Puzzle:

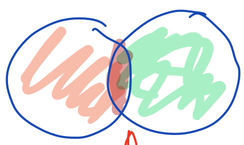
$$\int \frac{dx}{dx} = \frac{1}{d} \log|x| + C$$

Last time: Sets, operations with sets

U - "universal" set. Theorem: U does not exist. Zermelo

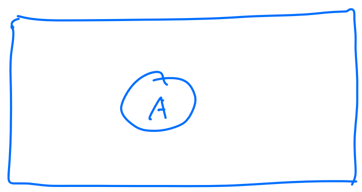
S - set $\quad \mathcal{P}(S)$ - power set $\quad |\mathcal{P}(S)| > |S|$

Set operations $\quad A \cup B$



$A \cap B$

U



$\bar{A} = U - A$

$U = \mathbb{Z}$ - integers

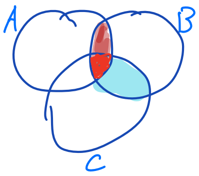
$A = 2\mathbb{Z} = \{0, \pm 2, \pm 4, \dots\}$

$\bar{A} = 2\mathbb{Z}+1 = \{\pm 1, \pm 3, \dots\}$

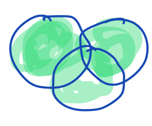
Theorem:

- $A \cap \emptyset = \emptyset$
- $A \cup \emptyset = A$
- $A \cap A = A$
- $A \cup A = A$

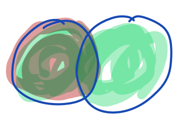
- $\bar{\bar{A}} = A$
- $A \cap B = B \cap A$
- $A \cup B = B \cup A$
- $(A \cap B) \cap C = A \cap (B \cap C)$



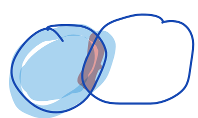
- $(A \cup B) \cup C = A \cup (B \cup C)$



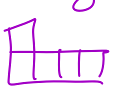
- $A \cap \bar{A} = \emptyset$



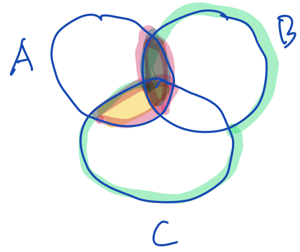
- $A \cap (A \cup B) = A$
- $A \cup (A \cap B) = A$



Young



- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



Hint $a \times (b + c) = a \cdot b + a \cdot c$

Proof: $LHS \subseteq RHS$, $\supseteq$

$\subseteq$: $x \in A \cap (B \cup C)$

$(x \in A) \wedge (x \in B \cup C)$

$\Rightarrow (x \in A) \wedge (x \in B \vee x \in C)$

$\Rightarrow \underbrace{(x \in A) \wedge (x \in B)}_{} \vee \underbrace{(x \in A) \wedge (x \in C)}_{}$

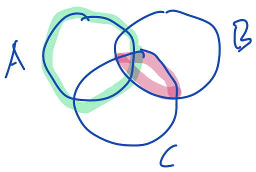
$(x \in A \cap B) \cup (x \in A \cap C)$

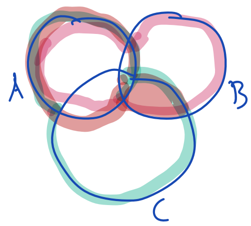
$\Downarrow$

$x \in (A \cap B) \cup (A \cup C)$

$\supseteq$: reverse all steps.

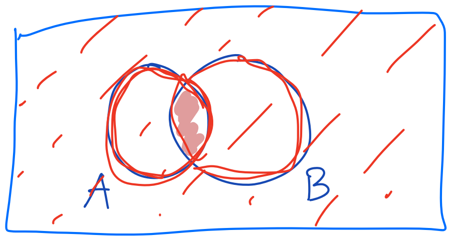
- $A \cup (B \cap C) = (A \cup C) \cap (A \cup B)$

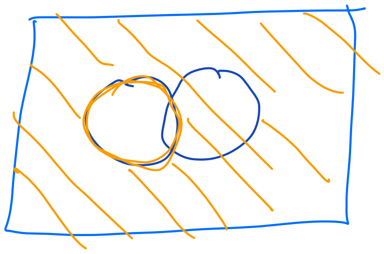


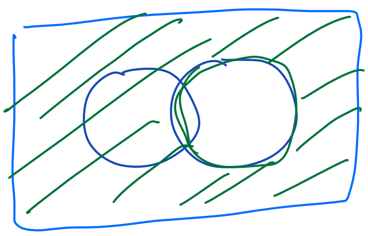


- De Morgan's law

1) $\overline{A \cap B} = \overline{A} \cup \overline{B}$







Proof:

