Math 53: Quiz 5
October 7, 2014

1. (1 point) Suppose that $f$ is a differentiable function of a single variable and that

$$u = f(x - ut)$$

implicitly defines $u$ as a function of $x$ and $t$. Show that

$$u_t + uu_x = 0.$$

We have

$$u_t = f'(x - ut)(-tu_t - u) \Rightarrow u_t(1 + tf'(x - ut)) = -uf'(x - ut)$$

and

$$u_x = f'(x - ut)(1 - tu_x) \Rightarrow u_x(1 + tf'(x - ut)) = f'(x - ut).$$

Therefore,

$$u_t(1 + tf'(x - ut)) = -uu_x(1 + tf'(x - ut)) \Rightarrow u_t = -uu_x$$

where we can cancel the factor $(1 + tf'(x - ut))$ on both sides since its being zero for all $u, x, t$ implies that $f' \equiv 0$ by the $u_x$ equation above and hence $1 = 0$, a contradiction.

2. (1 point) Consider the surface $x^4 - 4xy + y^4 - z^2 = 0$. Determine the equation of the tangent plane at $(1, 2, 3)$.

Differentiate implicitly with respect to $x$ to find $\frac{\partial z}{\partial x}$:

$$4x^3 - 4y - 2z \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{2x^3 - 2y}{z}.$$

Similarly, find $\frac{\partial z}{\partial y}$:

$$-4x + 4y^3 - 2z \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = \frac{2y^3 - 2x}{z}.$$

At $(1, 2, 3)$, the normal vector to the surface therefore is $\vec{n} = \left( -\frac{2}{3}, \frac{14}{3}, -1 \right)$ so the equation of the tangent plane at this point is

$$-\frac{2}{3}x + \frac{14}{3}y - z = \frac{17}{3} \Rightarrow -2x + 14y - 3z = 17.$$
3. (1 point) Let \( f(x, y, z) = (x - y) \cos(xy + yz) \). Approximate \( f(1.01, 1.02, -1.01) \).

We have

\[
f(1.01, 1.02, -1.01) \approx f(1, 1, -1) + (0.01) \left. \frac{\partial f}{\partial x} \right|_{(1,1,-1)} + (0.02) \left. \frac{\partial f}{\partial y} \right|_{(1,1,-1)} + (-0.01) \left. \frac{\partial f}{\partial z} \right|_{(1,1,-1)}.
\]

Note that \( f(1, 1, -1) = 0 \) and that

\[
\left. \frac{\partial f}{\partial x} \right|_{(1,1,-1)} = (\cos(xy + yz) + (x - y)(-y) \sin(xy + yz)) \bigg|_{(1,1,-1)} = \cos(0) = 1
\]
\[
\left. \frac{\partial f}{\partial y} \right|_{(1,1,-1)} = (-\cos(xy + yz) + (x - y)(x + z) \sin(xy + yz)) \bigg|_{(1,1,-1)} = -\cos(0) = -1
\]
\[
\left. \frac{\partial f}{\partial z} \right|_{(1,1,-1)} = ((x - y)(-y) \sin(xy + yz)) \bigg|_{(1,1,-1)} = 0.
\]

Thus,

\[
f(1.01, 1.02, -1.01) \approx 0 + (0.01) - (0.02) + 0 = -0.01
\]

which coincidentally turns out to be the exact value as well.