Tetrahedron Instantons

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Based on work with

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and work in progress

Plan for the talk

- Motivations:
    1. A generalization of Yang-Mills instantons
    2. New moduli spaces
    3. Nontrivial tests of M-theory/type IIA duality
- Properties of tetrahedron instantons:
    1. Construction in string theory
    2. Instanton moduli space
    3. Instanton partition function
- The index of M-theory and matching with instanton partition function

Yang-Mills instantons

- The Euclidean action of 4d Yang-Mills theory is

$$S = -\frac{1}{2g^2}\int d^4x\, \mathrm{tr} F_{\mu\nu}^2, \quad F_{\mu\nu} = F_{\mu\nu}^a T_a, \quad [T_a, T_b] = f_{ab}^c T_c,$$

which can be written as

$$S = -\frac{1}{4g^2}\int d^4x\, \mathrm{tr}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})^2 \pm \frac{1}{2g^2}\int d^4x\, \mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}) \geq \frac{8\pi^2 |k|}{g^2},$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$, and k is a topological invariant (instanton number).

- In addition to the perturbative vacua of the theory, there are other minima with finite action, the (anti-)instantons:

$$\tilde{F}_{\mu\nu} = \pm F_{\mu\nu}, \quad k = \frac{1}{16\pi^2}\int d^4x\, \mathrm{tr} F_{\mu\nu}\tilde{F}_{\mu\nu} \in \mathbb{Z}$$

Yang-Mills instantons

$$\tilde{F}_{\mu\nu} = \pm F_{\mu\nu}, \quad k = \frac{1}{16\pi^2}\int d^4x \, \mathrm{tr} F_{\mu\nu}\tilde{F}_{\mu\nu} \in \mathbb{Z}$$

- Example: BPST instanton ($\mathrm{SU}(2), k = 1$ in regular gauge): [Belavin, Polyakov, Schwarz, Tyupkin, 1975]

$$A_\mu(x) = \frac{2\sigma_{\mu\nu}(x-z)_\nu}{(x-z)^2 + \rho^2}.$$

  - This instanton solution is characterized by eight free parameters: the position z_μ, the size ρ, and the global SU(2) gauge orientation.
- In general, the space of instanton solutions up to local gauge transformations is a smooth manifold, the moduli space of instantons $\mathscr{M}_{G,k}$.

Yang-Mills instantons: ADHM construction

- All SU(n) instantons with instanton number k can be constructed from the ADHM data:

1. Two $k \times k$ complex matrices B_1, B_2, one $k \times n$ complex matrix I, and one $n \times k$ complex matrix J
2. Moment maps $\mu^{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J, \quad \mu^{\mathbb{C}} = [B_1, B_2] + IJ$
3. U(k) symmetry: $\left(B_a, I, J\right) \sim \left(gB_a g^{-1}, gI, Jg^{-1}\right), \quad g \in \mathrm{U}(k)$

[Atiyah, Drinfeld, Hitchin, Manin, 1978]

- The moduli space $\mathscr{M}_{n,k} \cong \left\{ \left(B_1, B_2, I, J\right) \middle| \mu^{\mathbb{R}} = \mu^{\mathbb{C}} = 0 \right\} \Big/ \mathrm{U}(k)$

- To avoid the non-compactness of $\mathscr{M}_{n,k}$ due to small instantons, Nakajima introduced a smooth manifold $\widetilde{\mathscr{M}}_{n,k}$, which can be obtained from the Uhlenbeck compactification of $\mathscr{M}_{n,k}$ by resolving the singularities

[Nakajima, 1994]

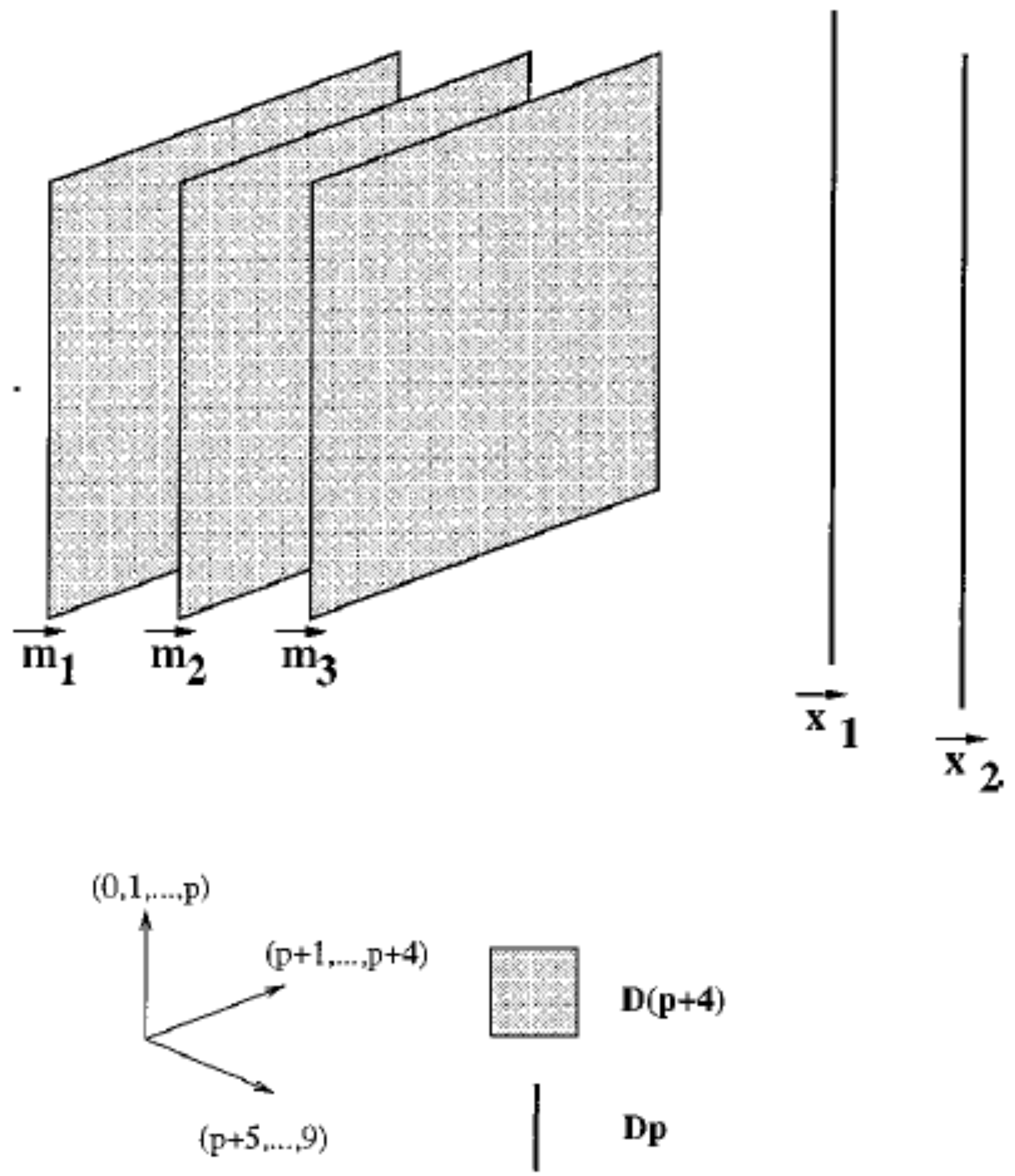
$$\widetilde{\mathscr{M}}_{n,k} \cong \left\{ \left(B_1, B_2, I, J\right) \middle| \mu^{\mathbb{R}} - r \cdot \mathbb{I}_k = \mu^{\mathbb{C}} = 0 \right\} \Big/ \mathrm{U}(k), \quad r > 0$$

Yang-Mills instantons: String theory

- k Dp-branes probing a stack of n coincident D(p+4)-branes in type II string theory ⇒ SU(n) instantons with instanton number k [Witten, 1994; Douglas 1995]

$\mathcal{M}_{n,k}$ ≅ Higgs branch of supersymmetric gauge theory on Dp-branes.