$\overline{A \cap B} = \{x \mid x \notin (A \cap B)\}$

$= \{x \mid \neg(x \in (A \cap B))\}$

$= \{x \mid \neg(x \in A \wedge x \in B)\}$

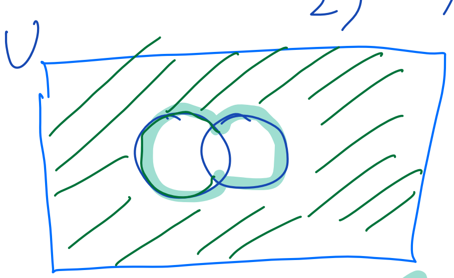
$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$

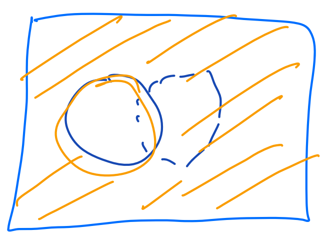
$= \{x \mid (x \notin A) \vee (x \notin B)\}$

$= \{x \mid x \in \overline{A} \cup \overline{B}\}$

$= \overline{A} \cup \overline{B}.$

2) $\overline{A \cup B} = \overline{A} \cap \overline{B}$



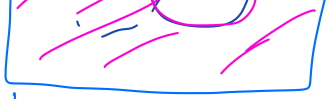


$U = \mathbb{Z}$

$A = 3\mathbb{Z} = \{0, \pm 3, \pm 6 \dots\}$

$\{0, \pm 6, \pm 12 \dots\}$

$B = 6 \cdot \mathbb{Z}$



$A \cup B = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$

$\overline{A \cup B} = \{\pm 1, \pm 2, \pm 4, \pm 5, \ldots\}$

$-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7$ ← LHS $\overline{A \cup B}$

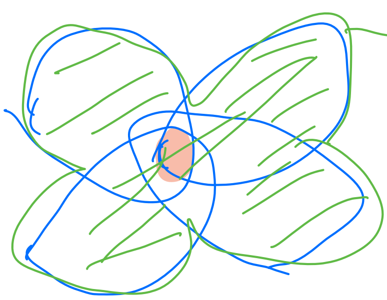
$-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7$ ← RHS $\overline{A}$

$-7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7$ ← RHS $\overline{B}$

Def: $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i = \{x \mid \exists i \in I \, (x \in A_i)\}$

$I = \{1, 2, \ldots n\}$ ~ indexing set

Def: $A_1 \cap A_2 \cap A_3 \ldots \cap A_n = \bigcap_{i=1}^{n} A_i = \{x \mid \forall i \in I \, (x \in A_i)\}$



(2.3)

Functions: Last time $A \times B \supseteq R$ - relation from A to B

Def: A function f from set A to set B is a relation with domain A and codomain B s.t.

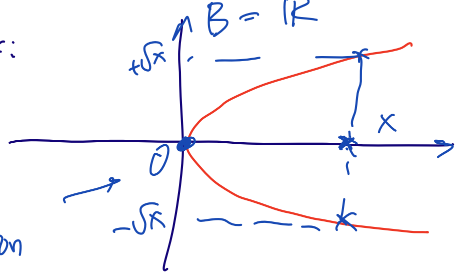
1) $\forall x \in A \quad \exists y \in B \quad$ s.t. $\quad (x, y) \in f$

2) single valuedness property: $\forall x \in A$ and $y, z \in B$

$$(x, y) \in f \land (x, z) \in f \rightarrow y = z$$

Remark: 2) can be removed. We can study multi valued functions

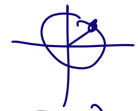
Ex:



$x = \pm\sqrt{y}$

$A = \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$

ramification point



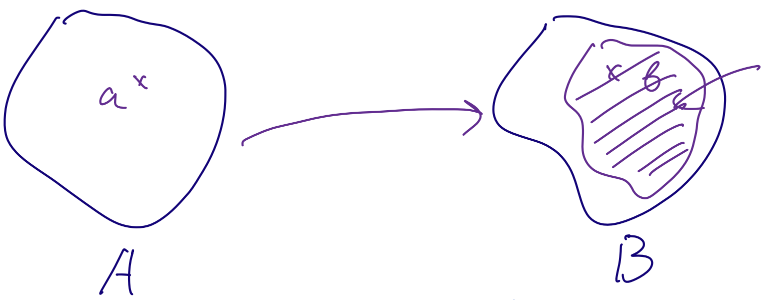
Ex: $y = e^x$, $x \in \mathbb{C}$ $\qquad e^{i\varphi} = e^{i(\varphi + 2\pi n)}$

