NÉRON-TATE PROJECTION OF ALGEBRAIC POINTS

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ABSTRACT. Let X be a geometrically irreducible closed subvariety of an abelian variety A over a number field k such that X generates A. Let V be a finite-dimensional subspace of $A(\overline{k}) \otimes \mathbf{R}$, and let $\pi : A(\overline{k}) \to V$ be the orthogonal projection relative to a Néron-Tate pairing $\langle\ ,\ \rangle : A(\overline{k}) \times A(\overline{k}) \to \mathbf{R}$. For $V = A(k) \otimes \mathbf{R}$, we prove that $\pi(X(\overline{k})) = A(k) \otimes \mathbf{Q}$, and moreover, there exist $c, c' > 0$ such that for any $a \in A(k) \otimes \mathbf{Q}$, $\{ x \in X(\overline{k}) : \pi(x) = a \text{ and } h(x) < ch(a) + c' \}$ is Zariski dense in X.

1. Introduction

Let k be a number field, and let $\overline{k}$ be its algebraic closure. Let A be an abelian variety over k, and let X be a geometrically irreducible closed subvariety of A. Several results describe the location of the rational or algebraic points of X within A. For example, the "Mordell-Lang conjecture" states that if Γ is a finite rank subgroup of $A(\overline{k})$ and if X is not a translate of an abelian subvariety, then $X(\overline{k}) \cap \Gamma$ is not Zariski dense in X. This version of the statement was proved by Hindry [Hi], after earlier work of Faltings, Raynaud, Vojta and others. A generalization to semiabelian varieties was proved by McQuillan [McQ].

If one defines the Néron-Tate canonical height $h : A(\overline{k}) \to \mathbf{R}_{\geq 0}$ associated to a symmetric ample line sheaf on A, one can also state the "generalized Bogomolov conjecture:" If X is not a translate of an abelian subvariety by a torsion point, there exists $\epsilon > 0$ such that $\{ x \in X(\overline{k}) : h(x) < \epsilon \}$ is not Zariski dense in X. The conjecture was proved by Zhang [Zh1], using ideas from an important special case (the original Bogomolov conjecture) proved by Ullmo [Ul] using an equidistribution theorem of Szpiro, Ullmo, and Zhang [SUZ]. There is also the combined "Mordell-Lang plus Bogomolov" result of [Po] and the further distribution result of [Zh2]. Moriwaki [Mo],[Mo2] has proved generalizations of most of these statements with k replaced by a finitely generated field extension of $\mathbf{Q}$.

Define a *Néron-Tate pairing* for A to be a bilinear form $\langle\ ,\ \rangle : A(\overline{k}) \times A(\overline{k}) \to \mathbf{R}$ such that $\langle x, x \rangle = h(x)$ for a height function h as above. We may consider $\langle\ ,\ \rangle$ also as an inner product on the vector space $A(\overline{k})_{\mathbf{R}} := A(\overline{k}) \otimes \mathbf{R}$, which is infinite dimensional if $\dim A > 0$. For any field extension L of k, define also $A(L)_{\mathbf{Q}} := A(L) \otimes \mathbf{Q}$ and $A(L)_{\mathbf{R}} := A(L) \otimes \mathbf{R}$.

In this article, we study the image of $X(\overline{k})$ under the orthogonal projection $\pi : A(\overline{k}) \to V$ where V is a finite-dimensional subspace of $A(\overline{k})_{\mathbf{R}}$. After possibly enlarging k, we have $V \subseteq A(k)_{\mathbf{R}}$, and for our purposes, we lose no information in enlarging V to $A(k)_{\mathbf{R}}$. Also, by translating X to assume $0 \in X$ (enlarging k if necessary), and then replacing A by the image of the Albanese homomorphism $\operatorname{Alb} X \to A$, we may reduce to the case in which X *generates* A, i.e., in which $\operatorname{Alb} X \to A$ is surjective, or equivalently, the differences $P - Q$

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with $P, Q \in X(\overline{k})$ generate the group $A(\overline{k})$. We make these hypotheses, to simplify the statement of our result.

Theorem 1. *Let X be a geometrically irreducible closed subvariety of an abelian variety A over a number field k. Assume that X generates A. Let $\pi : A(\overline{k}) \to A(k)_{\mathbf{R}}$ denote the orthogonal projection relative to a Néron-Tate pairing for A. Then*

(a) $\pi(X(\overline{k})) = A(k)_{\mathbf{Q}}$.

(b) *There exist $c, c' > 0$ such that for any $a \in A(k)_{\mathbf{Q}}$,*

$$\{\, x \in X(\overline{k}) : \pi(x) = a \text{ and } h(x) < ch(a) + c' \,\}$$

is Zariski dense in X.

