Eigenfunctions of a discrete elliptic integrable particle model with hyperoctahedral symmetry*

Jan Felipe van Diejen

Universidad de Talca, Chile
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Normalized Jacobi theta functions

Let

$$[z]_1 = [z;\alpha,p]_1 = \frac{\vartheta_1(\frac{\alpha}{2}z;p)}{\sin(\frac{\alpha}{2})\vartheta_1'(0;p)}, \quad [z]_r = [z;\alpha,p]_r = \frac{\vartheta_r(\frac{\alpha}{2}z;p)}{\vartheta_r(0;p)} \ (r=2,3,4),$$

with $z \in \mathbb{C}$, $0 < \alpha < 2\pi$, and $0 < |p| < 1$.

The normalization chosen is such that:

$$\lim_{p\to 0}[z;\alpha,p]_1 = \frac{\sin(\frac{\alpha}{2}z)}{\sin(\frac{\alpha}{2})} \quad \text{and} \quad \lim_{\alpha\to 0}[z;\alpha,p]_1 = z.$$

Hamiltonian (vD '94)

$$H = \sum_{1\leqslant j\leqslant n} B_j(x)T_j + B_j(-x)T_j^{-1}$$

with

$$B_j(x) = \Big(\prod_{\substack{1\leqslant k\leqslant n\\ k\neq j}} \frac{[x_j-x_k+g]_1}{[x_j-x_k]_1}\frac{[x_j+x_k+g]_1}{[x_j+x_k]_1}\Big)\prod_{1\leqslant r\leqslant 4}\frac{[x_j+g_r]_r}{[x_j]_r}$$

and $(T_j f)(x_1,\ldots,x_n) = f(x_1,\ldots,x_{j-1},x_j+1,x_{j+1},\ldots,x_n)$

Features

- After a gauge trafo, H recovers the quantization of Inozemtsev's ´89 Calogero model by scaling the step size to 0:

$$H \to \sum_{1\leqslant j\leqslant n} \partial_j^2 - g(g-1) \sum_{1\leqslant j\neq k\leqslant n} \Big(\wp(x_j - x_k) + \wp(x_j + x_k)\Big)$$
$$- \sum_{1\leqslant j\leqslant n} \Big(g_0(g_0-1)\wp(x_j) + g_1(g_1-1)\wp(x_j+\omega_1) +$$
$$g_2(g_2-1)\wp(x_j+\omega_2) + g_3(g_3-1)\wp(x_j+\omega_1+\omega_2)\Big)$$

- At $p = 0$, H degenerates to the Macdonald-Koornwinder difference operator.
- H is quantum integrable (Komori & Hikami '97).
- H enjoys remarkable reflection symmetries in the parameter space (Ruijsenaars '04).
- H arises in the study of BC_n elliptic hypergeometric integrals and biorthogonal functions, and connects to the Sklyanin algebra and the elliptic DAHA (Rains '06, '10, '20).
- The quantum integrals of H are generated by a quantum Lax matrix (Chalykh '19).
- Some explicit eigenfunctions of H are known (see e.g. Atai '20, Atai & Noumi '22, Ruijsenaars '15, Spiridonov '07).
- H arises for $n = 1$ as a reduction of the Lax operator for Sakai's elliptic difference Painlevé equation (Noumi, Ruijsenaars, & Yamada '20).
- H describes surface defects in compactifications of conformal matter theories on a punctured Riemann surface (see e.g. Nazzal, Nedelin, & Razamat '21).

II. Lattice model on bounded partitions

Discretization

We restrict H onto a lattice of shifted partitions:

$$\rho + \Lambda^{(n)}$$

where $\rho = (\rho_1, \ldots, \rho_n)$ with $\rho_j = (n-j)g + g_1$ and

$$\Lambda^{(n)} = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}.$$

Truncation at level m

Upon picking the real period of the form

$$\boxed{\alpha = \frac{\pi}{m + (n-1)g + g_1 + g_2} \quad \text{with } m \in \mathbb{N}}$$