$x = \log y$

$f : A \to B$ map, mapping, transformation

$f(a) = b$ — b is an image of a

a is a preimage of b

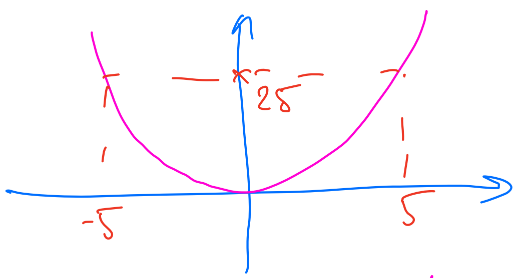


range of f

Range of f = set of images of all $a \in A$ = $\{y \in B \mid y = f(x)$ for some $x \in A\}$

(image of A)

Ex:

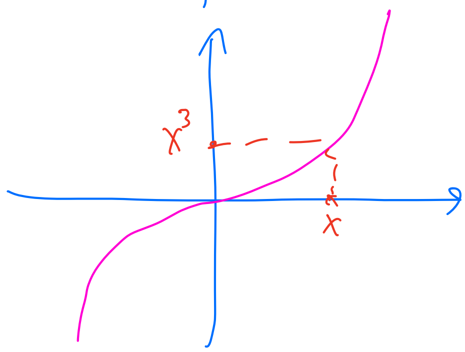


$f : \mathbb{R} \to \mathbb{R}$

$x \in \mathbb{R}$, $y \in \mathbb{R}$ $\qquad$ range $= \mathbb{R}_+$

$f(x) = x^2$

$10x^2 + 13x^3$



$f(x) = x^3$

Range $= \mathbb{R}$

If $B = \mathbb{Z}$ $\quad f$ is integer-valued

$\mathbb{R}$ $\quad$ real-valued

$\mathbb{C}$ $\quad$ complex-valued

⋮

Operations on functions

$f: A \to B$

$f+g : x \mapsto f(x)+g(x)$

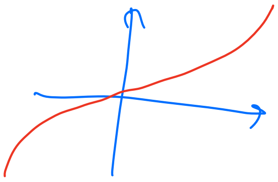
$f \cdot g \quad x \mapsto f(x)\cdot g(x)$

$\frac{f}{g}$

$\alpha f + \beta g \qquad \alpha \in B$

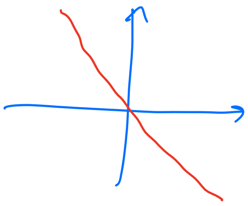
Def: A function f is called one-to-one (injective) iff
$$f(a) = f(b) \to a = b$$

Def: let $a, b \in A$. $f: A \to B$ is called strictly increasing
increasing



if $\quad a < b \to f(a) < f(b)$

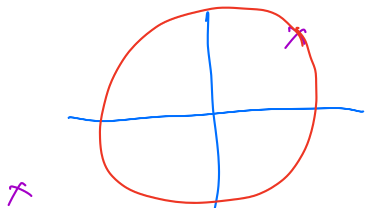
$\to f(a) \leq f(b)$ strictly decreasing
decreasing



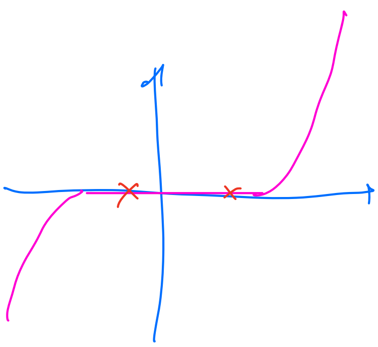
if $\quad a < b \to f(a) > f(b)$

$f(a) \geq f(b)$

(Assumes set B is ordered: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$)



A function f is strictly monotonic if it is
monotonic
either strictly increasing or strictly decreasing
increasing decreasing.