- Nekrasov and Schwarz interpreted $\widetilde{\mathcal{M}}_{n,k}$ as the moduli space of U(n) instantons on $\mathbb{C}^2_\Theta$. [Nekrasov, Schwarz, 1998]
- $\widetilde{\mathcal{M}}_{n,k}$ in string theory: turn on a nonzero constant background B-field. [Seiberg, Witten, 1999]



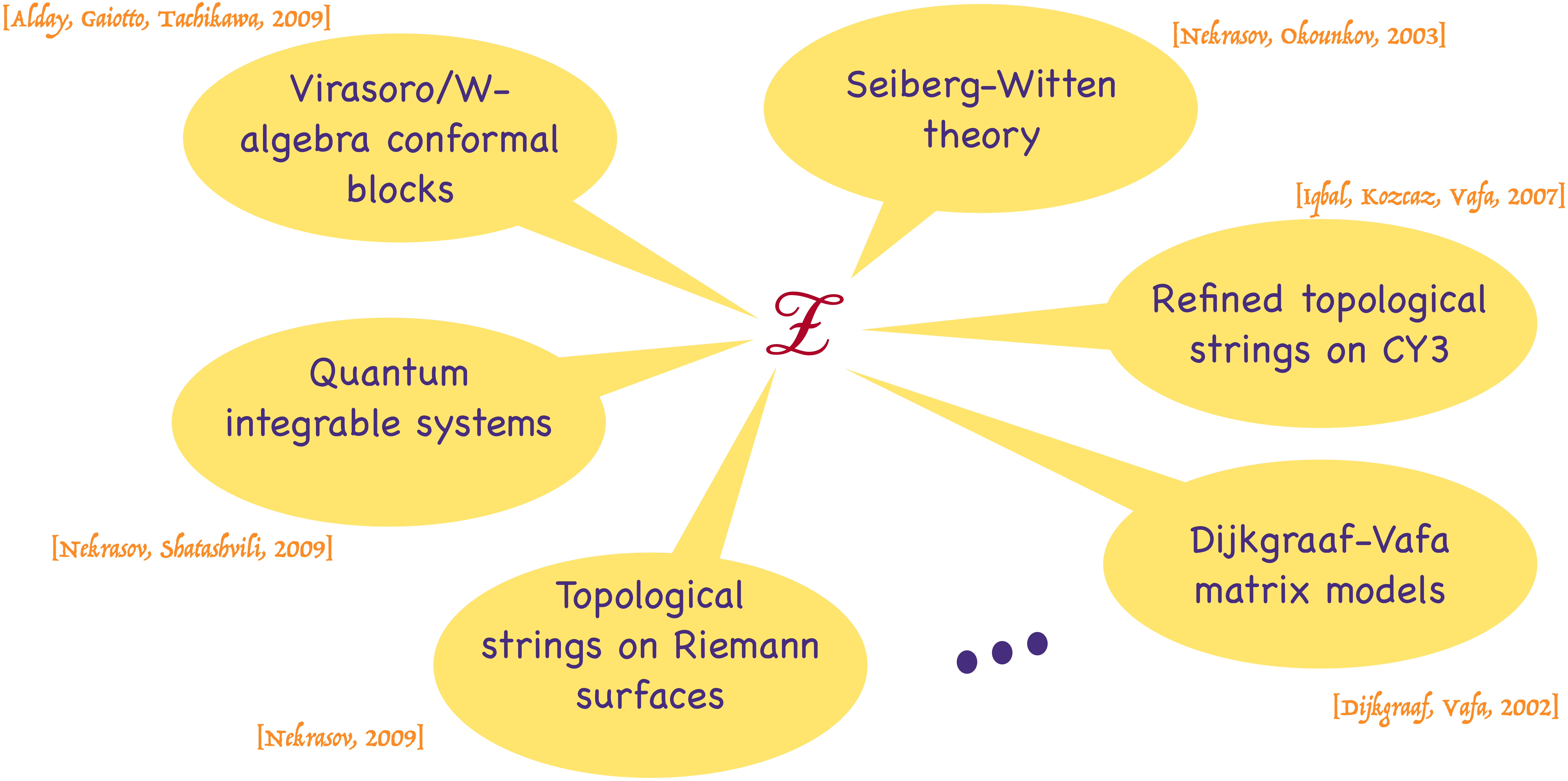
Instanton partition function

Nekrasov introduced the instanton partition function [Nekrasov, 2002]

$$\mathcal{Z} = \sum_{k\geq 0} \mathsf{q}^k \mathcal{Z}_k, \quad \mathcal{Z}_k = \int_{\widetilde{\mathcal{M}}_{n,k},\mathbf{T}} \cdots$$

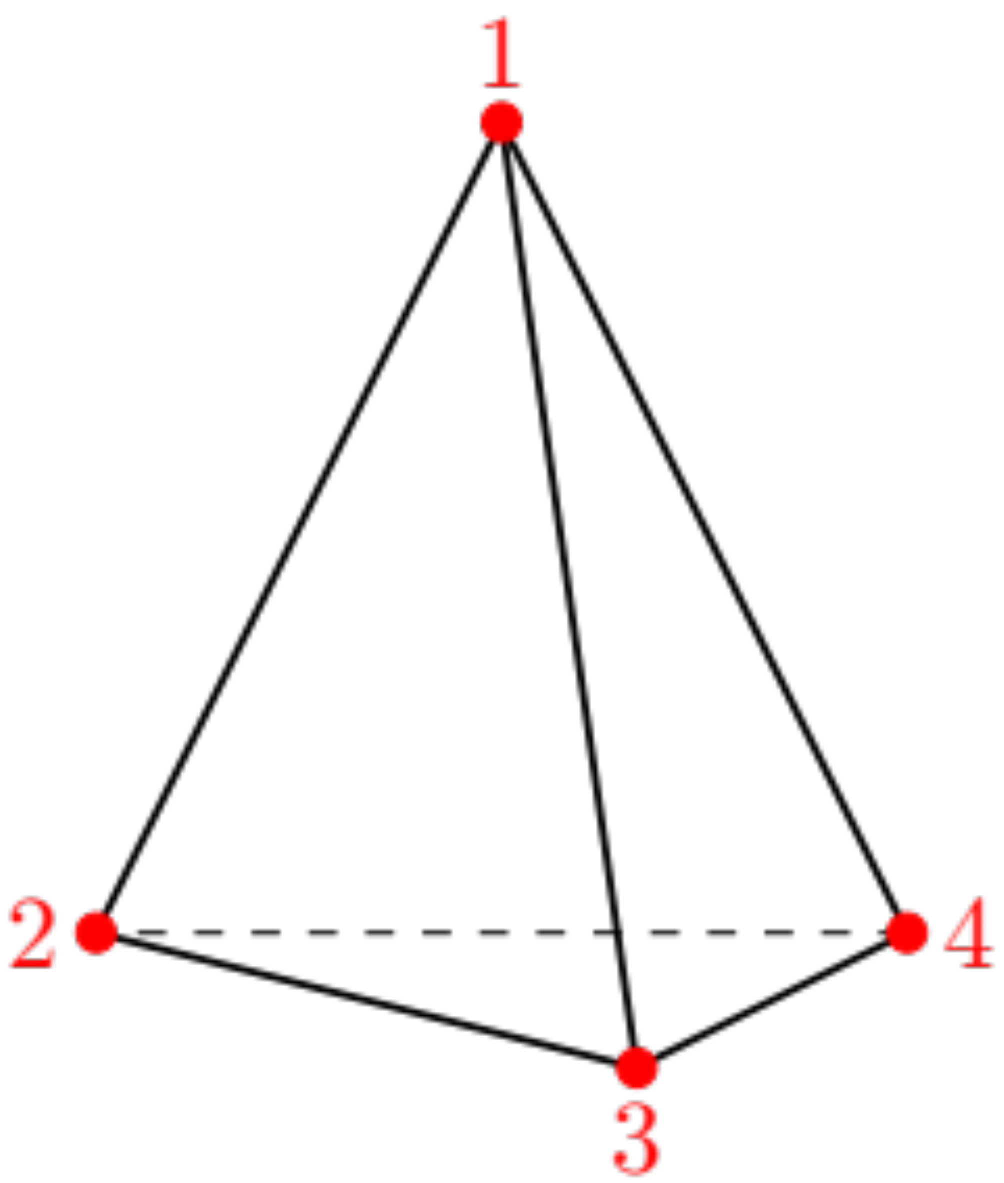
- The equivariant group $\mathbf{T}$: a maximal torus of $\mathrm{U}(1)^2 \times \mathrm{U}(n)$, which rotate the spacetime $\mathbb{C}^2$ and the gauge orientation at infinity.
- $\mathcal{Z}$ is the non-perturbative part of the supersymmetric partition function of a 4d $\mathcal{N} = 2$ theory (or its higher-dimensional lift) in the Omega background.
- $\mathcal{Z}$ can be evaluated exactly using localization techniques. The result can be expressed as a statistical sum over a collection of random partitions.

Instanton partition function



Tetrahedron instantons

Aim:Study D0-branes probing a configuration of intersecting D6-branes.