$(g, g_r > 0)$, we implement a (Racah type) truncation that restricts H to the finite lattice of bounded shifted partitions:

$$\rho + \Lambda^{(n,m)}$$

with

$$\Lambda^{(n,m)} = \{\lambda \in \mathbb{Z}^n \mid m \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}.$$

Hilbert space

Let $\ell^2(\Lambda^{(n,m)}, \Delta)$ be the $\binom{n+m}{n}$-dimensional space of functions $f : \Lambda^{(n,m)} \to \mathbb{C}$ endowed with the inner product:

$$\langle f, g\rangle_\Delta = \sum_{\lambda \in \Lambda^{(n,m)}} f_\lambda \overline{g_\lambda} \Delta_\lambda$$

$$\Delta_\lambda = \prod_{1\leqslant j\leqslant n} \frac{[2\rho_j + 2\lambda_j]_1}{[2\rho_j]_1} \prod_{1\leqslant r\leqslant 4} \frac{[\rho_j + g_r]_{r,\lambda_j}}{[\rho_j + 1 - g_r]_{r,\lambda_j}}$$

$$\times \prod_{1\leqslant j<k\leqslant n} \frac{[\rho_j \pm \rho_k + \lambda_j \pm \lambda_k]_1}{[\rho_j \pm \rho_k]_1} \frac{[\rho_j \pm \rho_k + g]_{1,\lambda_j \pm \lambda_k}}{[\rho_j \pm \rho_k + 1 - g]_{1,\lambda_j \pm \lambda_k}},$$

where $[z]_{r,l} = \prod_{0\leqslant k<l} [z+k]_r$ (with $[z]_{r,0} = 1$).

Self-adjointness

Proposition

The action of H in $\ell^2(\Lambda^{(n,m)},\Delta)$:

$$(Hf)_\lambda = \sum_{\substack{1\leqslant j\leqslant n,\ \varepsilon=\pm 1\\ \lambda+\varepsilon e_j\in\Lambda^{(n,m)}}} B_{\lambda,\varepsilon j} f_{\lambda+\varepsilon e_j}$$

with

$$B_{\lambda,\varepsilon j} = \prod_{1\leqslant r\leqslant 4} \frac{[\rho_j+\lambda_j+\varepsilon g_r]_r}{[\rho_j+\lambda_j]_r} \prod_{\substack{1\leqslant k\leqslant n\\ k\neq j}} \frac{[\rho_j\pm\rho_k+\lambda_j\pm\lambda_k+\varepsilon g]_1}{[\rho_j\pm\rho_k+\lambda_j\pm\lambda_k]_1},$$

is self-adjoint

$$\forall f,g\in\ell^2(\Lambda^{(n,m)},\Delta):\qquad \langle Hf,g\rangle_\Delta = \langle f,Hg\rangle_\Delta.$$

Spectrum

Proposition (Eigenvalues)

(i) The eigenvalues of the difference operator H in $\ell^2(\Lambda^{(n,m)}, \Delta)$ are given by *real-analytic* functions E_ν, $\nu \in \Lambda^{(n,m)}$ in $p \in (-1, 1)$ that specialize at $p = 0$ to

$$E_\nu|_{p=0} = 2 \sum_{1 \leqslant j \leqslant n} \cos \tfrac{\alpha}{2}(\hat{\rho}_j + \nu_j),$$

where $\hat{\rho}_j = (n - j)g + \frac{1}{2}(g_1 + g_2)$, $j = 1, \ldots, n$.

(ii) For generic coupling values, the eigenvalues E_ν, $\nu \in \Lambda^{(n,m)}$ from part *(i)* are distinct as analytic functions of $p \in (-1, 1)$.

