Math 16B: Homework 5 Solutions

Due: July 30

1. Find the nth Taylor polynomial for each of the following functions at the given point:

   (a) $n = 2$: $f(x) = x^{1/3}$ at $x = 27$:
   We have $f'(x) = \frac{1}{3x^{2/3}}$ and $f''(x) = \frac{-2}{9x^{5/3}}$. Since $f(27) = 3$, $f'(27) = \frac{1}{27}$ and $f''(27) = \frac{-2}{2187}$, the 2nd Taylor polynomial of f centered at $x = 27$ is

$$\begin{aligned} P(x) &= 3 + \frac{(\frac{1}{27})}{1}(x-27) + \frac{(\frac{-2}{2187})}{2}(x-27)^2 \\ &= 3 + \frac{x-27}{27} - \frac{(x-27)^2}{2187}. \end{aligned}$$

   (b) $n = 2$: $f(x) = \tan(x)$ at $x = \frac{\pi}{4}$:
   We have $f'(x) = \sec^2(x)$ and $f''(x) = 2\sec(x)(\sec(x)\tan(x)) = 2\sec^2(x)\tan(x)$. Since $f(\pi/4) = 1$, $f'(\pi/4) = (\sqrt{2})^2 = 2$ and $f''(\pi/4) = 2(2)(1) = 4$, the 2nd Taylor polynomial of f centered at $x = \frac{\pi}{4}$ is

$$\begin{aligned} P(x) &= 1 + \frac{(2)}{1}\left(x - \frac{\pi}{4}\right) + \frac{(4)}{2}\left(x - \frac{\pi}{4}\right)^2 \\ &= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2. \end{aligned}$$

   (c) $n = 3$: $f(x) = \frac{1}{7-x}$ at $x = 5$:
   We have $f'(x) = \frac{1}{(7-x)^2}$, $f''(x) = \frac{2}{(7-x)^3}$ and $f'''(x) = \frac{6}{(7-x)^4}$. Since $f(5) = \frac{1}{2}$, $f'(5) = \frac{1}{4}$, $f''(5) = \frac{2}{8} = \frac{1}{4}$ and $f'''(5) = \frac{6}{16} = \frac{3}{8}$, the 3rd Taylor polynomial of f centered at $x = 5$ is

$$\begin{aligned} P(x) &= \frac{1}{2} + \frac{(\frac{1}{4})}{1}(x-5) + \frac{(\frac{1}{4})}{2}(x-5)^2 + \frac{(\frac{3}{8})}{3!}(x-5)^3 \\ &= \frac{1}{2} + \frac{(x-5)}{4} + \frac{(x-5)^2}{8} + \frac{(x-5)^3}{16}. \end{aligned}$$

   (d) $n = 4$: $f(x) = \ln(x)$ at $x = 1$:
   We have $f'(x) = \frac{1}{x}$, $f''(x) = \frac{-1}{x^2}$, $f'''(x) = \frac{2}{x^3}$ and $f^{(4)}(x) = \frac{-6}{x^4}$. Since $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$ and $f^{(4)}(1) = -6$ the 4th Taylor polynomial of f centered at $x = 1$ is

$$\begin{aligned} P(x) &= 0 + \frac{(1)}{1}(x-1) + \frac{(-1)}{2}(x-1)^2 + \frac{(2)}{3!}(x-1)^3 + \frac{(-6)}{4!}(x-1)^4 \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4. \end{aligned}$$

2. For each of the following problems, write a differential equation to describe the physical system, solve it and use the solution to address the queries.

   (a) The population of a town grows at a rate proportional to the population present at time t. The initial population of 500 increases by 15% in 10 years. What will be the population in 30 years?

   The population P after t years obeys the differential equation

$$\frac{dP}{dt} = kP$$

   where k is a positive constant; the initial condition is $P(0) = 500$. To solve this, we use separation of variables:

$$\begin{aligned} \int \frac{1}{P}\, dP &= \int k\, dt \\ \ln|P| &= kt + C \\ |P| &= e^C e^{kt} \\ P &= Ae^{kt}. \end{aligned}$$

   Using $P(0) = 500$ gives $500 = Ae^0 \Rightarrow A = 500$. Thus, $P = 500e^{kt}$. Furthermore, $P(10) = 500 \times 115\% = 575$ so

$$\begin{aligned} 575 &= 500e^{10k} \\ e^{10k} &= 1.15 \\ 10k &= \ln(1.15) \\ k &= \frac{\ln(1.15)}{10} \approx 0.0140. \end{aligned}$$

   Thus, $P = 500e^{0.014t}$. The population after 30 years is therefore

$$P = 500e^{0.014(30)} = 760.44.$$

   (b) A thermometer reading 70° F is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads 110° F after $\frac{1}{2}$ minute and 145° F after 1 minute. Assuming that the heating process obeys Newton's Law of Heating, how hot is the oven?