	$\mathbb{S}^1_t$	$\mathbb{C}_1$		$\mathbb{C}_2$		$\mathbb{C}_3$		$\mathbb{C}_4$		
	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
k D0	–	•	•	•	•	•	•	•	•	•
n_{123} $\mathrm{D6}_{123}$	–	–	–	–	–	–	–	•	•	•
n_{124} $\mathrm{D6}_{124}$	–	–	–	–	–	•	•	–	–	•
n_{134} $\mathrm{D6}_{134}$	–	–	–	•	•	–	–	–	–	•
n_{234} $\mathrm{D6}_{234}$	–	•	•	–	–	–	–	–	–	•

Vertex $a \in \underline{4} = \{1,2,3,4\} \leftrightarrow \mathbb{C}_a$

Face $A \in \underline{4}^\vee = \{(123),(124),(134),(234)\}$

$\leftrightarrow \mathbb{C}^3_A = \prod_{a\in A} \mathbb{C}_a \subset \mathbb{C}^4$

$\check{A} = \underline{4} \backslash A$

$$B = \sum_{a\in\underline{4}} b_a dx^{2a-1} \wedge dx^{2a}$$

$$e^{2\pi i v_a} = \frac{1+ib_a}{1-ib_a}, \quad -\frac{1}{2} < v_a < \frac{1}{2}$$

BPS bound states

- Supersymmetry is completely broken for generic v_a.
- When $v_1 = v_2 = v_3 = v_4 = \frac{1}{6} + \frac{r}{3}$, the scalar potential is given by

$$V = \mathrm{Tr}\left(\sum_{a\in\underline{4}}[B_a, B_a^\dagger] + \sum_{A\in\underline{4}^\vee} I_A I_A^\dagger - r\right)^2 + \sum_{A\in\underline{4}^\vee}\mathrm{Tr}\left|B_{\check{A}} I_A\right|^2 + \sum_{a<b\in\underline{4}}\mathrm{Tr}\left|[B_a, B_b]\right|^2 .$$

Here B_a and I_A are from D0-D0 strings and D0-D6$_A$ strings, respectively.

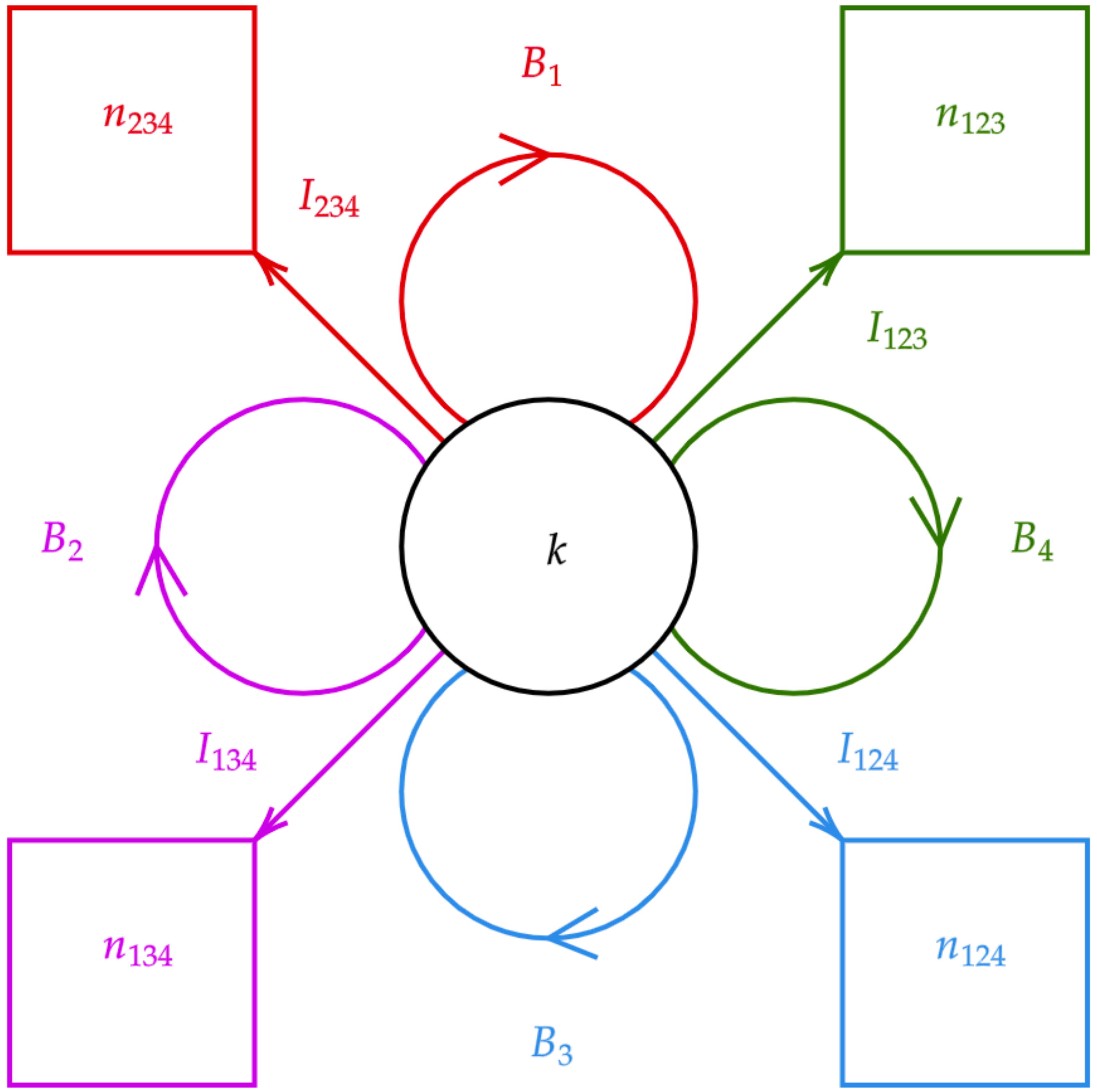
- In order to obtain an analogue of $\widetilde{\mathcal{M}}_{n,k}$, we need to take $r > 0$. In this case, the low-energy theory on k D0-branes can be described by a supersymmetric gauged matrix model with gauge group $\mathrm{U}(k)$, and it preserves two supercharges $Q_+, \bar{Q}_+$.

Tetrahedron instantons: Quivers

Quivers for the supersymmetric gauge theory on D0-branes

$$V = \mathrm{Tr}\left(\sum_{a\in\underline{4}}[B_a, B_a^\dagger] + \sum_{A\in\underline{4}^\vee} I_A I_A^\dagger - r\right)^2 + \sum_{A\in\underline{4}^\vee}\mathrm{Tr}\left|B_{\check{A}} I_A\right|^2 + \sum_{a<b\in\underline{4}}\mathrm{Tr}\left|[B_a, B_b]\right|^2$$

- $\widetilde{\mathscr{M}}_{n,k}$ is the Higgs branch of this quiver gauge theory.



Instanton moduli space

- The moduli space of tetrahedron instantons: the space of solutions to $V = 0$ modulo the gauge symmetry $\mathrm{U}(k)$,

$$\mathfrak{M}_{\vec{n},k} \cong \left\{ \left(B_a \in \mathrm{End}\left(\mathbb{C}^k\right), I_A \in \mathrm{Hom}\left(\mathbb{C}^{n_A}, \mathbb{C}^k\right) \right) \middle| \mu^{\mathbb{R}} - r \cdot \mathbb{1}_k = \mu^{\mathbb{C}} = \sigma = 0 \right\} \Big/ \mathrm{U}(k)$$

$$\mu^{\mathbb{R}} = \sum_{a \in \underline{4}} \left[B_a, B_a^\dagger\right] + \sum_{A \in \underline{4}^\vee} I_A I_A^\dagger, \ \mu^{\mathbb{C}} = \left(\mu^{\mathbb{C}}_{ab} = \left[B_a, B_b\right]\right)_{a,b \in \underline{4}}, \ \sigma = \left(\sigma_A = B_{\check{A}} I_A\right)_{A \in \underline{4}^\vee}$$

$$\left(B_a, I_A\right) \sim \left(g B_a g^{-1}, g I_A\right), \quad g \in \mathrm{U}(k)$$

- If $\vec{n} = \left(n_{123} = 1,0,0,0\right)$, $\mathfrak{M}_{\vec{n},k}$ reduces to the moduli space of certain torsion free sheaves on $\mathbb{C}^3$. [Cirafici, Sinkovics, Szabo, 2008] [Benini et al, 2018]

[Nekrasov, 2017; Nekrasov, Piazzalunga, 2018]

- If we drop σ-equations, $\mathfrak{M}_{\vec{n},k}$ becomes the moduli space of magnificent four model.
- The virtual dimension (# components of matrices - # constraints - # gauge) of $\mathfrak{M}_{\vec{n},k}$ is 0.

Instanton moduli space with k=1

- $\vec{n} = (n_{123} = n, 0, 0, 0)$: B_1, B_2, B_3 are unconstrained complex numbers, $I_{124} = I_{134} = I_{234} = 0$, and

$$B_4 I_{123} = 0, \quad \sum_{\alpha=1}^{n} \left| I_{123,\alpha} \right|^2 = r, \quad I_{123} \sim e^{i\theta} I_{123}.$$

Therefore, $\mathfrak{M}_{(n,0,0,0),1} \cong \mathbb{C}^3 \times \mathbb{CP}^{n-1}$.