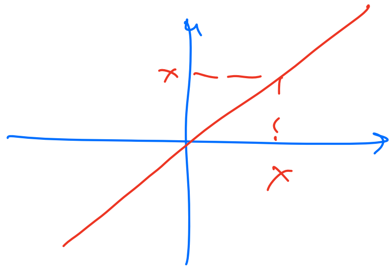
Fact: Every strictly monotonic function is one-to-one

Def: A function $f: A \to B$ is called onto (surjective) iff
(epimorphism)
$$\forall y \in B \ \exists x \in A \text{ s.t. } f(x) = y$$

Def: If f is one-to-one and onto the f is called a bijection (isomorphism)

Ex: • Identity function $\iota_A : A \to A$

$a \mapsto a$



Inverse functions

Def: Suppose f is an isomorphism (one-to-one and onto)

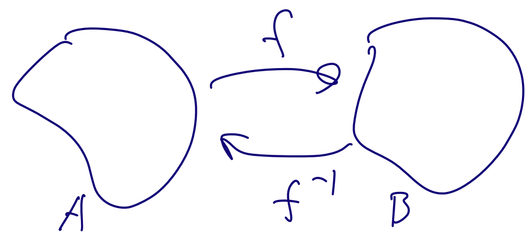
Then $f^{-1}(y)$ is defined s.t. $f(x)=y$

$f(x)=y \iff x=f^{-1}(y)$ Ex: $y=x^2$

$x=+\sqrt{y}$

$f: A \to B$

$f^{-1}: B \to A$

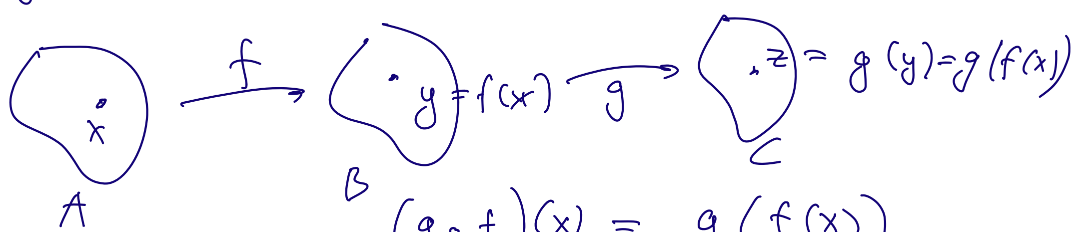


Def: The composition of f, g - functions

\circ

$f: A \to B$ $(g\circ f): A \to C$

$g: B \to C$



$(g\circ f)(x) = g(f(x))$

Ex: $f(x)=\sqrt{x}$

$g(y)=\dfrac{y+1}{y+2}$

Evaluate $(g\circ f)(5) =$

$f(5)=\sqrt{5}$

$g(f(5)) = \dfrac{\sqrt{5}+1}{\sqrt{5}+2}$

f_1, f_2, f_3

$f_1 : A_1 \to A_2$ $f_n\circ \dots \circ f_3\circ f_2\circ f_1 : A_1 \to A_{n+1}$

$f_2 : A_2 \to A_3$

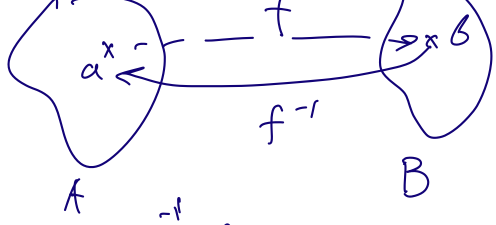
$\vdots$

$f_n : A_n \to A_{n+1}$

Fact: $f^{-1}\circ f = \iota_A$

$f: A \to B$

$f^{-1}: B \to A$



$f \circ f^{-1} = \iota_B : b \mapsto b \qquad f^{-1} \circ f : a \mapsto a$

1 . A, B as $\mathbb{Z}, \mathbb{R}$

Def: let $x \in \mathbb{R}$ define floor of x $\lfloor x \rfloor$: (dropping the decimal part)

$\lfloor x \rfloor$ is the unique integer n s.t. $n \le x < n+1$

$$\lfloor x \rfloor = n \iff n \le x < n+1$$

$\lfloor 5.3 \rfloor = 5$

$\lfloor 5.99 \rfloor = 5$

Def: let $x \in \mathbb{R}$, define ceiling of x $\lceil x \rceil$:

$$\lceil x \rceil = n \iff n-1 < x \le n .$$

$\lceil 5.99 \rceil = 6$

$\lceil 5.001 \rceil = 6$

Q: $\lfloor x+y \rfloor \overset{!}{=} \lfloor x \rfloor + \lfloor y \rfloor$

$3.6 + 3.6 = 7.2$

2.4

$\mathbb{N} \to B$

Def: A sequence is a function whose domain is $\mathbb{Z}$ (or subset of $\mathbb{Z}$) $\mathbb{Z}_+$, $\mathbb{N}$