Almost in contradiction with Theorem 1 we have the following, which is a formal consequence of the Mordell-Lang conjecture.

Theorem 2. *Let A be an abelian variety over $\overline{\mathbf{Q}}$ with $\dim A > 0$. Then there exists a nonzero linear functional $\pi : A(\overline{\mathbf{Q}})_{\mathbf{Q}} \to \mathbf{Q}$ such that for every geometrically irreducible closed subvariety $X \subseteq A$ not containing a translate of a positive-dimensional abelian subvariety of A,*

(a) $\pi(X(\overline{\mathbf{Q}}))$ *is a discrete subset of $\mathbf{Q}$ in the archimedean topology.*

(b) $\{\, x \in X(\overline{\mathbf{Q}}) : \pi(x) = a \,\}$ *is finite for every $a \in \mathbf{Q}$.*

There is no contradiction, however, since $\ker \pi$ in Theorem 2 need not be the orthogonal complement of a finite-dimensional subspace of $A(\overline{\mathbf{Q}})_{\mathbf{Q}}$.

Remarks.

(1) Borrowing terminology from the field of medical imaging, Theorem 1 implies that X cannot be recovered from its Néron-Tate CAT scan!

(2) Analogues of Theorems 1 and 2 where the number fields are replaced by any field finitely generated over $\mathbf{Q}$ can be formulated using the height functions defined by Moriwaki [Mo],[Mo2], and their proofs are the same as in the number field case.

2. Proofs

Lemma 3. *Let X be a geometrically irreducible projective variety over an infinite field k, with $\dim X \geq 1$. Then there exists a geometrically irreducible closed curve $Y \subseteq X$ such that the induced morphism $\operatorname{Alb} Y \to \operatorname{Alb} X$ is surjective. Moreover, the union of such Y is Zariski dense in X.*

Proof. Let $A = \operatorname{Alb} X$. Choose a prime ℓ not equal to the characteristic of k. Let $A[\ell]$ denote the kernel of multiplication by ℓ on A. For each $P \in A[\ell](\overline{k})$, choose a zero-cycle of degree zero on $X_{\overline{k}}$ representing P. Let $S' \subseteq X(\overline{k})$ be the set of points appearing in these zero-cycles together with one extra point $Q \in X(\overline{k})$. Let S be the image of S' in X.

The blow-up $\alpha : X' \to X$ at S is projective; embed X' in some $\mathbf{P}^N$. Bertini's theorem [Da, p. 249] gives a dense open subset U of the Grassmannian of linear subspaces $L \subseteq \mathbf{P}^N$ of codimension $\dim X' - 1$ such that any point of $U(k)$ corresponds to $L \subseteq \mathbf{P}^N$ for which $X' \cap L$ is geometrically irreducible. Choose such an L, and let $Y' = X' \cap L$. For dimension reasons, Y' meets every exceptional fiber of α. Let $Y = \alpha(Y')$. Then Y is a geometrically irreducible curve passing through the points of S', so the image of $\operatorname{Alb} Y \to A$ contains $A[\ell]$. The only abelian subvariety of A containing all $\ell^{2 \dim A}$ points of order dividing ℓ is A itself, so $\operatorname{Alb} Y \to A = \operatorname{Alb} X$ is surjective. (This last trick is due to O. Gabber [Ka].) The final statement follows, since Y also passes through $Q \in X(\overline{k})$, which was arbitrary. ☐

Remark. It follows from [Ka] or alternatively [Po2] that the conclusion of Lemma 3 holds even if k is a finite field.

Lemma 4. *Let A, k, $\langle\ ,\ \rangle$, and π be as in Theorem 1. Then $\pi(A(\overline{k})) \subseteq A(k)_{\mathbf{Q}}$.*

Proof. Given $P \in A(\overline{k})$, let L be a Galois extension of k such that $P \in A(L)$. Any $\sigma \in \operatorname{Gal}(L/k)$ acts as an isometry of $A(L)_{\mathbf{R}}$ with $\langle\ ,\ \rangle$ and preserves $A(k)_{\mathbf{R}}$, so $\pi({}^\sigma P) = \pi(P)$. Thus

$$\pi(P) = \frac{1}{[L:k]}\, \pi\left(\sum_{\sigma \in \operatorname{Gal}(L/k)} {}^\sigma P \right) \in A(k)_{\mathbf{Q}}.$$

☐

If X is a curve over a perfect field k and $n \geq 1$, denote by $X^{(n)}$ the quotient of X^n by the action of the symmetric group $\mathcal{S}_n$ permuting the coordinates. Points in $X^{(n)}(k)$ will be identified with G_k-stable unordered n-tuples of points in $X(\overline{k})$, where $G_k := \operatorname{Gal}(\overline{k}/k)$.

