Math 10B: Worksheet 6 Solutions

March 1

1. A car hire firm hires out cars on a daily basis. The number of cars demanded per day follows a Poisson distribution with mean 3.

   (a) Find the probability that exactly two cars are hired out on any one day.

   Let D be the number of cars demanded on a given day. Then, $D \sim \text{Po}(3)$ so we have

   $$\mathbb{P}(D = 2) = e^{-3}\frac{3^2}{2!} = \boxed{\frac{9}{2e^3}} \approx 0.224.$$

   (b) Find the probability that at most two cars are in use on exactly three days of a five-day week.

   First note that

   $$\mathbb{P}(D \leq 2) = e^{-3} + 3e^{-3} + \frac{9}{2}e^{-3} = \frac{17}{2e^3}.$$

   Let X be the number of days out of five on which at most two cars are in use. Then, $X \sim \text{Bin}(5, \frac{17}{2e^3})$ so we conclude that

   $$\mathbb{P}(X = 3) = \boxed{\binom{5}{3}\left(\frac{17}{2e^3}\right)^3\left(1 - \frac{17}{2e^3}\right)^2} \approx 0.2522.$$

2. Jeremy is able to complete the newspaper crossword puzzle 60% of the time.

   (a) Find the expected value of the number of completed puzzles during a particular week.

   Let C be the number of puzzles in a week completed successfully. Then, $C \sim \text{Bin}(7, 0.6)$ so

   $$E[C] = 7 \times 0.6 = 4.2.$$

   (b) Find the probability that he completes at most two crosswords during a particular week.

   We have

   $$\begin{aligned}\mathbb{P}(C \leq 2) &= \mathbb{P}(C = 0) + \mathbb{P}(C = 1) + \mathbb{P}(C = 2) \\ &= (0.4)^7 + \binom{7}{1}(0.6)(0.4)^6 + \binom{7}{2}(0.6)^2(0.4)^5 \\ &= \boxed{0.096256}.\end{aligned}$$

(c) Find the probability that, in a period of four weeks, he completes two or less in only one of the four weeks.

Let X be the number of weeks out of four in which Jeremy completes at most puzzles. Then, $X \sim \text{Bin}(4, 0.096256)$ so that

$$\mathbb{P}(X = 1) = \boxed{\binom{4}{1}(0.096256)(1 - 0.096256)^3} \approx 0.2842.$$

3. An aircraft has 120 seats. The airline has found that on average 3% of people who have purchased tickets for a flight do not show up. The airline sells 125 tickets for a particular flight.

(a) Using a suitable approximation, find the probability that more than 120 people arrive for the flight.

Let X be the number of people who do not show up. Then, $X \sim \text{Bin}(125, 0.03)$. Since the number of trials is large and probability of success is small, we can approximate X by $\text{Po}(3.75)$ (note that $\lambda = 125 \times 0.03 = 3.75$ is the mean number of people who will not show up.).
The event that more than 120 people arrive corresponds to $X < 5$ so we have

$$\begin{aligned}
\mathbb{P}(X < 5) &= \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4) \\
&= e^{-\lambda} + \lambda e^{-\lambda} + e^{-\lambda}\frac{\lambda^2}{2!} + e^{-\lambda}\frac{\lambda^3}{3!} + e^{-\lambda}\frac{\lambda^4}{4!} \\
&= e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!}\right) \\
&= \boxed{e^{-3.75}\left(1 + 3.75 + \frac{3.75^2}{2!} + \frac{3.75^3}{3!} + \frac{3.75^4}{4!}\right)} \\
&\approx 0.6775.
\end{aligned}$$

(b) Find the probability that there are empty seats on the plane.

The event that there are empty seats corresponds to $X > 5$ so we have

$$\begin{aligned}
\mathbb{P}(X > 5) &= 1 - \mathbb{P}(X \leq 5) \\
&= \boxed{1 - e^{-3.75}\left(1 + 3.75 + \frac{3.75^2}{2!} + \frac{3.75^3}{3!} + \frac{3.75^4}{4!} + \frac{3.75^5}{5!}\right)} \\
&\approx 0.1771.
\end{aligned}$$

4. A student is practicing her high jump. The probability that she succeeds at any one attempt is 0.2.

(a) Find the expected number of attempts before the first successful jump.

Let F be the number of failed attempts before the first success. Then, $F \sim \text{Geo}(0.2)$ so we have

$$E[F] = \frac{1}{0.2} - 1 = 5 - 1 = \boxed{4}.$$

(b) Find the expected value and standard deviation of the number of successful jumps out of six attempts.

Let J be the number of successful jumps out of six attempts. Then, $J \sim \text{Bin}(6, 0.2)$ so

$$E[J] = 6 \times 0.2 = \boxed{1.2}\,, \quad SD[J] = \sqrt{6 \times 0.2 \times 0.8} = \boxed{\sqrt{0.96}} \approx 0.9798.$$

5. Suppose the number of robberies in a neighbourhood can be modelled using a Poisson distribution. The probability that no robberies occur in a week is 0.135. Given that the mean number of robberies per week is an integer, find the probability that there are fewer than two robberies in week. (You may require $\ln(0.135) = -2$)

   Let R be the number of robberies per week. Then, $R \sim \text{Po}(\lambda)$ for some λ. Observe next that

$$\begin{aligned} 0.135 &= \mathbb{P}(R = 0) = e^{-\lambda} \\ \ln(0.135) &= -\lambda \\ \Rightarrow \lambda &= 2. \end{aligned}$$

   It follows that

$$\begin{aligned} \mathbb{P}(R < 2) &= \mathbb{P}(R = 0) + \mathbb{P}(R = 1) \\ &= 0.135 + 2(0.135) = \boxed{0.405}. \end{aligned}$$

6. Alice runs a stall at a fete in which the player is guaranteed to win \$10. Players pay a certain amount each time they throw a dice and must keep throwing the dice until a four occurs. When a four is obtained, Alice gives the player \$10. On average, Alice expects to make a profit of \$2 per game. How much does she charge for each throw?

   Let \$$C$ be the charge per throw. Since the game continues until a four turns up, the number of unsuccessful throws T follows a geometric distribution with probability of success $1/6$. The expected number of unsuccessful throws then is

$$E[T] = \frac{1}{1/6} - 1 = 6 - 1 = 5.$$

   Including the one successful throw, on average a player therefore takes six throws. The average profit per game is given by $6C - 10$ which equals 2; solving for C gives

$$6C - 10 = 2 \Rightarrow \boxed{C = 2}.$$