The Matrix of a Linear Transformation

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Let V be a finite-dimensional vector space and $B = \{b_1, b_2, \dots b_n\}$ be a basis of V. For a vector $v \in V$, Lay (4.4) introduces the notation

$$[v]_B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \tag{1}$$

for the coordinate vector of v in the basis B; it describes the unique way to represent v as a linear combination

$$v = x_1 b_1 + \cdots + x_n b_n \tag{2}$$

of the elements of the basis B.

Now suppose W is another finite-dimensional vector space and $C = \{c_1, c_2, \dots c_m\}$ is a basis of W. For a linear transformation $T : V \to W$, there is a unique $m \times n$ matrix M such that

$$[T(v)]_C = M[v]_B. \tag{3}$$

This matrix M can be described as the matrix whose columns are the C-coordinate vectors $[T(b_1)]_C, [T(b_2)]_C$, etc.

However, Lay does not introduce any notation to describe this matrix M, *as a function of* T, B, C, except in the special case that $W = V, C = B$. In this note I suggest that we should use the notation ${}_C[T]_B$, so that the defining property now reads

$$[T(v)]_C = {}_C[T]_B[v]_B. \tag{4}$$

This notation has many pleasant properties, which we describe below.

Matrix multiplication and composition. With T, V, W, B, C as above, suppose in addition that $S : U \to V$ is another linear transformation and $A = \{a_1, a_2, \dots a_k\}$ is a basis of U, then we can write down the composite linear transformation

$$T \circ S : U \ni u \mapsto T(S(u)) \in W \tag{5}$$

and we can also ask for the relationship between the three matrices ${}_C[T]_B, {}_B[S]_A, {}_C[T \circ S]_A$. The relationship is that

$$ {}_C[T \circ S]_A = {}_C[T]_{B\,B}[S]_A. \tag{6} $$

Note how the two appearances of B match up, and how the left and right of both sides of the equation start with C and end with A; this allows us to do "type-checking" to make sure that expressions we write using this notation are meaningful.

To see that the above is true, write

$$ \begin{aligned} {}_C[T \circ S]_A[u]_A &= [T(S(u))]_C & (7) \\ &= {}_C[T]_B[S(u)]_B & (8) \\ &= {}_C[T]_{B\,B}[S]_A[u]_A. & (9) \end{aligned} $$

Change of basis. With T, V, B, C as above, suppose now that we want to switch from using B, C to using new bases $B' = \{b'_1, b'_2, \ldots b'_n\}$ of V and $C' = \{c'_1, c'_2, \ldots c'_m\}$ of W respectively. We get a new matrix ${}_{C'}[T]_{B'}$. What is its relationship to our old matrix ${}_C[T]_B$? It turns out that they can be related by two change-of-coordinates matrices (Lay, 4.7), as follows. Recall that the change-of-coordinates matrix $\underset{B' \leftarrow B}{P}$, for two bases B, B' of the same vector space, is the unique matrix satisfying

$$ [v]_{B'} = \underset{B' \leftarrow B}{P}[v]_B. \tag{10} $$

Then the relationship between our new matrix and our old matrix is

$$ {}_{C'}[T]_{B'} = \underset{C' \leftarrow C}{P}\,{}_C[T]_B \underset{B \leftarrow B'}{P}. \tag{11} $$

Note again how the appearances of B and C match up, and how the left and right of both sides of the equation start and end with C' and B'.

To see that the above is true, write

$$ \begin{aligned} [T(v)]_{C'} &= {}_{C'}[T]_{B'}[v]_{B'} & (12) \\ &= {}_{C'}[T]_B[v]_B & (13) \\ &= {}_{C'}[T]_B \underset{B \leftarrow B'}{P}[v]_{B'} & (14) \\ &= \underset{C' \leftarrow C}{P}\,{}_{C'}[T]_B \underset{B \leftarrow B'}{P}[v]_{B'}. & (15) \end{aligned} $$

In words, the effect of changing the basis B to a new basis B' is to multiply the matrix ${}_C[T]_B$ by an invertible matrix on the right, and the effect of changing the basis C to a new basis C' is to multiply the matrix ${}_C[T]_B$ by an invertible matrix on the left. This allows us to give a conceptual interpretation of row and column operations: since these are given by multiplying by elementary matrices on the left and right, respectively, it follows that they can be interpreted as changing the bases C and B, respectively, in such a way that the matrix ${}_C[T]_B$ simplifies.