Lemma 5. *Let X be a smooth projective geometrically integral curve of genus $g \geq 1$ over $\mathbf{F}_q$. Let U be a dense open subset of $X^{(g)}$. Then there exist infinitely many $u \in U(\overline{\mathbf{F}}_q)$ such that $\operatorname{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q(u))$ acts transitively on the g-tuple corresponding to u.*

Proof. For $\mathbf{F}_r \supseteq \mathbf{F}_q$, let $\sigma : X \to X$ denote the r-th power Frobenius morphism. The set $S_r := X(\mathbf{F}_{r^g}) - \bigcup_{d|g, d<g} X(\mathbf{F}_{r^d})$ has size $r^g + o(r^g)$ as $r \to \infty$ through powers of q, by the Weil bounds. The map $S_r \to X^{(g)}(\mathbf{F}_r)$ sending x to the g-tuple $\{x, x^\sigma, \dots, x^{\sigma^{g-1}}\}$ (on which $\operatorname{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_r)$ acts transitively) is a g-to-one map, so the image has $r^g/g + o(r^g)$ points as $r \to \infty$. By the Weil bounds again, at most $O(r^{g-1})$ of these lie outside U. Hence there remain $r^g/g + o(r^g)$ points in $U(\mathbf{F}_r)$ corresponding to desired g-tuples. Finally, $r^g/g + o(r^g)$ is unbounded as $r \to \infty$. ☐

Lemma 6. *Let $f : X \to X'$ be a finite morphism between quasiprojective varieties over a number field k, and let h and h' denote height functions on $X(\overline{k})$ and $X'(\overline{k})$, respectively, defined (up to $O(1)$) using embeddings of X and X' in projective spaces. Then there exist constants $c_1, c_2 > 0$ such that $h(x) \leq c_1 h'(f(x)) + c_2$ for all $x \in X(\overline{k})$.*

Proof. If we change the embedding of X, then h and the new height $\tilde{h}$ are bounded by linear polynomials in each other, since the isomorphisms between the two copies of X are given locally by rational functions. Hence the question is independent of embeddings. In particular, we may reduce to the case where $X = \operatorname{Spec} B$ is embedded in $\mathbf{A}^m$ and $X' = \operatorname{Spec} A$ is embedded in $\mathbf{A}^n$ for some $m, n \geq 0$. By finiteness, each of the m coordinate functions t on X satisfies a monic polynomial

$$t^r + a_1 t^{r-1} + \cdots + a_r = 0 \tag{1}$$

with $a_i \in A$. For $x \in X(\overline{k})$, (1) shows that the height of $t(x)$ is bounded by a linear polynomial in the heights of the $a_i(f(x))$, which in turn are bounded by a linear polynomial in $h'(f(x))$. ☐

Proof of Theorem 1. By Lemma 4, $\pi(X(\overline{k})) \subseteq A(k)_{\mathbf{Q}}$, so it remains to prove that for any $a \in A(k)_{\mathbf{Q}}$, $\{\, x \in X(\overline{k}) : \pi(x) = a \,\}$ is Zariski dense in X.

Lemma 3 lets us reduce to the case where X is a geometrically integral curve. (I learned this method for reducing to curves from Shou-Wu Zhang.) We may enlarge k in order to assume that $X(k)$ contains a smooth point P_0 of X. Translating X and a by $-P_0$, we may assume that $P_0 = 0$ in A. Let J be the Albanese (Jacobian) variety of the normalization $\tilde{X}$ of X. Then we have a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\ j\ } & J \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ X & \longrightarrow & A \end{array}$$

where ϕ is a surjection and j is the Abel map sending the point $\tilde{P}_0 \in \tilde{X}(k)$ above P_0 to $0 \in J(k)$. Choose a quotient abelian variety B of J such that the induced homomorphism $J \to A \times B$ is an isogeny. Define $\langle\ ,\ \rangle_J$ and π_J for J by tensoring the pullbacks of symmetric ample line sheaves on A and B. Then we have isomorphisms

$$J(\overline{k})_{\mathbf{Q}} \cong A(\overline{k})_{\mathbf{Q}} \oplus B(\overline{k})_{\mathbf{Q}},$$
$$J(\overline{k})_{\mathbf{R}} \cong A(\overline{k})_{\mathbf{R}} \oplus B(\overline{k})_{\mathbf{R}},$$

respecting the pairings. If we find a Zariski dense set of points S in $\tilde{X}(\overline{k})$ with $\pi_J(S) = \{(a, 0)\}$ under the isomorphism above, then $\alpha(S)$ is a Zariski dense set of points in $X(\overline{k})$ with $\pi(\alpha(S)) = \{a\}$, and the heights of the latter points are bounded by a linear polynomial in the heights of the former points, as is true for images under any morphism. Hence from now on, we may assume that X is a geometrically integral smooth projective curve of genus $g \geq 1$ embedded in its Jacobian A by the Albanese map determined by $P_0 \in X(k)$.