- $\vec{n} = (n_{123} = n, n_{124} = m, 0, 0)$: B_1, B_2 are unconstrained complex numbers, $I_{134} = I_{234} = 0$, and

$$B_3 I_{124} = B_4 I_{123} = 0, \quad \sum_{\alpha=1}^{n} \left| I_{123,\alpha} \right|^2 + \sum_{\alpha=1}^{m} \left| I_{124,\alpha} \right|^2 = r, \quad I_A \sim e^{i\theta} I_A$$

Therefore, $\mathfrak{M}_{(n,m,0,0),1} \cong \mathbb{C}^2 \times \left(\mathbb{C}^* \times \mathbb{CP}^{n-1} \cup \mathbb{C}^* \times \mathbb{CP}^{m-1} \cup \mathbb{CP}^{n+m-1} \right)$.

In general, $\mathfrak{M}_{\vec{n},k}$ consists of several smooth manifolds with different actual dimensions.

Instanton partition function

We define the (K-theoretical) instanton partition function as

$$Z = \sum_{k=0}^{\infty} (-p)^k Z_k = \sum_{k=0}^{\infty} (-p)^k \hat{A}_{\mathbf{T}} \left(\mathfrak{M}_{\vec{n},k} \right) = \sum_{k=0}^{\infty} (-p)^k \mathrm{Tr}_{\mathscr{H}_k} \left[(-1)^F \prod_{a \in \underline{4}} q_a^{\mathscr{J}_a} \prod_{A \in \underline{4}^\vee} \prod_{\alpha=1}^{n_A} t_{A,\alpha}^{T_{A,\alpha}} \right]_{\prod_{a\in\underline{4}} q_a = 1}$$

$\mathbf{T}$: maximal torus of $\mathrm{U}(1)^3 \times \prod_{A \in \underline{4}^\vee} \mathrm{U}(n_A)$

$\mathscr{H}_k$: the Hilbert space of the worldvolume theory with k D0-branes

$\mathscr{J}_a$: the generator of the $\mathrm{U}(1)_a$ rotation, satisfying $\left[\mathscr{J}_a, Q_+\right] = -Q_+$, $\left[\mathscr{J}_a, \bar{Q}_+\right] = \bar{Q}_+$

$T_{(A,\alpha)}$: the Cartan generators of the symmetry group $\mathrm{U}\left(n_A\right)$

.

Expectation value of codimension-two defects

Up to now, we treat all D6-branes on equal footing, but we can choose the physical spacetime to be $\mathbb{S}^1_t \times \mathbb{C}^3_{123}$, so that the bound states of D0- and D6$_{123}$-branes give rise to instantons on $\mathbb{C}^3_{123}$, while the remaining D6-branes will produce codimension-two defects.

$$Z = \sum_{k=0}^{\infty} \frac{(-p)^k}{k!} \int \prod_{i=1}^{k} d\phi_i \left[\left(Z_k^{0-0} Z_k^{0-6_{123}} \right) \left(\prod_{A \in \underline{4}^\vee \setminus \{(123)\}} Z_k^{0-6_A} \right) \right] = \left\langle \prod_{A \in \underline{4}^\vee \setminus \{(123)\}} \mathcal{O}_A \right\rangle_{\mathrm{DT}},$$

where $Z_{\mathrm{DT}} = \sum_{k=0}^{\infty} \frac{(-p)^k}{k!} \int \prod_{i=1}^{k} d\phi_i Z_k^{0-0} Z_k^{0-6_{123}}$.

- There is also an elliptic version if we perform a T-duality and studying D1-D7 system.

Instanton partition function

Applying the supersymmetric localization techniques, we can express Z_k as

$$Z_k = q_4^{k^2}\left(\frac{(1-q_1q_2)(1-q_1q_3)(1-q_2q_3)}{\prod_{a\in\underline{4}}(1-q_a)}\prod_{A\in\underline{4}^\vee} q_A^{n_A/2}\right)^k \times$$

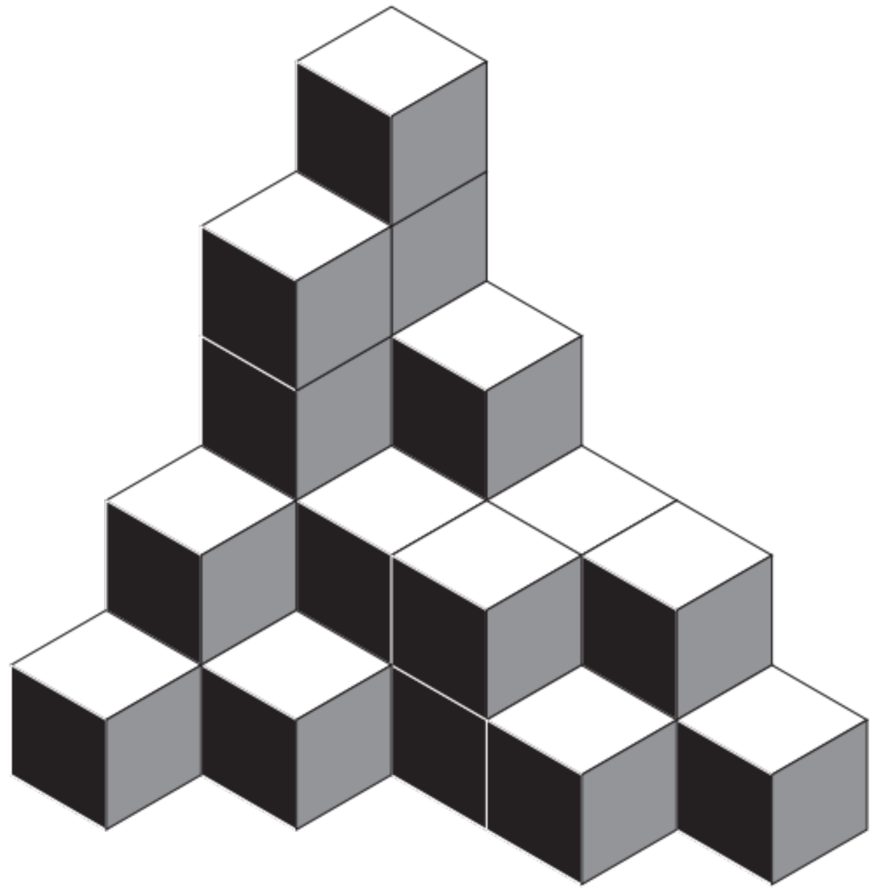
$$\times\frac{1}{k!}\int\prod_{i=1}^{k}\frac{dx_i}{x_i}\prod_{\substack{i,j=1\\ i\neq j}}^{k}\frac{(x_j-x_i)(x_j-q_1q_2x_i)(x_j-q_1q_3x_i)(x_j-q_2q_3x_i)}{\prod_{a\in\underline{4}}(x_j-q_ax_i)}\times\prod_{i=1}^{k}\prod_{A\in\underline{4}^\vee}\prod_{\alpha=1}^{n_A}\frac{(x_i-q_A^{-1}t_{A,\alpha})}{(x_i-t_{A,\alpha})}.$$

The contour integral is evaluated using the *Jeffrey-Kirwan residue prescription*, and the poles are labeled by a collection of plane partitions $\vec{\pi} = \left\{\pi^{(A,\alpha)}\right\}$. [Jeffrey, Kirwan, 1993]

The poles:

$$\{x_i\} = \left\{t_{A,\alpha}q_a^{1-s_X}q_b^{1-s_Y}q_c^{1-s_Z},\ (s_X,s_Y,s_Z)\in\pi^{(A,\alpha)},\ A=(abc)\in\underline{4}^\vee,\ \alpha=1,\cdots,n_A\right\}$$

Each plane partition $\pi = \begin{pmatrix}\pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \cdots\\ \pi_{2,1} & \pi_{2,2} & \pi_{2,3} & \cdots\\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix},\quad \pi_{x,y}\geq\pi_{x+1,y},\pi_{x,y+1}\geq 0.$



Plethystic exponential form

Remarkably, the instanton partition function allows a plethystic expression

$$Z = \sum_{k=0}^{\infty} (-p)^k Z_k = \mathrm{PE}_{\vec{q},p}\left\{ \frac{[q_1q_2]\,[q_1q_3]\,[q_2q_3]}{\prod_{a\in\underline{4}}[q_a]} \frac{[Q]}{[Q^{\frac{1}{2}}p]\,[Q^{\frac{1}{2}}p^{-1}]} \right\}, \quad Q = \prod_{A\in\underline{4}^\vee} q_{\check{A}}^{n_A}$$

$= \mathbb{F}_{\vec{n}}(\vec{q},p)$: single-particle seed

$[x] = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$. Plethystic exponent $\mathrm{PE}_{\vec{x}}\{f(x_1,\cdots,x_m)\} = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f(x_1^n,\cdots,x_m^n)\right]$.