Eigenfunctions

For generic values of the coupling parameters such that the discriminant

$$\Delta(H) = \prod_{\substack{\mu,\nu\in\Lambda^{(n,m)}\\ \mu\neq\nu}} (E_\nu - E_\mu) \neq 0$$

(as an analytic function of $p \in (-1,1)$), let

$$h^{(\nu)} = \Big(\prod_{\substack{\mu\in\Lambda^{(n,m)}\\ \mu\neq\nu}} \frac{H - E_\mu}{E_\nu - E_\mu} \Big) \chi$$

with

$$\chi_\lambda = \begin{cases} 1, & \text{if } \lambda = 0, \\ 0, & \text{if } \lambda \neq 0, \end{cases} \qquad (\lambda \in \Lambda^{(n,m)}).$$

Theorem (Eigenfunctions)

The following statements hold for generic positive parameters subject to the level m *truncation condition.*

(i) H *is diagonalized in the Hilbert space* $\ell^2(\Lambda^{(n,m)}, \Delta)$ *by an orthonormal basis of eigenfunctions* $f^{(\nu)}$, $\nu \in \Lambda^{(n,m)}$ *that depend analytically on* $p \in (-1,1)$ *such that* $Hf^{(\nu)} = E_\nu f^{(\nu)}$ *(with* E_ν *as before).*

(ii) For generic coupling parameters such that $\Delta(H) \neq 0$*, one has that*

$$Hh^{(\nu)} = E_\nu h^{(\nu)}$$

and

$$\langle h^{(\nu)}, h^{(\tilde{\nu})} \rangle_\Delta = \begin{cases} h_0^{(\nu)} & \text{if } \nu = \tilde{\nu}, \\ 0 & \text{if } \nu \neq \tilde{\nu}. \end{cases}$$

Moreover, the functions $h^{(\nu)} \in \ell^2(\Lambda^{(n,m)}, \Delta)$ *coincide with the orthonormal eigenfunctions in part (i) up to normalization:*

$$h^{(\nu)} = \overline{f_0^{(\nu)}} f^{(\nu)}.$$

Theorem (Trigonometric limit)

(iii) At $p = 0$ *the eigenbasis is given explicitly by*

$$h_\lambda^{(\nu)}|_{p=0} = \frac{C_{\lambda,q}}{N_{\nu,q}} P_\lambda(q^{\hat{\rho}+\nu}; q, t, a, b, c, d),$$

where

$$C_{\lambda,q} = \prod_{\substack{1\leqslant j\leqslant n \\ r=1,2}} \frac{[\rho_j]_{r,q,\lambda_j}}{[\rho_j + g_r]_{r,q,\lambda_j}} \prod_{1\leqslant j<k\leqslant n} \frac{[\rho_j \pm \rho_k]_{1,q,\lambda_j \pm \lambda_k}}{[\rho_j \pm \rho_k + g]_{1,q,\lambda_j \pm \lambda_k}}$$

and P_λ *denotes the monic Macdonald-Koornwinder polynomial with parameters*

$$q = e^{i\alpha}, \quad t = q^g,$$
$$a = q^{(g_1+g_2)/2}, \quad b = -q^{(g_1+g_2)/2}, \quad c = q^{(g_1-g_2+1)/2}, \quad d = -q^{(g_1-g_2+1)/2}.$$

Here

$$N_{\nu,q} = \sum_{\lambda\in\Lambda^{(n,m)}} C_{\lambda,q}^2 P_\lambda^2(q^{\hat{\rho}+\nu}; q, t, a, b, c, d)\Delta_{\lambda,q},$$

$$\Delta_{\lambda,q} = \prod_{1\leqslant j\leqslant n} \frac{[2\rho_j + 2\lambda_j]_{1,q}}{[2\rho_j]_{1,q}} \prod_{r=1,2} \frac{[\rho_j + g_r]_{r,q,\lambda_j}}{[\rho_j + 1 - g_r]_{r,q,\lambda_j}}$$
$$\times \prod_{1\leqslant j<k\leqslant n} \frac{[\rho_j \pm \rho_k + \lambda_j \pm \lambda_k]_{1,q}}{[\rho_j \pm \rho_k]_{1,q}} \frac{[\rho_j \pm \rho_k + g]_{1,q,\lambda_j \pm \lambda_k}}{[\rho_j \pm \rho_k + 1 - g]_{1,q,\lambda_j \pm \lambda_k}},$$

$$[z]_{1,q} = \frac{\sin(\frac{\alpha}{2} z)}{\sin(\frac{\alpha}{2})} = \frac{q^{\frac{z}{2}} - q^{-\frac{z}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \quad [z]_{2,q} = \cos(\tfrac{\alpha}{2} z) = \frac{q^{\frac{z}{2}} + q^{-\frac{z}{2}}}{2}.$$

