercise 11.4.** Where is the default quiver in Bernhard Keller’s applet on the above list? Can you mutate it to get to a standard form?
Some of the entries in the above list, such as $E_6, E_7, E_8$, are not only mutation-finite but of finite type (finitely many cluster variables). The affine ones $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ are not mutation-finite, but the number of clusters reachable after $n$ mutations is $O(n)$ rather than exponential for most quivers.

**Exercise 11.5.** *Mutate the punctured hexagonal quiver to obtain the $D_6$ quiver.*
Figure 11: Bernhard Keller’s default quiver.

Figure 12: The punctured hexagon and $D_6$. 