This is a generalization of the plethystic expression for Donaldson-Thomas invariants.

Singular when $p \to Q^{\pm\frac{1}{2}}$

[Nekrasov, 2009; Nekrasov, Okounkov, 2014]

Magnificent four and Tachyon condensation

Magnificent four model: a system of D0-branes probing a D8-brane and an anti-D8-brane, with a strong background B-field who's instanton partition function looks very similar.

[Nekrasov, 2017]

$$Z^{MF} = \mathrm{PE}_{\vec{q},p,\mu}\left\{ \frac{[q_1q_2]\,[q_1q_3]\,[q_2q_3]}{\prod_{a\in\underline{4}}[q_a]} \frac{[\mu]}{\left[\mu^{\frac{1}{2}}p\right]\left[\mu^{\frac{1}{2}}p^{-1}\right]} \right\}$$

where μ encodes the relative position of the D8-brane and the anti-D8-brane.

The tetrahedron instanton partition function $Z^{MF} = Z$ by substituting the following in the magnificent four model

$$\mu = Q = \prod_{A\in\underline{4}^{\vee}} q_{\check{A}}^{n_A}$$

- indicating that the annihilation of the D8-brane and the anti-D8-brane leaves behind a system of D6-branes.

Decomposition property

$\mathbb{F}_{\vec{n}}\left(\vec{q},p\right)$ is independent of $t_{A,\alpha}$. We can take all D6-branes to be widely separated.

$\Rightarrow$ decomposition property:

$$\mathbb{F}_{(n,0,0,0)}\left(\vec{q},p\right)=\sum_{a=1}^{n}\mathbb{F}_{(1,0,0,0)}\left(\vec{q},q_4^{a-\frac{n+1}{2}}p\right),$$

$$\mathbb{F}_{(n,m,0,0)}\left(\vec{q},p\right)=\sum_{a=1}^{n}\mathbb{F}_{(1,0,0,0)}\left(\vec{q},q_3^{\frac{m}{2}}q_4^{a-\frac{n+1}{2}}p\right)+\sum_{b=1}^{m}\mathbb{F}_{(0,1,0,0)}\left(\vec{q},q_3^{b-\frac{m+1}{2}}q_4^{-\frac{n}{2}}p\right),$$

$$\cdots\cdots$$

$\Rightarrow$ $\mathbb{F}_{\vec{n}}\left(\vec{q},p\right)$ has the structure of orbifold partition function. Crucial in matching M-theory index

Cohomological limit

We introduce $q_a = e^{\beta \varepsilon_a}$. Taking the limit $\beta \to 0$ while keeping ε_a and p fixed, we have

$$\mathbb{F}_{\vec{n}}\left(\vec{q}, p\right) \to \frac{p}{(1-p)^2} \sum_{A \in \underline{4}^\vee} r_A n_A, \quad r_A = -\frac{\prod_{a<b\in A}\left(\varepsilon_a + \varepsilon_b\right)}{\prod_{a\in A} \varepsilon_a}$$

The cohomological instanton partition function (D-instanton probing intersecting D5-branes):

$$\mathcal{Z}^\flat\left(\vec{\varepsilon}, p\right) = \lim_{\beta \to 0} Z\left(\vec{q}, p\right) = \prod_{A \in \underline{4}^\vee} \mathcal{M}_3(p)^{r_A n_A},$$

where $\mathcal{M}_3(p)$ is the MacMahon function

$$\mathcal{M}_3(p) = \sum_{k=0}^{\infty} \mathrm{PL}(k) p^k = \prod_{m=1}^{\infty} \frac{1}{\left(1-p^m\right)^m} = \mathrm{PE}\left[\frac{p}{(1-p)^2}\right]$$

It is interesting that the all-genus A-model topological string partition function of $\mathbb{C}^3_{123}$ is

$$Z^{\mathrm{top}}\left(\varepsilon_1, \varepsilon_2, \varepsilon_3, \hbar\right) = \exp \sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g\left(\varepsilon_1, \varepsilon_2, \varepsilon_3\right) = \left(\mathcal{M}_3\left(-e^{\mathrm{i}\hbar}\right)\right)^{\frac{(\varepsilon_1+\varepsilon_2)(\varepsilon_1+\varepsilon_3)(\varepsilon_2+\varepsilon_3)}{\varepsilon_1\varepsilon_2\varepsilon_3}}$$

Free field representation

- Free massless multi-component scalar field φ on $\mathbb{T}^2$:

$$\left\langle \varphi_i(z,\bar{z})\,\varphi_j(0,0)\right\rangle_{\mathbb{T}^2} = -\log\left|\frac{\theta_1(z|\,\tau)}{2\pi\eta(\tau)^3}\exp\left(-\frac{\pi\,(\mathrm{Im}z)^2}{\mathrm{Im}\tau}\right)\right|^2 \delta_{i,j}.$$

- Introduce a vertex operator

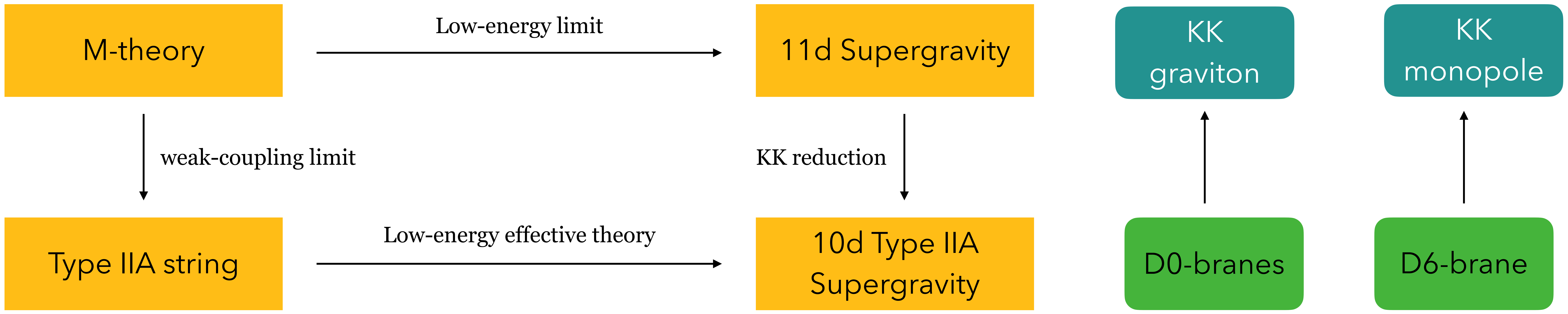
$$\mathscr{V}_{\alpha,\rho}(z,\bar{z}) =: \exp\left[\mathrm{i}\sum_{i=1}^{7}\alpha_i\varphi_i\left(z+\rho_i,\bar{z}+\rho_i\right)\right] :: \exp\left[-\mathrm{i}\sum_{i=1}^{7}\alpha_i\varphi_i\left(z-\rho_i,\bar{z}-\rho_i\right)\right]:,$$

where $\alpha = \left(\mathrm{i},\mathrm{i},\mathrm{i},\mathrm{i},1,1,1\right)$ and $\rho = \frac{1}{2}\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_{12},\varepsilon_{13},\varepsilon_{23}\right)$.

- Introduce a linear source operator $\Upsilon = \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma} dz \sum_{A\in\underline{4}^\vee}\varpi_A(z)\partial_z\varphi_{\check{A}}(z)$, where Γ is a loop arount $z=0$ encircling all $\pm\rho_i$, and $\varpi_A(z) = \sum_{\alpha=1}^{n_A}\log\theta_1\left(z-\mathrm{a}_{A,\alpha}-\frac{1}{2}\varepsilon_A\,\middle|\,\tau\right)$.

- The instanton partition function admits a free field representation: $Z = \left\langle e^{\Upsilon} e^{\mathfrak{q}\oint_{\mathscr{C}}\mathscr{V}_{\alpha,\rho}(z)dz}\right\rangle^{\mathrm{hol}}_{\mathbb{T}^2}.$

M-theory/Type IIA duality



A bound state of a D6-brane and k D0-branes on $\mathbb{S}^1$ can be lifted to an 11d bound state of k KK gravitons on $\mathbb{S}^1 \times \mathbb{C}^3 \times \mathbb{TN}$.

$$\mathscr{Z}^{\mathrm{M}}(\vec{g})\left[\mathbb{S}^1 \times \mathbb{C}^3 \times \mathbb{TN}\right] = \sum_{k=0}^{\infty} Z_{D6-k\,D0}(\vec{\mu})$$

Geometric parameters

Fugacities of global symmetry

[Nekrasov, 2009;
Nekrasov, Okounkov, 2014;
Benini, Bonelli, Poggi, Tanzini, 2018]

M-theory Index

A system of intersecting D6-branes ⇒ superposition of KK monopoles, described by a non-compact Calabi-Yau fivefold $\mathscr{X}$.