- $\{1, 2, 3, 4, 5, 6, \ldots\} \subseteq \mathbb{Z}$

  $\mathbb{N}$

$f: \mathbb{N} \longrightarrow B$

$n \longmapsto a_n$ n-th element of the sequence

Ex:
- $a_n = n^2$

  $1, 4, 9, 16, \ldots$

- geometric sequence with ratio r

  $a \in \mathbb{R}$ $\qquad a, ar, ar^2, ar^3, \ldots$

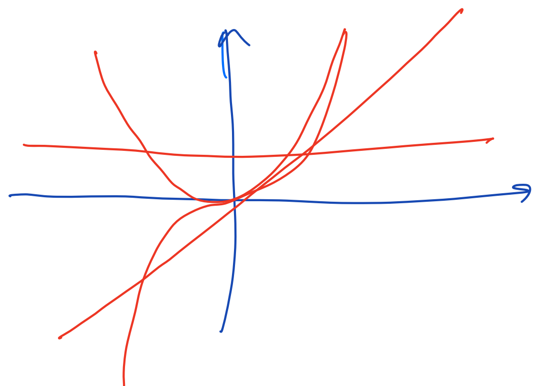
a_0, a_1, a_2, a_3

- arithmetic sequence w/ difference d

$c \in \mathbb{R}$ $\qquad \underset{a_0}{\overset{c}{\|}},\ \underset{a_1}{\overset{c+d}{\|}},\ \underset{a_2}{\overset{c+2d}{\|}},\ \underset{a_3}{\overset{c+3d}{\|}} \ldots$

- functional sequence

$\{x^n\}$ $\qquad n \mapsto x^n$



Fibonacci sequence

$\underset{F_1}{1}, \underset{F_2}{1}, \underset{F_3}{2}, 3, \underset{F_5}{5}, 8, 13$

$\underbrace{F_1, F_2}_{\text{initial conditions}}$

$$\left.\begin{aligned} F_3 &= F_1 + F_2 \\ F_4 &= F_2 + F_3 \\ &\vdots \\ F_n &= F_{n-2} + F_{n-1} \end{aligned}\right\} \text{Recursive relations}$$

Def: Let $\{a_n\}$ be a sequence. A relation which connects a_n with a subset of $\{a_1 \ldots a_{n-1}\}$ is called a recursive relation.

Fibonacci-type sequence

$\underset{F_1}{2}, \underset{F_2}{-1}, 1, 0, 1, 1, 2, 3$

$$F_5 = F_4 + F_3 = F_3 + F_2 + F_2 + F_1 = F_2 + F_1 + F_2 + F_2 + F_1$$
$$= 3F_2 + 2F_1 = 5$$

$$F_n \overset{?}{=} f(n)$$

Claim: $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

F_4: $\dfrac{1 + 4\sqrt{5} + 6\cdot 5 + 4\cdot 5\sqrt{5} + 25}{8}$

$$\frac{1}{\sqrt5}\left[\left(\frac{1+\sqrt5}{2}\right)^2-\left(\frac{1-\sqrt5}{2}\right)^2\right]\left[\left(\frac{1+\sqrt5}{2}\right)+\left(\frac{1-\sqrt5}{2}\right)\right]$$

$$\frac{\cancel{1}+2\sqrt5+\cancel{5}-\cancel{1}+2\sqrt5-\cancel{5}}{4}$$

$$\cancel{\sqrt5}$$

$$\frac{12}{4}=3$$

Sums: Geometric sum

$$\sum_{i=1}^{n} x^i = 1+x+x^2+x^3+\dots+x^n = \frac{1-x^{n+1}}{1-x}$$

1

$$1-x^2 = (1-x)(1+x)$$

$$1-x^3 = (1-x)(1+x+x^2)$$

$$\vdots$$

$$1-x^n = (1-x)(1+x+x^2+\dots+x^{n-1}) = 1+\cancel{x}+\cancel{x^2}+\dots+\cancel{x^{n-1}} \; \cancel{-x}\cancel{-x^2}-\dots-\cancel{x^{n-1}}-x^n$$

$n\to\infty$ $\quad x^n\to0$ $\quad |x|<1$

0.1, 0.01, 0.001

If $|x|<1 \Rightarrow \displaystyle\sum_{i=1}^{n} x^i \longrightarrow \frac{1}{1-x}$

$$\boxed{1+x+x^2+\dots\dots = \frac{1}{1-x}, \quad |x|<1}$$

sum of geometric series

$$S_N = \sum_{n=0}^{N} a_n x^n \qquad \{S_N\}$$

- $$\sum_{i=0}^{n} ar^i = \begin{cases} a\dfrac{1-r^{n+1}}{1-r}, & r\neq1 \\ a(n+1), & r=1 \end{cases}$$