We can also express change-of-coordinates matrices using the notation ${}_C[T]_B$, as follows. Let's use I to denote the identity transformation $I : V \to V$, which is given by $I(v) = v$, and let B, C both be bases of V. Then it's not hard to see that

$$\underset{C \leftarrow B}{P} = {}_C[I]_B. \tag{16}$$

In words, the change-of-coordinates matrix $\underset{C \leftarrow B}{P}$ is the matrix representation of the identity transformation, but with respect to two different bases of V. Note that if $C = B$, then ${}_B[I]_B$ is always the identity matrix.

Square matrices, similarity and diagonalization. Suppose that $T : V \to V$ is a linear transformation from V to itself and that $B = \{b_1, b_2, \ldots b_n\}$ is a basis of V (so $W = V, C = B$). Then we can consider the square matrix ${}_B[T]_B$, where we use the same basis for both the inputs and the outputs. One reason to do this is that it relates taking powers of T, the linear transformation, to taking powers of square matrices: for every positive integer $k \geq 1$ we have

$${}_B[T^k]_B = ({}_B[T]_B)^k. \tag{17}$$

Another reason is that it relates eigenvectors of linear transformations to eigenvectors of square matrices: if v is an eigenvector of T, so that $T(v) = \lambda v$, then

$${}_B[T]_B[v]_B = [T(v)]_B = [\lambda v]_B = \lambda[v]_B \tag{18}$$

so $[v]_B$ is an eigenvector of ${}_B[T]_B$. In order to maintain this relationship between linear transformations $T : V \to V$ and square matrices it's essential that we use the same basis for both inputs and outputs. Note that Lay (5.4) does introduce notation for this special case; he calls this square matrix $[T]_B$.

What if we wanted to use a different basis B', again for both the inputs and the outputs? Specializing our general formula from earlier, we get

$${}_{B'}[T]_{B'} = \underset{B' \leftarrow B}{P} {}_B[T]_B \underset{B \leftarrow B'}{P}. \tag{19}$$

Alternatively, writing $M =_B [T]_B$ for our old matrix, $M' = {}_{B'}[T]_{B'}$ for our new matrix, and $P = \underset{B' \leftarrow B}{P}$ for the first change of coordinates matrix, we get

$$M' = PMP^{-1}. \tag{20}$$

So M' and M are **similar**. In general, similar square matrices can be thought of as representing the same linear transformation $T : V \to V$ in different bases. The significance of the fact that we're multiplying first by P and then by P^{-1} is that we're changing coordinates back and forth.

Now for simplicity suppose that $V = \mathbb{R}^n$, and let $E = \{e_1, e_2, \dots\}$ be the standard basis. $T : \mathbb{R}^n \to \mathbb{R}^n$ has some matrix $A = {}_E[T]_E$ with respect to the standard basis. Suppose that $\mathbb{R}^n$ has a basis of eigenvectors $B = \{v_1, v_2, \dots\}$ of T (equivalently, of A), with eigenvalues $\lambda_1, \lambda_2, \dots$. By definition, this means that

$$T(v_i) = Av_i = \lambda_i v_i \tag{21}$$

for all i, which in turn means that the matrix representing T in the eigenvector basis B is diagonal: that is,

$${}_B[T]_B = D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}. \tag{22}$$

This lets us represent our old matrix A in terms of a diagonal matrix D as follows:

$$\begin{aligned} A &= {}_E[T]_E && (23) \\ &= \underset{E \leftarrow B}{P}\, {}_B[T]_B \underset{B \leftarrow E}{P} && (24) \\ &= PDP^{-1} && (25) \end{aligned}$$

where $P = \underset{E \leftarrow B}{P}$ is the change-of-coordinates matrix from the eigenvector basis B to the standard basis and $P^{-1} = \underset{B \leftarrow E}{P}$ is the change-of-coordinates matrix in the other direction. P is none other than the matrix $\begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$ whose columns are the entries of the eigenvectors v_i, or equivalently the coordinate vectors $[b_i]_E$ of the eigenvectors v_i in the standard basis.

As in Lay, a square matrix A that can be written this way is **diagonalizable**.

Exercise 0.1. *Recall that $\mathbb{P}_n$ denotes the vector space of polynomials of degree at most n. Let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be the linear transformation*

$$T(p(t)) = \frac{d}{dt} p(t). \tag{26}$$

Let $B = \{1, t, t^2, t^3\}$ be the standard basis of $\mathbb{P}_3$. Find ${}_B[T]_B$. Using this matrix, what can you say about the eigenvalues of T? Is T diagonalizable?