Compute the twisted index of M-theory

$$\mathscr{Z}^{\mathrm{M}}\left[\mathbb{S}^1_t \rtimes_g \mathscr{X}\right](v_1,\cdots,v_5) = \mathrm{PE}_{\overrightarrow{v}}\left\{\mathscr{F}^{\mathrm{M}}(v_1,\cdots,v_5)\right\}$$

SU(5) **isometry of** $\mathscr{X}$**,** $\prod_{i=1}^{5} v_i = 1$

Mapping $(v_1,\cdots,v_5) \to (\overrightarrow{q}, p)$, we find $\mathscr{F}^{\mathrm{M}}(v_1,\cdots,v_5) = \mathbb{F}(\overrightarrow{q},p) + \mathscr{P}(\overrightarrow{q})$.

$\mathscr{P}(\overrightarrow{q})$: the contribution without D0-branes.

$$\mathscr{Z}^{\mathrm{M}}\left[\mathbb{S}^1_t \rtimes_g \mathscr{X}\right](v_1,\cdots,v_5) = Z^{\mathrm{pert}}(\overrightarrow{q})\, Z(\overrightarrow{q},p)$$

Matching M-theory Index with instant partition function

Basic example: $\vec{n} = (n_{123} = 1, n_{124} = n_{134} = n_{234} = 0)$:

M-theory side: $\mathscr{F}^{\mathrm{M}}\left(v_1, \cdots, v_5\right)\left[\mathbb{S}^1_t \rtimes_g \mathbb{C}^5\right] = -\dfrac{\sum_{i=1}^5 [v_i^2]}{\prod_{i=1}^5 [v_i]}$

Type IIA string theory side: $\mathbb{F}_{(1,0,0,0)}\left(\vec{q}, p\right) = \dfrac{[q_1q_2]\,[q_1q_3]\,[q_2q_3]}{\prod_{a\in\underline{4}}[q_a]}\dfrac{[q_4]}{\left[q_4^{\frac{1}{2}}p\right]\left[q_4^{\frac{1}{2}}p^{-1}\right]}$

We can find the dictionary:

$$\mathscr{F}^{\mathrm{M}}\left(v_1 = q_1, v_2 = q_2, v_3 = q_3, v_4 = q_4^{\frac{1}{2}}p, v_5 = q_4^{\frac{1}{2}}p^{-1}\right)\left[\mathbb{S}^1_t \rtimes_g \mathbb{C}^5\right] = \mathbb{F}_{(1,0,0,0)}\left(\vec{q}, p\right) + \frac{[q_4]}{[q_1]\,[q_2]\,[q_3]}$$

Using the decomposition property, we can also find the correspondence for general $\vec{n}$.

$$\mathbb{F}_{(n,0,0,0)}\left(\vec{q}, p\right) = \sum_{a=1}^{n}\mathbb{F}_{(1,0,0,0)}\left(\vec{q}, q_4^{a-\frac{n+1}{2}}p\right) \qquad \ldots\ldots$$

Summary

- We introduced the tetrahedron instantons, which can be realized in string theory by D0-branes probing a configuration of intersecting D6-branes with a suitable background B-field.
- We studied the moduli space of tetrahedron instantons.
- The instanton partition function can be computed exactly, and allows a plethystic expression.
- Lifting the type IIA configuration to M-theory, the instanton partition function can be reproduced from the M-theory index.

Outlook

Our discussion can be extension in several directions:

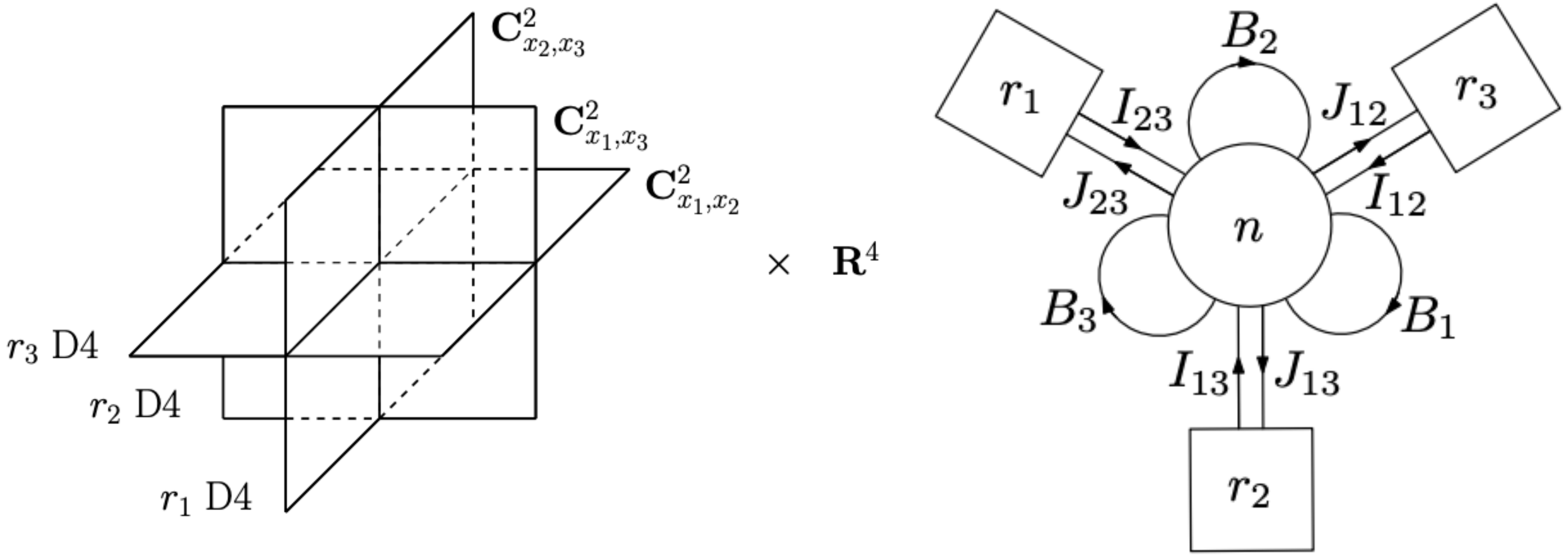
① Other spacetime $\mathbb{S}^1_t \times \mathrm{CY}_4 \times \mathbb{R}$ in IIA string theory.

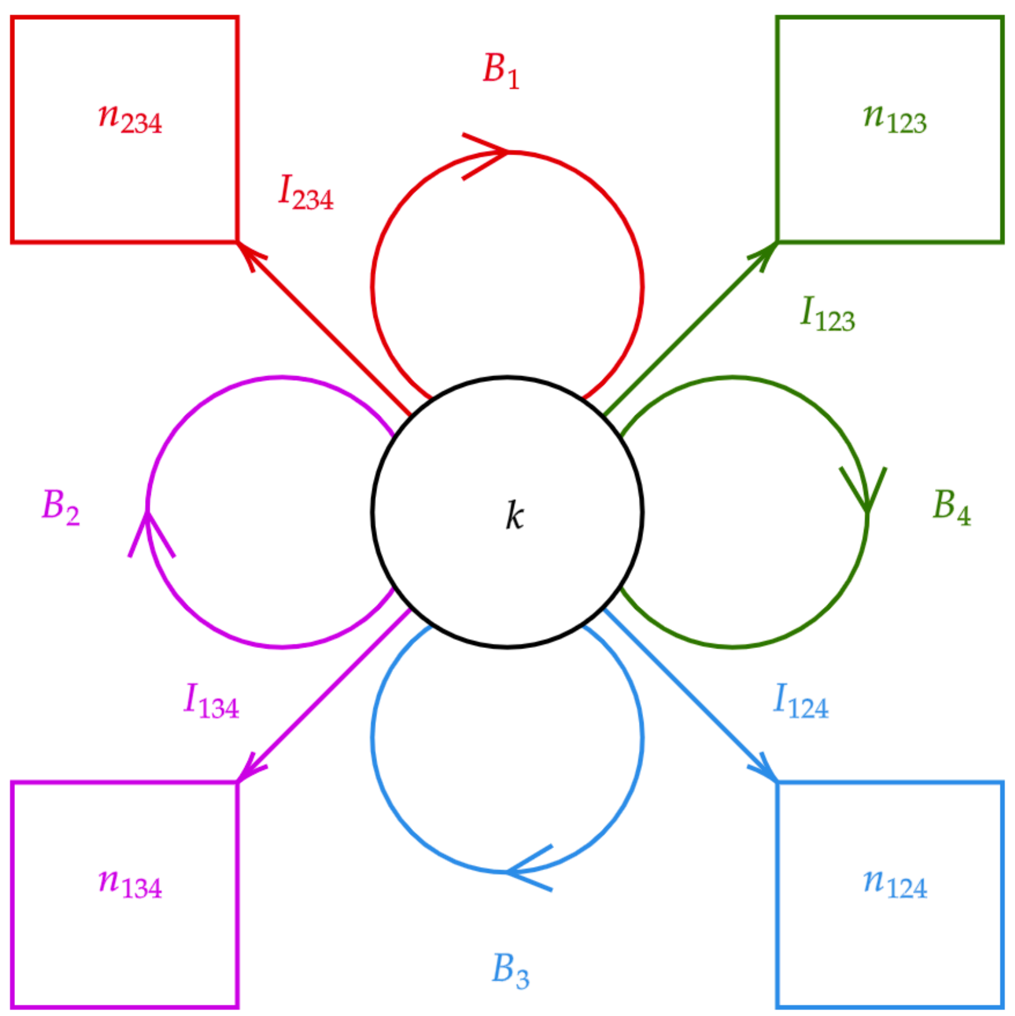
② Adding D2- and D4-branes. The M-theory index will also receive contributions from M-branes.