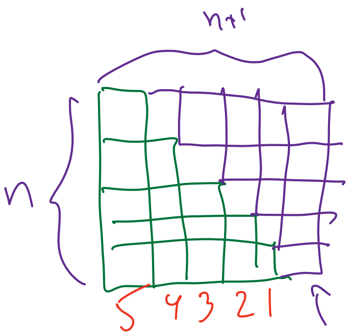
- $$\sum_{k=1}^{n} k = 1+2+3+\dots+n = \frac{n(n+1)}{2}$$

n-1

$$\int_0^a x = \frac{a^2}{2}$$

even: $\underbrace{(n+1)+(n+1)\dots(n+1)}_{\frac{n}{2}\text{ pairs}} = \boxed{\frac{n}{2}\times(n+1)}$

$$1,2,\dots,\;\dots,(n-1),n = \left(\frac{n-1}{2}\right)\times(n+1)+\frac{n+1}{2}$$



$\frac{n+1}{2}$ $\quad$ $\frac{n-1}{2}$ pairs $(n+1)$ $= \boxed{\frac{n}{2}(n+1)} \cancel{+\frac{n+1}{2}}$

$\cancel{+\frac{n+1}{2}}$

$$\frac{n(n+1)}{2}$$

- $1+3+5+7+\dots = ?$
- $1+5+9+13+\dots = ?$
- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(n+\alpha)}{3}$

$\int x^2 = \frac{x^3}{3}$

$n=2$
$n=3$

$n=2$: $1^2+2^2=5 = \frac{2\cdot\cancel{3}\cdot(2+\alpha)}{\cancel{3}}$

$\frac{5}{2} = 2+\alpha \Rightarrow \alpha = \frac{1}{2}$

$n=3$: $1^2+2^2+3^2 = 14 = \frac{\cancel{3}\cdot 4\cdot(3+\alpha)}{\cancel{3}}$

$\frac{14}{4} = 3+\alpha$

$3.5 = 3+\alpha$

Mathematical induction

Family of statements, $F_n = 0$

$$\boxed{\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}, \quad n=0,1,2,\dots$$

1) Base of induction: test the statement for some value of n.
2) Assume that $F_n=0$ holds for some n
3) Inductive step: verify that $F_{n+1}=0$ true.

Need to show that:

$$1^2+2^2+\dots+n^2+(n+1)^2 = \sum_{k=0}^{n+1} k^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

$$\sum_{k=0}^{n} k^2 + (n+1)^2$$

by assumption

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$(n^2+n)(2n+1) + 6n^2+12n+6$$

$$= \cancel{2n^3} + 3n^2 + \cancel{n} + 6n^2 + \cancel{12n} + \cancel{6}$$

$$(n^2+3n+2)(2n+3)$$

$$\cancel{2n^3} + 6n^2 + \cancel{4n}$$

$$+\,3n^2 + \cancel{9n} + \cancel{6}$$

- $$\sum_{k=0}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

HW: Prove by math. induction

$$\left(\sum_{k=0}^{n} k\right)^2 = \sum_{k=0}^{n} k^3$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad |x|<1 \qquad \leftarrow$$

differentiate w.r.t x both sides

$$\sum_{k=0}^{\infty} k\,x^{k-1} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$

$$\boxed{\sum_{k=0}^{\infty} k\,x^{k-1} = \frac{1}{(1-x)^2}, \qquad |x|<1}$$

$$F_n = F_{n-1} + F_{n-2} \qquad F_1 = 1, \quad F_2 = 1$$

Generating function for Fibonacci sequence

$F_0 = 0$

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \underset{F_0}{\cancel{0}} + \underset{F_1}{x} + \underset{F_2}{x^2} + \underset{F_3}{2x^3} + 3x^4 + \dots$$

$$F(x) = x + \sum_{n=2}^{\infty} F_n x^n = x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2})\,x^n$$

$$= x + \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n$$

$n-1 = m$

$n = m+1$

$n-2 = \ell$

$$= x + \underbrace{x \sum_{n=2}^{\infty} F_{n-1} x^{n-1}}_{\underbrace{x \sum_{m=1}^{\infty} F_m x^m}_{F(x)}} + \underbrace{x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2}}_{\underbrace{x^2 \sum_{n=0}^{\infty} F_\ell x^\ell}_{F(x)}}$$

$$\sum_{n=2} a_{n-1} x^n$$

$$\to a_1 x^2 + a_2 x^3 + a_3 x^4$$

$$= x(\underbrace{a_1 x + a_2 x^2 + a_3 x^3}_{x \sum_{n=1} a_n x^n})$$

$$= x + x \cdot F(x) + x^2 \cdot F(x)$$

Solve for $F(x)$

$$(x^2 + x - 1) F(x) = -x$$

$$F(x) = \frac{x}{1 - x - x^2}$$

Taylor series for small x:

$$F(x) = \frac{x}{1 - x - x^2} = \frac{A}{x - \alpha} + \frac{B}{x - \beta} = \frac{Ax - \beta A + Bx - \alpha B}{(x-\alpha)(x-\beta)}$$

$$\frac{-x}{x^2 + x - 1} = \frac{(A+B)x - (\beta A + \alpha B)^{=0}}{x^2 + x - 1}$$

