274 Microlocal Geometry, Lecture 13

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13 Microlocal support

Last time we associated to every reasonable space X the dg-category $\mathrm{Sh}(X)$ of derived constructible sheaves, and we associated to every morphism $f : X \to Y$ a pair of adjunctions (f^*, f_*) and $(f_!, f^!)$. The last two of Grothendieck's six operations are sheaf tensor and sheaf hom. We also wrote down some identities between these operations, including base change.

One thing we get for free here is an equivalence $D_X : \mathrm{Sh}(X) \to \mathrm{Sh}(X)^{op}$ called Verdier duality with D_X^2 naturally isomorphic to the identity. It is given by $D_X F \cong \mathrm{Hom}(F, \mathbb{D}_X)$ where $\mathbb{D}_X = \pi^! \mathbb{C}_{\mathrm{pt}}$ is the Verdier dualizing sheaf (where $\pi : X \to \mathrm{pt}$ is the unique map), or Borel-Moore chains shifted by $\dim X$. We have $D_X C_X \cong \mathbb{D}_X$ and $D_X \mathbb{D}_X \cong C_X$. Verdier duality conjugates the above two adjunctions in the sense that

$$D_Y f_* D_X = f_! \tag{1}$$

$$D_X f^* D_Y = f^! \tag{2}$$

and in fact Verdier duality intertwines the adjunctions as well. So understanding (f^*, f_*) and understanding Verdier duality means we understand $(f_!, f^!)$ as well.

Exercise 13.1. *Show that $D_X IC_X \cong IC_X$ (where X has even-dimensional strata and its smooth locus is oriented).*

(This is a manifestation of the idea that IC_X sits between C_X and $\mathbb{D}_X$.)

Let $F \in \mathrm{Sh}(X)$. The *support* $\mathrm{supp}(F)$ of F is the complement of the set of points such that, on an open ball around such a point, F vanishes (is quasi-isomorphic to zero). There is also a function $\chi(F)$ given by taking the Euler characteristic of the stalk at a point. This is a constructible (integer-valued) function (locally constant on strata).

Exercise 13.2. *Show that χ induces an isomorphism between the Grothendieck group $K(Sh(X))$ and constructible (integer-valued) functions $Fun(T^*(X))$ on X.*

In microlocal geometry we will consider more refined versions of these invariants. For X an ambient smooth manifold, the refined version of support is the *microsupport* or *singular support* $\mu\mathrm{supp}(F)$. This will be a closed, $\mathbb{R}_+$-conic, and Lagrangian subspace of $T^*(X)$. The refined version of χ is the *characteristic cycle* $\mathrm{cc}(F) \in \mathrm{Fun}(T^*(X))$. These refine the above in the sense that

$$\mathrm{supp}(F) = X \cap \mu\mathrm{supp}(F) \tag{3}$$

$$\chi(F) = \mathrm{cc}(F)|_X. \tag{4}$$

Consider the example $X = \mathbb{R}$, so $T^*(X) \cong \mathbb{R} \times \mathbb{R}^\vee$. A constructible sheaf can be described by describing some set of points and some restriction maps from those points to the intervals bordering them. This is a local but not a microlocal description.

Now instead of describing sheaves at a point x we want to describe sheaves at (x, ξ) where $\xi \neq 0$ is a covector. We want to measure the change in the topology in the positive direction (where ξ tells us which direction is positive). Pick a ball $B_x \ni x$ and a function $f : B_x \to \mathbb{R}$ such that $df_x = \xi$, and define

$$N_{x,\xi} = \{f(y) = -\epsilon\} \subset B_x, \epsilon > 0. \tag{5}$$

We want to consider sections $F(B_x, N_{x,\xi})$ relative to $N_{x,\xi}$ (in a derived sense). In terms of the six operations, let $i : B_x \to X$ be the inclusion. We first restrict F to the interval $\{f > -\epsilon\}$ in B_x, getting a sheaf i^*F. Letting $j : \{f > -\epsilon\} \to \{f \geq -\epsilon\}$, we then take $j_!$, getting a sheaf $j_! i^* F$. Said another way, we are looking at the cone of the map $\Gamma(B_x, F) \to \Gamma(N_{x,\xi}, F)$.

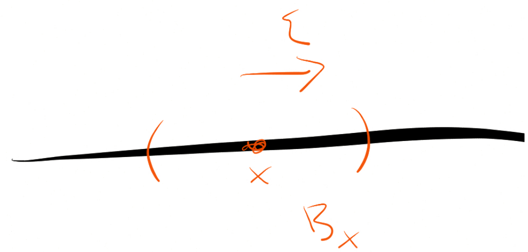

Figure 1: Change in the positive direction.

Example Let $X = \mathbb{R}$ and let F be a local system. Taking sections on a small ball and sections on a smaller ball in the negative half-space, we are looking at the cone of a quasi-isomorphism, which is quasi-isomorphic to zero. Hence $F_{x,\xi}$ vanishes for all $\xi \neq 0$. The singular support or microsupport is X, and the characteristic cycle of L is $\chi(L)$.

Example Let $X = \mathbb{R}$ and let $F = \mathbb{C}_{\text{pt}}$ be a skyscraper sheaf at a point pt. Because F is concentrated at pt, the restriction to any open subset not containing pt is zero, so we have $F_{x,\xi} \cong \mathbb{C}$ if ξ is based at pt and it vanishes otherwise. The microsupport consists of pairs (pt, ξ).