IV. One particle case: $n = 1$

Level m truncated difference Heun equation

For $\lambda = 0, \ldots, m$:

$$\boxed{f_{\lambda+1} \prod_{1\leqslant r\leqslant 4} \frac{[\lambda+g_1+g_r]_r}{[\lambda+g_1]_r} + f_{\lambda-1} \prod_{1\leqslant r\leqslant 4} \frac{[\lambda+g_1-g_r]_r}{[\lambda+g_1]_r} = E f_\lambda}$$

(with $\alpha = \frac{\pi}{m+g_1+g_2}$ and $g_r > 0$).

Orthogonality weights

$$\boxed{\Delta_\lambda = \prod_{1\leqslant j\leqslant n} \frac{[2g_1+2\lambda]_1}{[2g_1]_1} \prod_{1\leqslant r\leqslant 4} \frac{[g_1+g_r]_{r,\lambda}}{[g_1+1-g_r]_{r,\lambda}}}$$

Elliptic Racah polynomials

For $k = 0, 1, 2, \ldots$ we define the elliptic Racah polynomials:

$$P_k(E) = \det \begin{bmatrix} E & -b_0^+ & 0 & \cdots & 0 \\ -b_1^- & E & \ddots & & \vdots \\ 0 & -b_2^- & \ddots & -b_{k-3}^+ & 0 \\ \vdots & & \ddots & E & -b_{k-2}^+ \\ 0 & \cdots & 0 & -b_{k-1}^- & E \end{bmatrix}$$

(so $P_0(E) = 1$) with

$$b_k^+ = \prod_{1 \leqslant r \leqslant 4} \frac{[g_1 + g_r + k]_r}{[g_1 + k]_r} \quad \text{and} \quad b_k^- = \prod_{1 \leqslant r \leqslant 4} \frac{[g_1 - g_r + k]_r}{[g_1 + k]_r}.$$

Eigenvalues

The eigenvalues of H are given by the roots

$$E_0 > E_1 > \cdots > E_m$$

of $P_{m+1}(E)$.

Eigenfunctions

$$h_\lambda^{(\nu)} = \tfrac{C_\lambda}{N_\nu} P_\lambda(E_\nu) \qquad (0 \leqslant \lambda, \nu \leqslant m)$$

with

$$C_\lambda = \prod_{1 \leqslant r \leqslant 4} \frac{[g_1]_{r,\lambda}}{[g_1 + g_r]_{r,\lambda}}$$

and

$$N_\nu = \sum_{0 \leqslant \lambda \leqslant m} C_\lambda^2 P_\lambda^2(E_\nu) \Delta_\lambda = C_m^2 \Delta_m P_m(E_\nu) \prod_{\substack{0 \leqslant j \leqslant m \\ j \neq \nu}} (E_\nu - E_j).$$

V. The case g=1: free fermions

For g=1 the gauge transformation

$$H \to \tilde{H} = V_\lambda H V_\lambda^{-1} \quad \text{with} \quad V_\lambda = \prod_{1\leqslant j<k\leqslant n} \frac{[\rho_j \pm \rho_k + \lambda_j \pm \lambda_k]_1}{[\rho_j \pm \rho_k]_1}$$

transforms the particle model to n free fermions on the open lattice $\{0, 1, 2, \ldots, m+n-1\}$ placed in an external field of elliptic Racah type:

$$(\tilde{H}\tilde{f})_\lambda = \sum_{\substack{1\leqslant j\leqslant n,\\ \lambda+e_j\in\Lambda^{(n,m)}}} b^+_{n-j+\lambda_j} \tilde{f}_{\lambda+e_j} + \sum_{\substack{1\leqslant j\leqslant n,\\ \lambda-e_j\in\Lambda^{(n,m)}}} b^-_{n-j+\lambda_j} \tilde{f}_{\lambda-e_j}$$

(where $\tilde{f}_\lambda = V_\lambda f_\lambda$).