$x^2 + x - 1 = 0$

$\alpha = \frac{-1+\sqrt{5}}{2} = -\frac{\sqrt{5}+1}{2}$

$\beta = \frac{-1-\sqrt{5}}{2} = \frac{\sqrt{5}-1}{2}$

$$\frac{(1+\sqrt{5})(1-\sqrt{5})}{4} = \frac{1-5}{4} = -1$$

$$\begin{cases} A + B = -1 & \big| \times(-\beta) \\ \beta A + \alpha B = 0 \end{cases}$$

$$-\beta A - \beta B = \beta$$
$$+\ \beta A + \alpha B = 0$$

$$(\alpha - \beta) B = \beta$$

$$B = \frac{\beta}{\alpha - \beta} = -\frac{\beta}{\sqrt{5}}$$

$$A = -\frac{\alpha}{\beta} B = \frac{\alpha}{\sqrt{5}}$$

$$F(x) = \frac{1}{\sqrt{5}} \frac{\alpha}{x - \alpha} - \frac{1}{\sqrt{5}} \frac{\beta}{x - \beta}$$

		F_{-5}	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6	
13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13

$$F_{n+2} = F_n + F_{n+1}$$

$$\text{wt } \ell > 0 \qquad F_{-\ell} = (-1)^{\ell+1} F_\ell$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$F(x) = \frac{1}{\sqrt5}\frac{\alpha}{x-\alpha} - \frac{1}{\sqrt5}\frac{\beta}{x-\beta}$$

$$= -\frac{1}{\sqrt5}\frac{1}{1-\frac{x}{\alpha}} + \frac{1}{\sqrt5}\frac{1}{1-\frac{x}{\beta}} = -\frac{1}{\sqrt5}\frac{1}{1+\beta x} + \frac{1}{\sqrt5}\frac{1}{1+\alpha x}$$

$$\alpha = -\gamma = \frac{1+\sqrt5}{2} \qquad \frac{1}{\alpha} = -\frac{2}{1+\sqrt5} = -\frac{2}{1+\sqrt5}\frac{(1-\sqrt5)}{(1-\sqrt5)} = -\frac{(1-\sqrt5)}{-2} = -\beta$$

$$\beta = -\delta = \frac{1-\sqrt5}{2} \qquad \frac{1}{\beta} = -\frac{2}{1-\sqrt5} = -\frac{2}{1-\sqrt5}\frac{1+\sqrt5}{1+\sqrt5} = -\frac{2(1+\sqrt5)}{-4} = -\alpha$$

$$= \frac{1}{\sqrt5}\left(\frac{1}{1-\gamma x} - \frac{1}{1-\delta x}\right) = \frac{1}{\sqrt5}\left(\sum_{n=0}^{\infty}\gamma^n x^n - \sum_{n=0}^{\infty}\delta^n x^n\right)$$

$$= \sum_{n=0}^{\infty}\underbrace{\frac{1}{\sqrt5}(\gamma^n - \delta^n)}_{F_n} x^n$$

$$F_n = \frac{1}{\sqrt5}\left[\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2}\right)^n\right]$$

Cardinality of sets:

Def: The sets A and B have the same cardinality if there is a one-to-one correspondence between elements of A and B

$$|A| = |B|$$

$$A = \mathbb{Z}, \quad B = \mathbb{Q} = \left\{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}_+\right\} \qquad |A| \stackrel{?}{=} |B|$$

$$\{\ldots -2, -1, 0, 1, 2 \ldots\}$$

- $A = \mathbb{Z}$, $\quad$ $\cdot\, B = \mathbb{R}$ $\qquad\qquad |A| \overset{?}{<} |B|$

  $\cdot\, B =$ space of continuous functions on $[0,1]$

<u>Def</u>: If a set is either finite or if has the same cardinality as positive integers $\mathbb{Z}_+$ then it is called <u>countable</u>. Otherwise a set is called uncountable.