Since $A(\overline{k})$ is divisible, $a \in A(k)_{\mathbf{Q}}$ is represented by a point in $A(\overline{k})$, which we again call a. Enlarge k to assume that $a \in A(k)$. (This changes π as well, but it only makes the problem harder.) Choose a prime $\mathfrak{p}$ of good reduction for A, and let $\mathbf{F}_q$ denote the residue field. Extend $\mathfrak{p}$ to a place of $\overline{k}$. Let $\bar{X}$, $\bar{a} \in \bar{A}(\mathbf{F}_q)$, etc. denote the mod $\mathfrak{p}$ reductions of X, $a \in A(k)$, etc. Let ϕ denote the birational morphism $\bar{X}^{(g)} \to \bar{A}$ sending $\{x_1, \ldots, x_g\}$ to $x_1 + \cdots + x_g$, using the embedding $\bar{X} \hookrightarrow \bar{A}$. Let U and V denote dense open subsets of $\bar{X}^{(g)}$ and $\bar{A}$, respectively, such that ϕ induces an isomorphism $U \to V$. Let $\bar{u} \in U(\overline{\mathbf{F}}_q)$ be one of the infinitely many points given by Lemma 5, let $\bar{b} = \phi(\bar{u}) - g\bar{a} \in A(\overline{\mathbf{F}}_q)$, and lift $\bar{b}$ to a *torsion* point $b \in A(\overline{k})$.

By choice of U, we can write $ga + b = x_1 + \cdots + x_g$ for $x_i \in X(\overline{k})$, which are uniquely determined up to permutation. Moreover, $\operatorname{Gal}(\overline{k}/k(b))$ acts transitively on the x_i, since the choice of $\bar{u}$ guarantees Galois-transitivity on the reductions. Hence $\pi(x_i) = \pi(x_1)$ for all i, and

$$\pi(x_1) = \frac{1}{g} \sum_{i=1}^{g} \pi(x_i) = \frac{1}{g}\, \pi(ga + b) = a,$$

since $\pi(a) = a$ and $\pi(b) = 0$. There were infinitely many choices for $\bar{u}$, hence infinitely many distinct possibilities for $\bar{b}$, for b, and for x_1. In particular, the x_1 with $\pi(x_1) = a$ are Zariski dense in X. Finally, let E denote the largest open subset of A above which the summing morphism $s : X^g \to A$ is finite, i.e., above which $X^{(g)} \to A$ is an isomorphism.

Lemma 6 applied to $s^{-1}(E) \to E$ shows that $h(x_1)$ is bounded by a linear polynomial in $h(ga+b) = g^2 h(a)$. □

Proof of Theorem 2. Let $X_1, X_2, \dots$ be a complete list of the countably many possibilities for X. Choose a flag of subspaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \dots$$

of $A(\overline{\mathbf{Q}})_{\mathbf{Q}}$ such that $\dim V_n = n$ and $\bigcup V_n = A(\overline{\mathbf{Q}})_{\mathbf{Q}}$. Let $S_n(X_j)$ denote the set of $x \in X_j(\overline{\mathbf{Q}})$ whose image in $A(\overline{\mathbf{Q}})_{\mathbf{Q}}$ lies in V_n. The Mordell-Lang conjecture guarantees that $S_n(X_j)$ is finite for each $n \geq 0$ and $j \geq 1$. Starting with the zero map $\pi_0 : V_0 \to \mathbf{Q}$, by induction on $n \geq 1$, we can define $\mathbf{Q}$-linear maps $\pi_n : V_n \to \mathbf{Q}$ such that $\pi_n|_{V_{n-1}} = \pi_{n-1}$ and $|\pi_n(x)| \geq n$ for any x in the finite set $\bigcup_{j \leq n} (S_n(X_j) - S_{n-1}(X_j))$.

The π_n glue to give $\pi : A(\overline{\mathbf{Q}})_{\mathbf{Q}} \to \mathbf{Q}$. For each $j, n \geq 1$, $\{x \in X_j(\overline{\mathbf{Q}}) : \pi(x) \in (-n, n)\}$ is contained in $S_{n-1}(X_j)$, so it is finite. This implies, for each $j \geq 1$, that $\pi(X_j(\overline{\mathbf{Q}}))$ is discrete and that $\{x \in X_j(\overline{\mathbf{Q}}) : \pi(x) = a\}$ is finite for each $a \in \mathbf{Q}$. □

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