   Let the temperature of the oven be $S°F$. By Newton's Law of Heating, the thermometer reading H obeys

$$\frac{dH}{dt} = k(S - H)$$

where k is some constant; we are given that $H(0) = 70, H(1/2) = 110$ and $H(1) = 145$. We can solve this by separation of variables:

$$\begin{aligned}
\int \frac{1}{S-H}\,dH &= \int k\,dt \\
-\ln|S-H| &= kt + C \\
|S-H| &= e^{-C}e^{-kt} \\
S-H &= Ae^{-kt} \\
H &= S - Ae^{-kt}
\end{aligned}$$

Use $H(0) = 70$:

$$70 = S - Ae^0 \Rightarrow A = S - 70.$$

Thus, $H = S - (S-70)e^{-kt}$. Next, use $H(1/2) = 110$:

$$\begin{aligned}
110 &= S - (S-70)e^{-k/2} \\
(S-70)e^{-k/2} &= S - 110 \\
e^{-k/2} &= \frac{S-110}{S-70} \\
k &= -2\ln\left(\frac{S-110}{S-70}\right)
\end{aligned}$$

Thus, $H = S - (S-70)e^{2t\ln\left(\frac{S-110}{S-70}\right)}$. Finally, use $H(1) = 145$:

$$\begin{aligned}
145 &= S - (S-70)e^{2t\ln\left(\frac{S-110}{S-70}\right)} \\
145 &= S - (S-70)e^{\ln\left(\frac{S-110}{S-70}\right)^2} \\
S - 145 &= (S-70)\left(\frac{S-110}{S-70}\right)^2 \\
S - 145 &= \frac{(S-110)^2}{S-70} \\
(S-145)(S-70) &= (S-110)^2 \\
S^2 - 145S - 70S + 10150 &= S^2 - 220S + 12100 \\
5S &= 1950 \\
S &= 390
\end{aligned}$$

The oven is therefore at a temperature of 390° F.

(c) A dead body was found within a closed room of a house where the temperature was a constant 70° F. At the time of discovery the core temperature of the body was determined to be 85° F. One hour later a second measurement showed that the core temperature of the body was 80° F. Assume that the core temperature of the body at the beginning was 98.6° F. Determine how many hours elapsed before the body was found.

Let B denote the body's temperature (in °F) at time after t hours and suppose the body was found at time $t = T$. Then, the following relationship is obeyed:

$$\frac{dB}{dt} = k(70 - B)$$

where k is some constant. From the given information, we know that $B(0) = 98.6$, $B(T) = 85$ and $B(T+1) = 80$. The equation can be solved using separation of variables:

$$\begin{aligned}
\int \frac{1}{70 - B}\, dB &= \int k\ dt \\
-\ln|70 - B| &= kt + C \\
|70 - B| &= e^{-C} e^{-kt} \\
70 - B &= Ae^{-kt} \\
B &= 70 - Ae^{-kt}
\end{aligned}$$

Use $B(0) = 98.6$:

$$98.6 = 70 - Ae^{0} \Rightarrow A = -28.6.$$

Thus, $B = 70 + 28.6e^{-kt}$. Next, use $B(T) = 85$:

$$\begin{aligned}
85 &= 70 + 28.6e^{-kT} \\
e^{-kT} &= \frac{15}{28.6} \qquad (1)
\end{aligned}$$

Finally, use $B(T+1) = 80$:

$$\begin{aligned}
80 &= 70 + 28.6e^{-k(T+1)} \\
e^{-kT-k} &= \frac{10}{28.6} \\
e^{-kT} e^{-k} &= \frac{10}{28.6}
\end{aligned}$$

Use (1) to get:

$$\begin{aligned}
\left(\frac{15}{28.6}\right) e^{-k} &= \frac{10}{28.6} \\
e^{-k} &= \frac{10}{15} = \frac{2}{3} \qquad (2)
\end{aligned}$$

Finally, plug (2) into (1) to get:

$$\begin{aligned}
(e^{-k})^{T} &= \frac{15}{28.6} \\
\left(\frac{2}{3}\right)^{T} &= \frac{15}{28.6} \\
T &= \frac{\ln(15/28.6)}{\ln(2/3)} \approx 1.59
\end{aligned}$$

Thus, the body was found after 1.59 hours or 1 hour, 35.5 minutes.

