1. Let \( f \) be a function. Define carefully: \( \lim_{x \to \infty} f(x) = \infty \) if and only if...

**Solution:** ...for all \( N > 0 \) there exists \( M > 0 \) such that if \( x > M \) then \( f(x) > N \).

2. Let \( f(x) = \arctan(\ln x) \).
   (a) Explain why \( f \) is one-to-one.
   (b) What is the domain of \( f \)? What is the domain of \( f^{-1} \)?
   (c) Find a formula for \( f^{-1} \).

**Solution:**
(a) \( f \) is one-to-one if and only if \( f(x) = f(y) \) implies \( x = y \). If \( \arctan(\ln x) = \arctan(\ln y) \), then by taking the tangent of both sides we get \( \ln x = \ln y \), and by taking the exponential of both sides we get \( x = y \).

Alternatively, \( \arctan \) is an increasing function and so is \( \ln \), so their composition is increasing too, and increasing functions are always one-to-one.

(b) Since \( \arctan \) has domain all of \( \mathbb{R} \), \( \arctan \ln x \) is undefined if and only if \( \ln x \) is undefined. So the domain of \( f \) is the same as the domain of \( \ln x \), or the positive reals \((0, \infty)\).

The domain of \( f^{-1} \) is the same as the range of \( f \). Since \( \ln x \) takes on all real values, \( \arctan \ln x \) takes on all the values that \( \arctan x \) does. So the range of \( f \) is the same as the range of \( \arctan x \), or the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\).

(c) We can do this by writing \( x = \arctan \ln y \) and solving for \( y \). This gives \( \tan x = \ln y \) and hence

\[
y = f^{-1}(x) = e^{\tan x}
\]  
(1)

(But note that we have to restrict the domain of this function to \((-\frac{\pi}{2}, \frac{\pi}{2})\) for it to really be the inverse. The function \( e^{\tan x} \) has a larger domain than \( f^{-1} \) does.)
3. Compute the following limits (be specific about infinite limits).

(a) \( \lim_{x \to 0} \tan \frac{2x}{x} \)

(b) \( \lim_{x \to -\infty} \frac{x^3 + 1}{\sqrt{x^6 + 6x}} \)

(c) \( \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + x - x}} \)

**Solution:**

(a) We can write \( \tan \frac{2x}{x} = \frac{\sin \frac{2x}{x}}{\cos \frac{2x}{x}} \), so that the limit is

\[
\lim_{x \to 0} \frac{\sin \frac{2x}{x}}{\cos \frac{2x}{x}}.
\]

Next, recalling that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \), we can rewrite this limit as

\[
\lim_{x \to 0} 2 \frac{\sin \frac{2x}{x}}{2x} \cdot \frac{1}{\cos \frac{2x}{x}}.
\]

Using the fact that the product of limits is the limit of the products, the fact that \( \cos \) is continuous, and the fact that taking \( x \to 0 \) is the same as taking \( 2x \to 0 \), we conclude that the limit is

\[
2 \left( \lim_{x \to 0} \frac{\sin 2x}{2x} \right) \left( \lim_{x \to 0} \frac{1}{\cos 2x} \right) = 2(1)(1) = 2.
\]

(b) Dividing the numerator and denominator by \( x^3 \) gives

\[
\lim_{x \to -\infty} \frac{1 + \frac{1}{x^3}}{\sqrt{1 \frac{6x}{x^3}}}. \]

We would like to absorb \( x^3 \) into the square root, but since \( x \to -\infty \), \( x^3 \) will eventually be negative, so we need to introduce a negative sign to do this. The limit becomes

\[
\lim_{x \to -\infty} \frac{1 + \frac{1}{x^3}}{-\sqrt{1 \frac{6x}{x^3}}} = \lim_{x \to -\infty} \frac{1 + \frac{1}{x^3}}{-\sqrt{1 + \frac{6}{x^3}}}. \]

But now the numerator has limit 1 and the denominator has limit \(-1\), so using the fact that the quotient of limits is the limit of the quotients (if the denominator has a nonzero limit) we conclude that the original limit is \(-1\).

(c) Multiplying the numerator and denominator by the conjugate gives

\[
\lim_{x \to \infty} \frac{x(\sqrt{x^2 + x} + x)}{(x^2 + x) - x^2} = \lim_{x \to \infty} \frac{x(\sqrt{x^2 + x} + x)}{x} = \lim_{x \to \infty} (\sqrt{x^2 + x} + x). \]

This is a sum of two terms going to \( +\infty \), so goes to \( +\infty \).
4. Let \( f(x), g(x) \) be two functions such that \( g(1) = 2, f(1) = 3, g'(1) = 4, f'(1) = 5. \) Compute the derivatives of the following functions at 1.

(a) \( f(x) + g(x) \)
(b) \( f(x)g(x) \)
(c) \( \frac{f(x)}{g(x)} \)

Solution:

(a) By the sum rule the derivative is \( f'(1) + g'(1) = 4 + 5 = 9 \).

(b) By the product rule the derivative is \( f'(1)g(1) + f(1)g'(1) = 5 \cdot 2 + 3 \cdot 4 = 22 \).

(c) By the quotient rule the derivative is \( \frac{f'(1)g(1) - f(1)g'(1)}{g(1)^2} = \frac{5 \cdot 2 - 3 \cdot 4}{2^2} = -\frac{1}{2} \).