Let $g=1$, $\alpha = \frac{\pi}{m+n-1+g_1+g_2}$ $(g_r > 0)$ and let

$$E_0 > E_1 > \cdots > E_{n+m-1}$$

denote the roots of $P_{n+m}(E)$.

Theorem (Diagonalization for $g=1$)

(i) The eigenvalues of H become in terms of the elliptic Racah roots:

$$E_\nu = \sum_{1 \leqslant j \leqslant n} E_{n-j+\nu_j} \qquad (\nu \in \Lambda^{(n,m)}).$$

(ii) The eigenfunctions are given by Schur polynomials of elliptic Racah type:

$$h_\lambda^{(\nu)} = \frac{c_\lambda}{N_\nu} s_\lambda^{(\nu)} \qquad (\lambda, \nu \in \Lambda^{(n,m)})$$

with

$$s_\lambda^{(\nu)} = A_\lambda^{(\nu)} / A_0^{(\nu)}, \quad A_\lambda^{(\nu)} = \det \left[P_{n-i+\lambda_i}(E_{n-j+\nu_j}) \right]_{1 \leqslant i,j \leqslant n}.$$

Here the normalizations are governed by

$$c_\lambda = \frac{1}{V_\lambda} \prod_{\substack{1 \leqslant j \leqslant n \\ 1 \leqslant r \leqslant 4}} \frac{[\rho_j]_{r,\lambda_j}}{[\rho_j + g_r]_{r,\lambda_j}}$$

and

$$N_\nu = \sum_{\lambda \in \Lambda^{(n,m)}} c_\lambda^2 (s_\lambda^{(\nu)})^2 \Delta_\lambda = \frac{1}{\left(A_0^{(\nu)}\right)^2} \prod_{1 \leqslant j \leqslant n} \frac{N_{n-j+\nu_j}}{\Delta_{n-j} c_{n-j}^2}.$$

VI. The level $m = 1$ case: one-column partitions

The lattice $\Lambda^{(n,1)}$ consists of one-column partitions

$$(1^k) = (\underbrace{1,\ldots,1}_{k},\underbrace{0,\ldots,0}_{n-k}) \quad (0 \leqslant k \leqslant n).$$

The level $m = 1$ eigenvalue equation for H becomes of triangular form:

$$B_{-k} f_{(1^{k-1})} + B_{k+1} f_{(1^{k+1})} = E f_{(1^k)} \quad (0 \leqslant k \leqslant n),$$

with

$$B_{-k} = B_{(1^k),-k} = \frac{[2\rho_{k+1}+2, \rho_0-\rho_k, \rho_k+\rho_{n+1}+1, 1]_1}{[\rho_1+\rho_k+2, \rho_k+\rho_{k+1}+1, \rho_k-\rho_n+1, g]_1} \prod_{1\leqslant r\leqslant 4} \frac{[\rho_k+1-g_r]_r}{[\rho_k+1]_r}$$

and

$$B_k = B_{(1^k),k+1} = \frac{[\rho_0+\rho_{k+1}+1, 2\rho_{k+1}, \rho_{k+1}-\rho_{n+1}, 1]_1}{[\rho_k+\rho_{k+1}+1, \rho_1-\rho_{k+1}+1, \rho_{k+1}+\rho_n, g]_1} \prod_{1\leqslant r\leqslant 4} \frac{[\rho_{k+1}+g_r]_r}{[\rho_{k+1}]_r}.$$

Upshot: al level $m = 1$ the diagonalization can again be performed by means of polynomials

$$P_{(1^k)}(E) = \det \begin{bmatrix} E & -B_1 & 0 & \cdots & 0 \\ -B_{-1} & E & \ddots & & \vdots \\ 0 & -B_{-2} & \ddots & -B_{k-2} & 0 \\ \vdots & & \ddots & E & -B_{k-1} \\ 0 & \cdots & 0 & -B_{-k+1} & E \end{bmatrix},$$

$k = 0, \ldots, n+1.$

Theorem (Diagonalization at level $m=1$)