(d) A large tank is partially filled with 100 gallons of fluid in which 10 pounds of salt is dissolved. A highly concentrated salt solution containing $\frac{1}{2}$ pound of salt per gallon is pumped into the tank at a rate of 6 gal/min. The well-mixed solution is then pumped out at a slower rate of 4 gal/min. Find the number of pounds of salt in the tank after 30 minutes.

Let S denote the quantity (in pounds) of salt in the tank after t minutes. We require a differential equation that describes the rate of change of S. Observe that the $\frac{1}{2}$ pound/gal solution is pumped in at 6 gal/min so effectively salt is added at the rate of $\frac{1}{2} \times 6 = 3$ pound/min. Next, note that the solution in the tank is pumped out at 4 gal/min. This shows firstly that the volume of the liquid in the tank increases at $6 - 4 = 2$ gal/min so the volume at time t is $100 + 2t$ gal. This liquid contains S pounds of salt; removing 4 gallons of the liquid implies that $4 \times \frac{S}{100+2t}$ pounds of salt are removed. Thus, the rate at which salt is removed is given by $\frac{4S}{100+2t}$ pound/min. Combining these inferences, we have

$$\begin{aligned} \frac{dS}{dt} &= [\text{rate at which salt is added}] - [\text{rate at which salt is removed}] \\ &= 3 - \frac{4S}{100+2t} \end{aligned}$$

This equation is a first-order linear differential equation; it can be solved by the integrating factor technique. Note also the initial condition $S(0) = 10$. Thus, we have

$$\frac{dS}{dt} + \frac{4}{100+2t} S = 3. \tag{3}$$

Then, $a(t) = \frac{4}{100+2t}$ so $A(t) = \int \frac{4}{100+2t}\, dt = 4\frac{\ln(100+2t)}{2} = 2\ln(100+2t)$ (since $t > 0$, we don't need to put in the absolute value bars). Then, the integrating factor is $I(t) = e^{2\ln(100+2t)} = (100+2t)^2$. Multiply (3) by $I(t)$ to get

$$\begin{aligned} (100+2t)^2\frac{dS}{dt} + 4(100+2t)S &= 3(100+2t)^2 \\ \frac{d}{dt}\left(S(100+2t)^2\right) &= 3(100+2t)^2 \\ S(100+2t)^2 &= \int 3(100+2t)^2\, dt \\ S(100+2t)^2 &= \frac{3(100+2t)^3}{(2)(3)} + C \\ S &= \frac{100+2t}{2} + C(100+2t)^{-2} \\ S &= 50 + t + C(100+2t)^{-2} \end{aligned}$$

Use $S(0) = 10$ to get

$$\begin{aligned} 10 &= 50 + C(100)^{-2} \\ C &= -400,000 \end{aligned}$$

Thus, $S = 50 + t - 400,000(100 + 2t)^{-2}$. At $t = 30$ then, we get $S = 50 + 30 - 400,000(160)^{-2} = 64.375$ pounds.

(e) A vat with 500 gallons of beer contains 4% alcohol (by volume). Beer with 6% alcohol is pumped into the vat at a rate of 5 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after an hour?

Let Q be the quantity (in gallons) of alcohol in the vat after t mins. Since the volume of the beer is a constant 500 gallons for all time, the percentage P of alcohol is given by $P = \frac{Q}{500} \times 100 = \frac{Q}{5}$. We shall derive a differential equation in terms of Q and then switch to P since it is convenient to think in terms of volumes rather than percentages.