Possible action of Cohomological Hall algebra

Action of Cohomological Hall algebra (COHA) on the cohomology of moduli space of spiked instantons

[Rapcak, Soibelman, Yang, Zhao, 2018]
[Gaiotto, Rapcak]





PLOTS FROM [Rapcak, Soibelman, Yang, Zhao, 2018]

Free field representation of tetrahedron instanton partition function might help.

Thank you!

Free field representation

- Free massless multi-component scalar field φ on $\mathbb{T}^2$:

$$\left\langle \varphi_i(z,\bar{z})\,\varphi_j(0,0)\right\rangle_{\mathbb{T}^2} = -\log\left|\frac{\theta_1(z|\,\tau)}{2\pi\eta(\tau)^3}\exp\left(-\frac{\pi(\mathrm{Im}z)^2}{\mathrm{Im}\tau}\right)\right|^2 \delta_{i,j}.$$

- Introduce a vertex operator

$$\mathscr{V}_{\alpha,\rho}(z,\bar{z}) =: \exp\left[\mathrm{i}\sum_{i=1}^{7}\alpha_i\varphi_i\left(z+\rho_i,\bar{z}+\rho_i\right)\right] :: \exp\left[-\mathrm{i}\sum_{i=1}^{7}\alpha_i\varphi_i\left(z-\rho_i,\bar{z}-\rho_i\right)\right]:,$$

where $\alpha = \left(\mathrm{i},\mathrm{i},\mathrm{i},\mathrm{i},1,1,1\right)$ and $\rho = \frac{1}{2}\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_{12},\varepsilon_{13},\varepsilon_{23}\right)$.

- Introduce a linear source operator $\Upsilon = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma dz \sum_{A\in\underline{4}^\vee}\varpi_A(z)\partial_z\varphi_{\check{A}}(z)$, where Γ is a loop arount $z=0$ encircling all $\pm\rho_i$, and $\varpi_A(z) = \sum_{\alpha=1}^{n_A}\log\theta_1\left(z-\mathrm{a}_{A,\alpha}-\frac{1}{2}\varepsilon_A\,\middle|\,\tau\right)$.

- The instanton partition function admits a free field representation: $Z = \left\langle e^{\Upsilon}e^{\mathfrak{q}\oint_{\mathscr{C}}\mathscr{V}_{\alpha,\rho}(z)dz}\right\rangle^{\mathrm{hol}}_{\mathbb{T}^2}$.

Generalized field theories

A generalized field theory is constructed by merging several ordinary field theories across defects. Its spacetime X contains several intersecting components, $X = \cup_A X_A$. The fields and the gauge groups $G_A = G|_A$ on different components can be different, and the matter fields living on the intersection $X_A \cap X_B$ transform in the bifundamental representation of the product group $G_A \times G_B$.

Example: Spiked instantons

The instanton partition function of spiked instantons provides a unified treatment of instanton partition functions of 4d $\mathcal{N} = 2$ theories, with local or surface defects.

[Nekrasov, 2015; Nekrasov, Prabhakar, 2016]

The D1–D5 system for spiked instantons.

$\mathbf{R}^{1,9}$	1	2	3	4	5	6	7	8	9	0
$\mathbf{C}^4 \times \mathbf{R}^{1,1}$	z^1		z^2		z^3		z^4		x	t
D1									×	×
$D5_{(12)}$	×	×	×	×					×	×
$D5_{(13)}$	×	×			×	×			×	×
$D5_{(14)}$	×	×					×	×	×	×
$D5_{(23)}$			×	×	×	×			×	×
$D5_{(24)}$			×	×			×	×	×	×
$D5_{(34)}$					×	×	×	×	×	×

$$B = \sum_{a=1}^{4} b_a dx^{2a-1} \wedge dx^{2a}$$

Research Overview

My research interests span the areas of quantum field theory, supersymmetry, string theory and mathematical physics. In addition to today's topic, I have been working on

① $\mathcal{N}=2$ supersymmetric gauge theories

- Derivation of Seiberg-Witten geometry via instanton counting
- Alday-Gaiotto-Tachikawa correspondence via non-perturbative Dyson-Schwinger equations [with Saebyeok Jeong]
- First-principle calculation of effective gravitational couplings [with Jan Manschot and Gregory Moore]
- Correspondence with 2d topological strings

② Donaldson invariants of four-manifolds

- K-theoretical/elliptic Donaldson invariants [with Heeyeon Kim, Jan Manschot, Gregory Moore and Runkai Tao]

③ $\mathcal{N}=1$ theories of class S_k [with Thomas Bourton and Elli Pomoni]

④ Supersymmetric localization computations in field/supergravity theories [with Jun Nian]

⑤ Generalization of the notion of symmetries beyond group theory [with Enrico Andriolo, Hanno Bertle, Elli Pomoni and Konstantinos Zoubos]

- Hidden quantum $\mathcal{N}=4$ superconformal symmetry in $\mathcal{N}=2$ superconformal theories

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
k D1	–	•	•	•	•	•	•	•	•	–
n_{123} $D7_{123}$	–	–	–	–	–	–	–	•	•	–
n_{124} $D7_{124}$	–	–	–	–	–	•	•	–	–	–
n_{134} $D7_{134}$	–	–	–	•	•	–	–	–	–	–
n_{234} $D7_{234}$	–	•	•	–	–	–	–	–	–	–

- Two quartets of matrices

$$\vec{B} = (B_a)_{a\in\underline{4}},\ B_a \in \mathrm{End}\left(\mathbb{C}^k\right),$$

$$\vec{I} = (I_A)_{A\in\underline{4}^\vee},\ I_A \in \mathrm{Hom}\left(\mathbb{C}^{n_A}, \mathbb{C}^k\right)$$

$$\mathbb{R}^{1,9} \cong \mathbb{S}^1_t \times \prod_{a\in\underline{4}} \mathbb{C}_a \times \mathbb{R}_9$$

Strings	$\mathcal{N}=(2,2)$	$\mathcal{N}=(0,2)$	$(\mathrm{U}(k), \mathrm{U}(n_A))$
D1-D1	Vector	Vector Υ	(Adj, 1)
		Chiral $\Phi_{\breve{A}} = B_{\breve{A}} + \cdots$	
	Chiral $(a \in A)$	Chiral $\Phi_a = B_a + \cdots$	
		Fermi $\Psi_{a,-} = \psi_{a,-} + \cdots$	
D1-D7$_A$	Chiral	Chiral $\Phi_A = I_A + \cdots$	$(k, \overline{n_A})$
		Fermi $\Psi_{A,-} = \psi_{A,-} + \cdots$	

$$Z = \sum_{k=0}^{\infty} \mathfrak{q}^k \chi_{\mathbf{T}}\left(\mathfrak{M}_{\vec{n},k}\right) = \sum_{k=0}^{\infty} \mathfrak{q}^k \mathrm{Tr}_{\mathcal{H}_k}\left[(-1)^F q^{H_L}\bar{q}^{H_R} \prod_{a\in\underline{4}} e^{2\pi i \varepsilon_a \mathcal{J}_a} \prod_{A\in\underline{4}^\vee} \prod_{\alpha=1}^{n_A} e^{2\pi i \mathrm{a}_{A,\alpha} T_{A,\alpha}}\right]_{\sum_{a\in\underline{4}} \varepsilon_a = 0}$$

[Seiberg, Witten, 1999]

Noncommutative Instantons

- Open strings connecting D-branes in the presence of a strong background B-field can usually be described by noncommutative field theory.
- The spacetime becomes $\mathbb{R}^{1,1} \times \mathbb{C}^4_\Theta$, with
  $\left[z_a, z_b\right] = \left[\bar{z}_a, \bar{z}_b\right] = 0, \quad \left[z_a, \bar{z}_b\right] = -2\Theta\delta_{ab}$
-