(i) The eigenvalues of H *are given by the simple roots of* $P_{(1^{n+1})}(E)$*:*

$$E_{(1^0)} > E_{(1^1)} > \cdots > E_{(1^n)}.$$

(ii) The eigenfunctions $h^{(1^l)}$, $0 \leqslant l \leqslant n$ *of* H *are given by*

$$h^{(1^l)}_{(1^k)} = \frac{c_{(1^k)}}{N_{(1^l)}} P_{(1^k)}(E_{(1^l)}) \qquad (0 \leqslant k, l \leqslant n)$$

with

$$\begin{aligned} N_{(1^l)} &= \sum_{0 \leqslant k \leqslant n} c^2_{(1^k)} P^2_{(1^k)}(E_{(1^l)}) \Delta_{(1^k)} \\ &= c^2_{(1^n)} \Delta_{(1^n)} P_{(1^n)}(E_{(1^l)}) \prod_{\substack{0 \leqslant j \leqslant n \\ j \neq l}} \left(E_{(1^l)} - E_{(1^j)} \right), \end{aligned}$$

where

$$c_{(1^k)} = \prod_{1 \leqslant j \leqslant k} B_j^{-1} \quad \textit{and} \quad \Delta_{(1^k)} = \prod_{1 \leqslant j \leqslant k} B_j B_{-j}^{-1}.$$

VII. References

- JFvD & T. Görbe, Eigenfunctions of a discrete elliptic integrable particle model with hyperoctahedral symmetry, Comm. Math. Phys. (to appear).
- JFvD & T. Görbe, Elliptic Racah polynomials, arXiv:2106.07394 [math.CA]
- F. Atai, *Source identities and kernel functions for the deformed Koornwinder-van Diejen models*, Comm. Math. Phys. **377** (2020), 2191–2216.
- F. Atai & M. Noumi, *Eigenfunctions of the van Diejen model generated by gauge and integral transformations*, arXiv:2203.00498 [nlin.SI]
- O. Chalykh, *Quantum Lax pairs via Dunkl and Cherednik operators*, Comm. Math. Phys. **369** (2019), 261–316.
- JFvD, *Integrability of difference Calogero-Moser systems*, J. Math. Phys. **35** (1994), 2983–3004.
- V. Inozemtsev, *Lax representation with spectral parameter on a torus for integrable particle systems*, Lett. Math. Phys. **17** (1989), 11–17.
- Y. Komori & K. Hikami, *Quantum integrability of the generalized elliptic Ruijsenaars models*, J. Phys. A **30** (1997), 4341–4364.
- B. Nazzal, A.Nedelin, & S. Razamat, *Minimal* (D, D) *conformal matter and generalizations of the van Diejen model*, arXiv:2106.08335 [hep-th]
- E. Rains, BC_n*-symmetric Abelian functions*, Duke Math. J. **135** (2006), 99–180.
- E. Rains, *Transformations of elliptic hypergeometric integrals*, Ann. of Math. (2) **171** (2010), 169–243.
- E. Rains, *Elliptic double affine Hecke algebras.*,SIGMA Symmetry Integrability Geom. Methods Appl. **16** (2020), Paper No. 111.
- S.Ruijsenaars, *Integrable* BC_N *analytic difference operators: hidden parameter symmetries and eigenfunctions.* In: *New Trends in Integrability and Partial Solvability*, NATO Sci. Ser. II Math. Phys. Chem. **132**, Kluwer Acad. Publ., Dordrecht, 2004, pp. 217–261.
- S.Ruijsenaars, *Hilbert-Schmidt operators vs. integrable systems of elliptic Calogero-Moser type IV. The relativistic Heun (van Diejen) case,* SIGMA Symmetry Integrability Geom. Methods Appl. **11** (2015), Paper 004, 78 pp.
- M. Noumi, S. Ruijsenaars, & Y. Yamada, *The elliptic Painlevé Lax equation vs. van Diejen's 8-coupling elliptic Hamiltonian*, SIGMA Symmetry Integrability Geom. Methods Appl. 16 (2020), Paper No. 063, 16 pp.
- V. Spiridonov, *Elliptic hypergeometric functions and models of Calogero-Sutherland type*, Teoret. Mat. Fiz. **150** (2007), 311–324.

Thank You!