The quantity of alcohol increases at $6\% \times 5 = 0.3$ gal/min. We remove the mixture at 5 gal/min; since the quantity of alcohol is Q gallons in a total of 500 gallons of liquid, the removal takes place at $\frac{Q}{500} \times 5 = \frac{Q}{100}$ gal/min. Thus, we have

$$\frac{dQ}{dt} = 0.3 - \frac{Q}{100}.$$

We can now switch to using P. Divide by 5 throughout to get

$$\begin{aligned} \frac{1}{5}\frac{dQ}{dt} &= \frac{0.3}{5} - \frac{Q/5}{100} \\ \frac{dP}{dt} &= 0.06 - 0.01P \end{aligned}$$

Using the initial condition, we have $P(0) = 4$. We can solve this equation by separation of variables:

$$\begin{aligned} \int \frac{1}{0.06 - 0.01P}\, dP &= \int 1\, dt \\ \frac{1}{(-0.01)} \ln|0.06 - 0.01P| &= t + C \\ \ln|0.06 - 0.01P| &= -0.01t - 0.01C \\ |0.06 - 0.01P| &= e^{-0.01t - 0.01C} \\ 0.06 - 0.01P &= Ae^{-0.01t} \\ P &= 6 - 100Ae^{-0.01t} \end{aligned}$$

Use $P(0) = 4$ to get

$$\begin{aligned} 4 &= 6 - 100Ae^{0} \\ A &= 0.02 \end{aligned}$$

Thus, $P = 6 - 2e^{-0.01t}$. To find the percentage after 1 hour, we plug in $t = 60$ to get $P = 6 - 2e^{-0.6} = 4.90$. Thus, the percentage of alcohol in the beer is 4.90% after 1 hour.

3. Sketch the solutions of the following differential equations for the given initial conditions. Also include the constant solutions:

(a) $y' = -y(y-3)$ with initial conditions $y(0) = -4$, $y(0) = 1$ and $y(0) = 2$:
Set $y' = 0$ to get the constant solutions $y = 0$ and $y = 3$. Note that:

- for $y(0) = -4$, the slope is $4(-7) = -28$. This curve goes off to $-\infty$.
- for $y(0) = 1$, the slope is $-1(-2) = 2$. This curve becomes an asymptote to $y = 3$. Note that there will be a concavity change as y passes over the value $\frac{3}{2}$ since the slope equation has a turning point here.
- for $y(0) = 2$, the slope is $-2(-1) = 2$. This curve becomes an asymptote to $y = 3$.

The solutions are shown in Figure 1.