Instanton partition function

Applying the supersymmetric localization techniques, $Z_k = \frac{1}{k!}\oint \prod_{i=1}^{k} d\phi_i \left(Z_k^{1-1} \prod_{A\in \underline{4}^\vee} Z_k^{1-7_A} \right)$, where

$$Z_k^{1-1} = \left[\frac{2\pi\eta(\tau)^3\theta_1(\varepsilon_{12}|\tau)\theta_1(\varepsilon_{13}|\tau)\theta_1(\varepsilon_{23}|\tau)}{\theta_1(\varepsilon_1|\tau)\theta_1(\varepsilon_2|\tau)\theta_1(\varepsilon_3|\tau)\theta_1(\varepsilon_4|\tau)}\right]^k \times$$

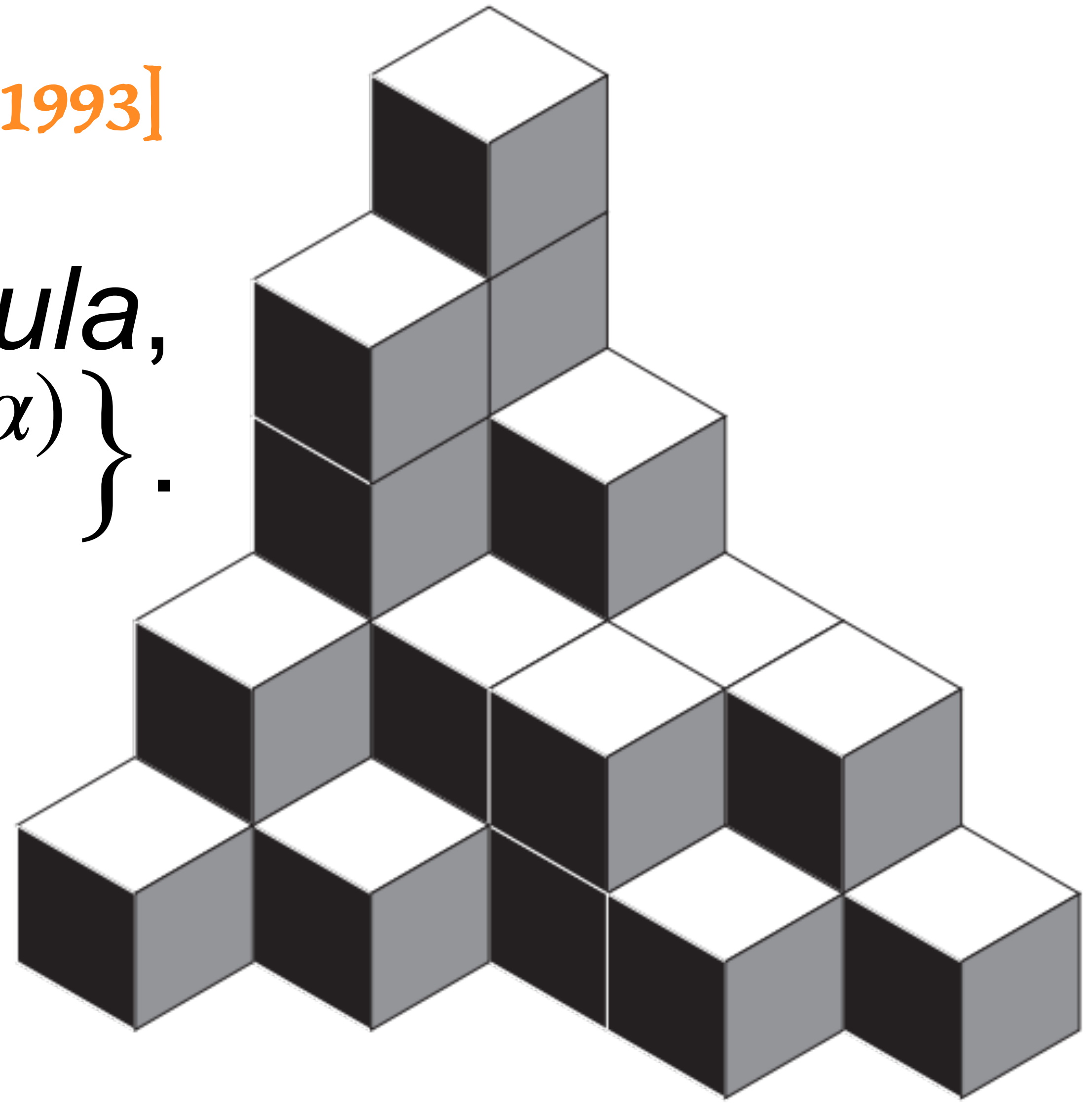
$$\times \prod_{\substack{i,j=1 \\ i\neq j}}^{k} \frac{\theta_1(\phi_{ij}|\tau)\theta_1(\phi_{ij}+\varepsilon_{12}|\tau)\theta_1(\phi_{ij}+\varepsilon_{13}|\tau)\theta_1(\phi_{ij}+\varepsilon_{23}|\tau)}{\theta_1(\phi_{ij}+\varepsilon_1|\tau)\theta_1(\phi_{ij}+\varepsilon_2|\tau)\theta_1(\phi_{ij}+\varepsilon_3|\tau)\theta_1(\phi_{ij}+\varepsilon_4|\tau)},$$

$$Z_k^{1-7_A} = \prod_{i=1}^{k}\prod_{\alpha=1}^{n_A} \frac{\theta_1(\phi_i - \mathsf{a}_{A,\alpha} - \varepsilon_A|\tau)}{\theta_1(\phi_i - \mathsf{a}_{A,\alpha}|\tau)},$$

[Jeffrey, Kirwan, 1993]

The contour integral is evaluated using the *Jeffrey-Kirwan residue formula*, and the poles are labeled by a collection of plane partitions $\vec{\pi} = \left\{\pi^{(A,\alpha)}\right\}$.

Each plane partition $\pi = \begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \cdots \\ \pi_{2,1} & \pi_{2,2} & \pi_{2,3} & \cdots \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$, $\quad \pi_{x,y} \geq \pi_{x+1,y}, \pi_{x,y+1} \geq 0.$



Dimensional reduction

- Performing a T-duality, we get a D0-D6 system in type IIA superstring theory.
- The K-theoretical instanton partition function is computed by the generalized Witten indices.

$$Z_k = \int_{(\mathfrak{M}_{\vec{n},k})} \hat{A}_{\mathbf{T}}\left(\mathfrak{M}_{\vec{n},k}\right) = \mathrm{Tr}_{\mathscr{H}_k}\left[(-1)^F \prod_{a\in\underline{4}} q_a^{\mathscr{J}_a} \prod_{A\in\underline{4}^\vee} \prod_{\alpha=1}^{n_A} t_{A,\alpha}^{T_{A,\alpha}}\right]_{\prod_{a\in\underline{4}} q_a=1}$$

$$q_a = e^{2\pi i \varepsilon_a}$$

$$t_{A,\alpha} = e^{2\pi i a_{A,\alpha}}$$

$$Z_k = q_4^{k^2}\left(\frac{(1-q_1q_2)(1-q_1q_3)(1-q_2q_3)}{\prod_{a\in\underline{4}}(1-q_a)} \prod_{A\in\underline{4}^\vee} q_A^{n_A/2}\right)^k \times$$

$$\times \frac{1}{k!}\int \prod_{i=1}^{k}\frac{dx_i}{x_i} \prod_{\substack{i,j=1\\ i\neq j}}^{k} \frac{(x_j-x_i)(x_j-q_1q_2x_i)(x_j-q_1q_3x_i)(x_j-q_2q_3x_i)}{\prod_{a\in\underline{4}}(x_j-q_ax_i)} \times$$

$$\times \prod_{i=1}^{k}\prod_{A\in\underline{4}^\vee}\prod_{\alpha=1}^{n_A}\frac{(x_i-q_A^{-1}t_{A,\alpha})}{(x_i-t_{A,\alpha})}$$

- Similar reduction to D(-1)-D5 system will give the rational instanton partition function.

Low-energy worldvolume theory

$v_1 = -v_2 = v_3 = -v_4$
for spiked instantons

Share a common (0,2) susy

When $v_1 = v_2 = v_3 = v_4 = \frac{1}{6} + \frac{r}{3}$, the scalar potential is given by

$$V = \mathrm{Tr}\left(\sum_{a \in \underline{4}} [B_a, B_a^\dagger] + \sum_{A \in \underline{4}^\vee} I_A I_A^\dagger - r \right)^2 + \sum_{A \in \underline{4}^\vee} \mathrm{Tr}\left| B_{\check{A}} I_A \right|^2 + \sum_{a < b \in \underline{4}} \mathrm{Tr}\left| [B_a, B_b] \right|^2 .$$

When $r > 0$, susy is broken in the original string theory vacuum, but is restored after transitioning to a nearby vacuum via tachyon condensation.

The moduli space of tetrahedron instantons: the space of solutions to $V = 0$ modulo the gauge symmetry $\mathrm{U}(k)$