When an infinite set is countable, we denote its cardinality $\aleph_0$ (aleph, Hebrew letter). $\quad |S| = \aleph_0$

<u>Ex1</u>: $|\mathbb{Z}| = \aleph_0$

$\mathbb{Z} = \{ \ldots -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5 \ldots \}$

$\mathbb{Z}_+ = \ldots$ $\{0, 1, -1, 2, -2, 3, -3 \ldots\}$

$f(0)\ f(1)\ f(2)\ f(3)$

<u>Ex2</u>: $\frac{p}{q} > 0$

$p+q=5$

$p+q=6$

$p+q=68$

$q \backslash p$	1	2	3	4	5
1	1	2	3	4	5
2	$\frac{1}{2}$	$\times$	$\frac{3}{2}$	$\times$	$\frac{5}{2}$
3	$\frac{1}{3}$	$\frac{2}{3}$	$\times$	$\frac{4}{3}$	$\frac{5}{3}$
4	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\times$	$\frac{5}{4}$
5	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\times$

$\mathbb{Q}_+ = \{1, 2, \frac{1}{2}, \frac{1}{3}, 3, 4, \frac{3}{2}, \frac{2}{3}, \frac{1}{4} \ldots$

$\Downarrow$

$\mathbb{Q} = \{0, 1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3} \ldots$

$|\mathbb{Z}_+| = |\mathbb{Q}|$

$\mathbb{Z}_+ \subsetneq \mathbb{Q}$

What about $|\mathbb{R}|$ - ?

$|\mathbb{R}| = |[0,1]|$

$\mathbb{R} \supset [0,1]$

0.543789

Assume $[0,1]$ is countable $\{r_1, r_2, r_3, r_4, \dots\}$

$$r_1 = 0.d_{11}\, d_{12}\, d_{13}\, d_{14}\, d_{15}$$
$$r_2 = 0.d_{21}\, d_{22}\, d_{23}\, d_{24}\, d_{25}$$
$$r_3 = 0.d_{31}\, d_{32}\, d_{33}\, d_{34}\, d_{35}$$
$$\vdots$$

$d_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}$

Construct a new real number $r = 0.d_1 d_2 d_3 d_4 d_5 \dots$

$$d_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{cases}$$

Ex:

$r_1 = 0.23794102$

$r_2 = 0.44590138$

$r_3 = 0.09118764$

$r_4 = 0.80553900$

$r = 0.4544\dots$

Claim: $r \neq r_n \quad \forall n$

So $[0,1]$ is uncountable

$|\mathbb{Z}| = \aleph_0$

$|\mathbb{R}| = \aleph_1$

$\aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 < \dots$ —

Fact: Consider power set of $\mathbb{Z}_+$ $\mathcal{P}(\mathbb{Z}_+)$

$|\mathcal{P}(\mathbb{Z}_+)| = \aleph_1$

$c := |\mathcal{P}(\mathbb{Z}_+)| = 2^{\aleph_0}$

$\aleph_1 = 2^{\aleph_0}$

$\aleph_2 = 2^{\aleph_1}$

$A = \{1,2,3\}$ $|A| = 3$

$|\mathcal{P}(A)| =$

$\emptyset \quad \{1\} \quad \{2\} \quad \{3\} \quad \{1,2\} \quad \{2,3\} \quad \{1,3\} \quad \{1,2,3\}$

$2^3 \leftarrow$ 1 3 3 1

$2^4 \leftarrow$ 1 4 6 4 1

⋮

$$\mathbb{Z},\ \underbrace{\text{Functions}(\mathbb{Z} \to \mathbb{Z})}_{\mathbb{P}_1},\ \underbrace{\text{Functions}(\mathbb{P}_1 \to \mathbb{P}_1)}_{\mathbb{P}_2}$$

The continuum hypothesis:

Zermelo-Fraenkel

$$\aleph_0 < ? < \aleph_1 \le 2^{\aleph_0}$$

n=5: $1 + 3 + 5 + 7 + 9 = 25$

1 2 3 4 5

n=4 $1+3+5+7 = 16$

$1+5 = 6$

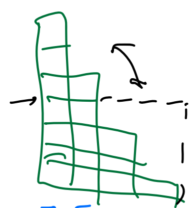
$1+5+9 = 15$

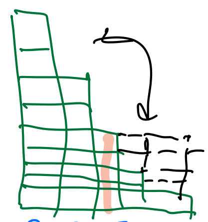
$1+5+9+13 = 28$

$1+5+9+13+17 = 45$

$+21 = 66$

$+25 = 91$





1 2 3

$\{1, 6, 15, 28, 45, 66 \ldots\}$

$n(2n-1)$