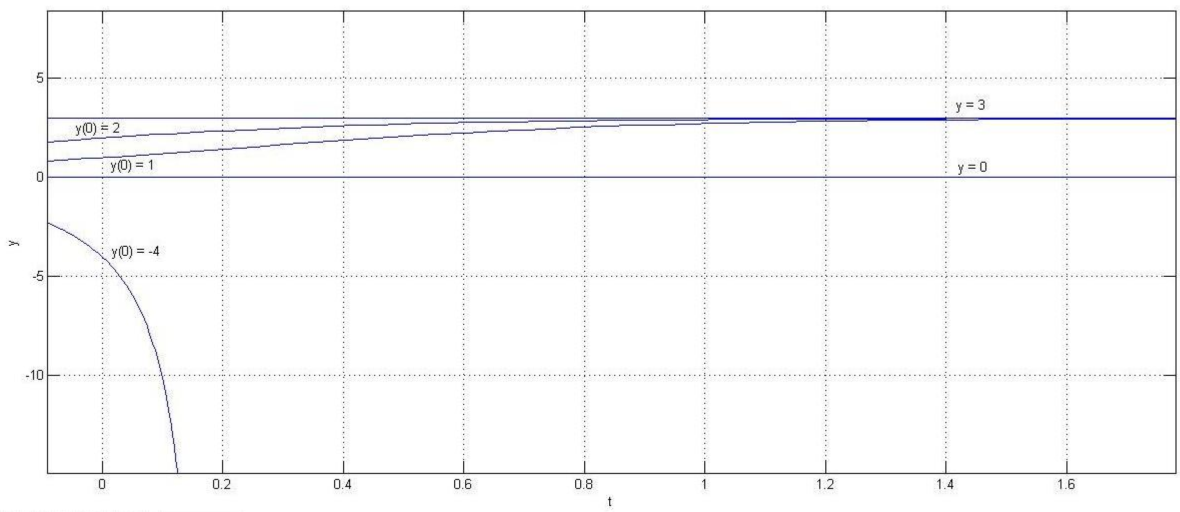

Figure 1: Solutions for $y' = -y(y-3)$

(b) $y' = \cos(y)$ with initial conditions $y(0) = -\frac{\pi}{3}$, $y(0) = \frac{\pi}{3}$ and $y(0) = \frac{5\pi}{3}$.
Set $y' = 0$ to get the constant solutions $y = -\frac{\pi}{2}$, $y = \frac{\pi}{2}$, $y = \frac{3\pi}{2}$ and $y = \frac{5\pi}{2}$ (constant solutions are of the form $\frac{n\pi}{2}$ where n is an odd integer; we are just considering the solutions that fall in our range). Note that:

- for $y(0) = -\frac{\pi}{3}$, the slope is $\cos(-\pi/3) = \frac{1}{2}$. This curve is therefore increasing and becomes asymptotic to $y = \frac{\pi}{2}$. Note that there will be a concavity change since there is a turning point of $\cos(y)$ at $y = 0$.
- for $y(0) = \frac{\pi}{3}$, the slope is $\cos(\pi/3) = \frac{1}{2}$. This curve is therefore increasing and becomes asymptotic to $y = \frac{\pi}{2}$.

- for $y(0) = \frac{5\pi}{3}$, the slope is $\cos(5\pi/3) = \frac{1}{2}$. This curve is therefore increasing and becomes asymptotic to $y = \frac{5\pi}{2}$. Note that there will again be a concavity change since there is a turning point of $\cos(y)$ at $y = 2\pi$.

The solutions are shown in Figure 2.

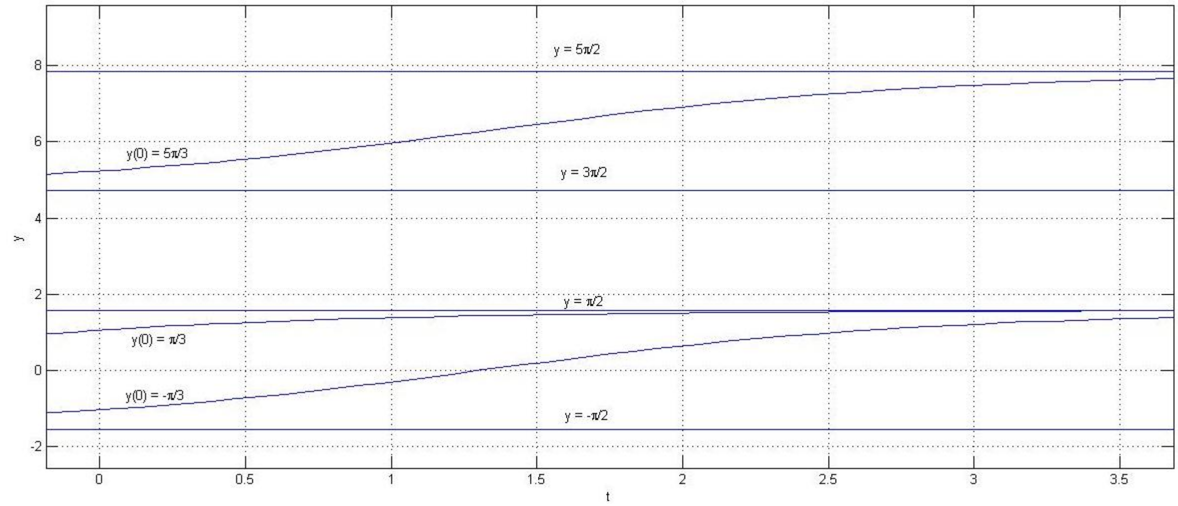

Figure 2: Solutions for $y' = \cos(y)$

(c) $y' = y^3 - y$ with initial conditions $y(0) = -3$, $y(0) = -0.5$ and $y(0) = 2$.

Set $y' = 0$ to get the constant solutions $y = 0$, $y = 1$ and $y = -1$. Note that:

- for $y(0) = -3$, the slope is $(-3)^3 - (-3) = -24$. This curve is therefore decreasing and goes off to $-\infty$
- for $y(0) = -0.5$, the slope is $(-0.5)^3 - (-0.5) = 0.375$. This curve is therefore increasing and becomes asymptotic to $y = 0$.
- for $y(0) = 2$, the slope is $2^3 - 2 = 6$. This curve is therefore increasing and goes off to $+\infty$

The solutions are shown in Figure 3.