Example Let $j : (a, b) \to \mathbb{R}$ be the inclusion of an open interval and let $F = j_* \mathbb{C}^\bullet_{(a,b)}$. The support of F is the closed interval $[a, b]$ and the corresponding constructible function takes value 1 on $[a, b]$ and value 0 otherwise.

Now for the microlocal picture. We want to compare how interesting the sheaf is in the past to the sheaf in the future. Inside (a, b) the future is exactly as interesting as the past (restriction maps are quasi-isomorphisms). At b, the restriction maps are interesting if positive points into the interval, and similarly at a. The microsupport is a jagged line and $\text{cc}(F)$ is equal to 1 on it.

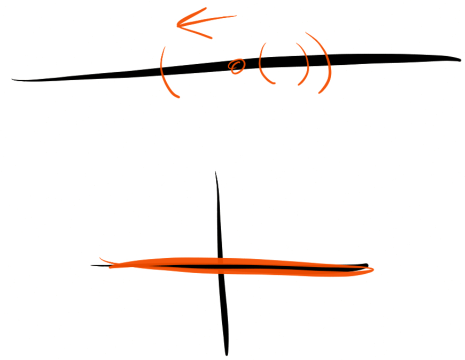

Figure 2: The singular support of a local system.

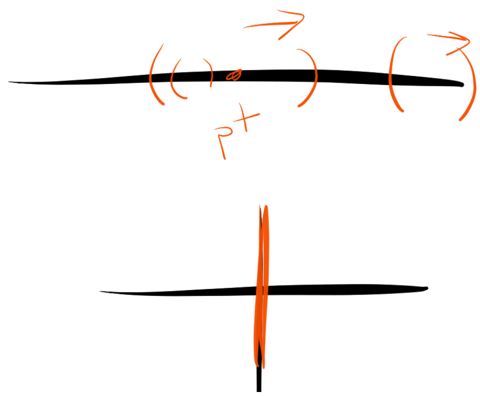

Figure 3: The singular support of a skyscraper sheaf.

Example Let $j : (a, b) \to \mathbb{R}$ be the inclusion of an open interval and let $F = j_! \mathbb{C}^\bullet_{(a,b)}$. The shriek extension from an open set is extension by zero, or sections with proper support. The support of F is again $[a, b]$, but now a section of F has to stay away from the boundary, so chains can be moved out of open neighborhoods of a and b and $\chi(F) = 0$ at a, b. The microsupport should be another jagged line.

Let $F \in \mathrm{Sh}_S(X)$, where $S = \{X_\alpha\}$ is a stratification. The singular support of F will be a closed, $\mathbb{R}_+$-conic, Lagrangian subset of

$$T^*_S(X) = \bigsqcup_\alpha T^*_{X_\alpha}(X) \subseteq T^*(X). \tag{6}$$

For example, if $X = \mathbb{R}$ and S is a stratification with a finite number of singular points, then $T^*_S(X)$ consists of cotangent vectors at the singular points.

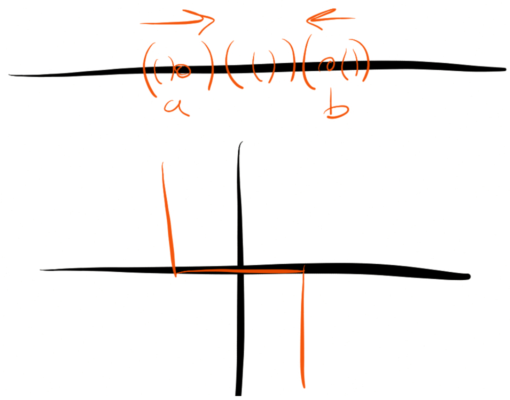

Figure 4: The singular support of the pushforward.

Let (x, ξ) be in the smooth locus of $T^*_S(X)$. Choose a small ball $B_x \ni x$ and a function $f : B_x \to \mathbb{R}$ such that $f(x) = 0, df_x = \xi$, and such that the graph of df is transverse to $T^*_S(X)$. (When $\xi = 0$ this means that f is locally a Morse function.) Define $F_f \cong F(B_x, N_f)$ where $N_f = \{f < 0\} \subset B_x$.

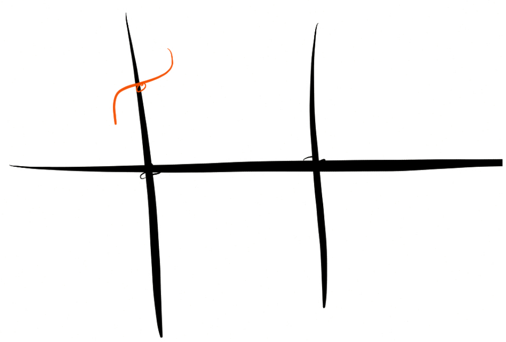

Figure 5: A transverse choice of f.

What is invariant about this construction (if we change f)? First, whether it is quasi-isomorphic or not is invariant. More can be invariant if we fix a convention about the quadratic part of f, e.g. ensure that f has a local minimum at x. If we do this then F_f is independent of the choice of f. This construction will give us both the microsupport and the characteristic cycle.