4. For each of the following problems, write a differential equation to describe the physical system and sketch its appropriate solutions to address the queries.

(a) The number of students boarding in a dorm is 200. At the beginning of a new term, suppose one student returns with a mild infection. Assume that the rate at which the number of infected students grows is proportional to the product of the infected number and the uninfected number. Assuming that no one uses any medication, argue that at least 199 of the students (including the original infected) will catch the infection.

Let I be the number of infected students at time t. Since the rate of change of I is proportional to the product of number of infected I and the uninfected $(200 - I)$,

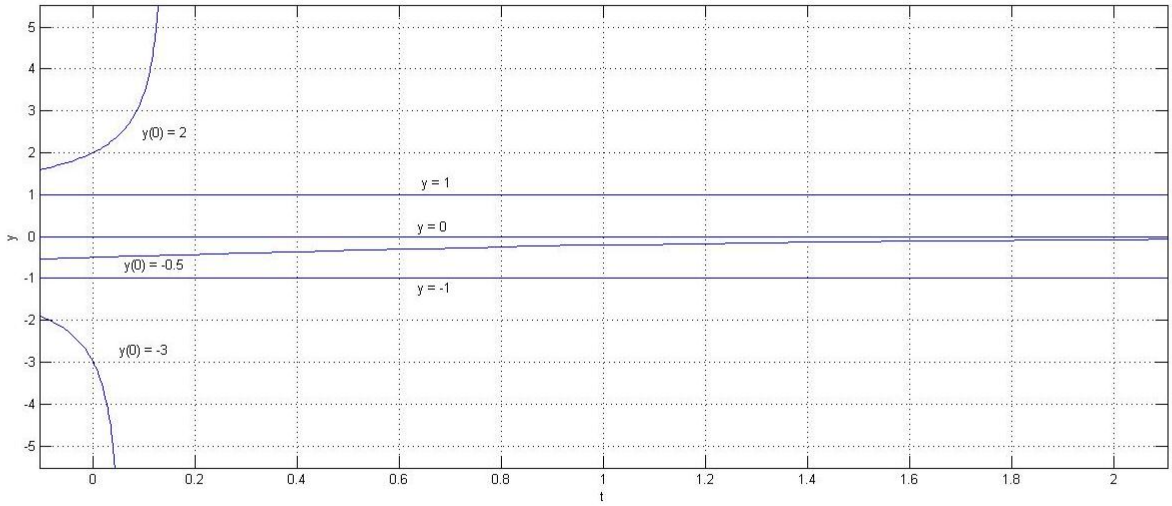

Figure 3: Solutions for $y' = y^3 - y$

we get the differential equation

$$\frac{dI}{dt} = kI(200 - I)$$

where k is some positive constant. We also have the initial condition $I(0) = 1$. We first find the constant solutions of this: set $\frac{dI}{dt} = 0$ to get $I = 0$ and $I = 200$. At $I = 1$, the slope is $k(1)(199) = 199k > 0$ so the solution is increasing and asymptotic to $I = 200$. See Figure 4 for the solution.

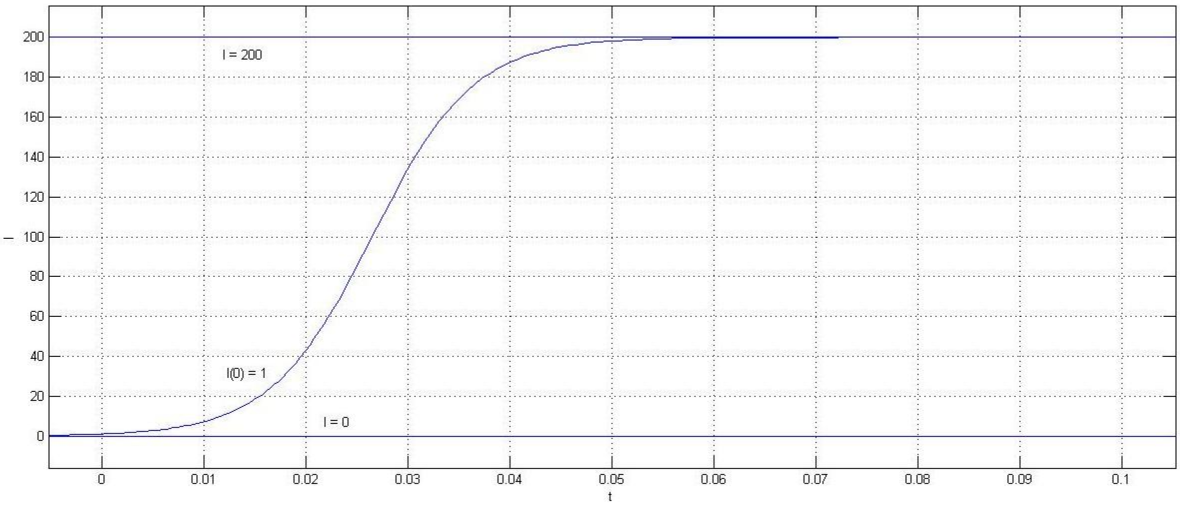

Figure 4: Solutions for infection spread

Since the number of infected students is asymptotic to 200, given enough time, it can be made to come as close to 200 as required. Hence, eventually the number of infected students exceeds 199 so at least this many students will be infected.

(b) A savings account needs to be set up that pays \$800 steadily per year. Suppose that the bank compounds continuously at 4%. What is the least initial amount that should be invested to ensure that the payments continue indefinitely?

Denote by S the amount (in \$) in the savings account at time t. Then, S increases due to the interest (at the rate $4\% \times S$) and decreases due to the steady withdrawals (of \$800 a year). Thus, S is governed by the differential equation

$$\frac{dS}{dt} = 0.04S - 800.$$

The only constant solution of this can be found by setting $\frac{dS}{dt} = 0 \Rightarrow 0.04S - 800 = 0 \Rightarrow S = 20,000$. For any value of S greater than $20,000$, the slope is positive while it is negative for any value less than $20,000$. The corresponding solutions are shown in Figure 5. If the initial investment is less than \$20,000, we see that the

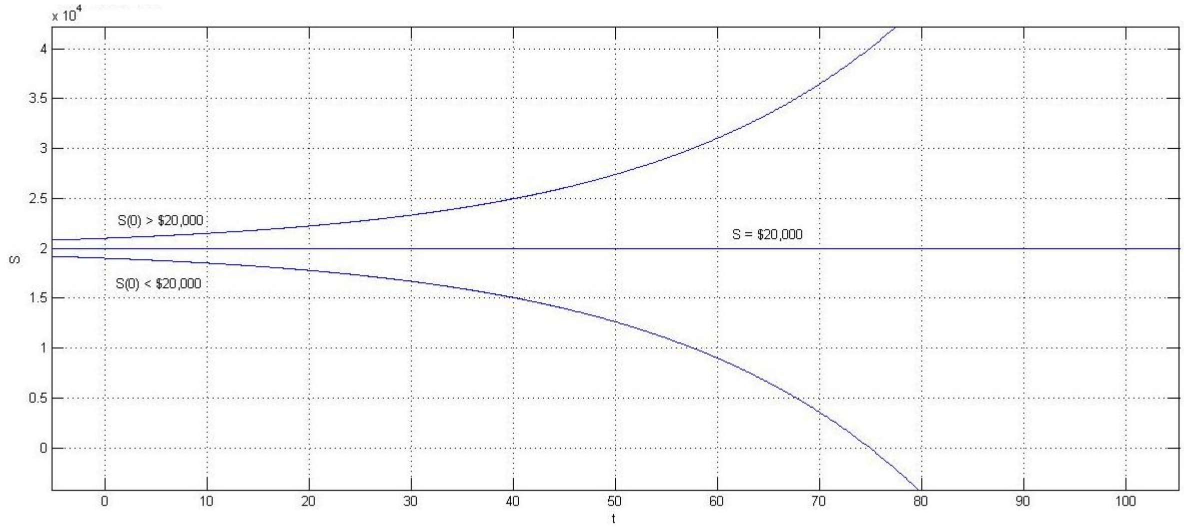

Figure 5: Solutions for savings account

money in the account eventually dries up. On the other hand, if the investment is greater than or equal to \$20,000, the payments continue indefinitely since some savings are always saved up in the account. Thus, the least initial investment is \$20,000.