q-Opers — what they are and what are they good for?

Peter Koroteev

Talk at M-Seminar at Kansas State University 03/31/2022

Literature

[arXiv:2108.04184]
q-Opers, QQ-systems, and Bethe Ansatz II: Generalized Minors
P. Koroteev, A. M. Zeitlin

[arXiv:2105.00588]
3d Mirror Symmetry for Instanton Moduli Spaces
P. Koroteev, A. M. Zeitlin

[arXiv:2007.11786] J. Inst. Math. Jussieu
Toroidal q-Opers
P. Koroteev, A. M. Zeitlin

[arXiv:2002.07344] J. Europ. Math. Soc.
q-Opers, QQ-Systems, and Bethe Ansatz
E. Frenkel, P. Koroteev, D. S. Sage, A. M. Zeitlin
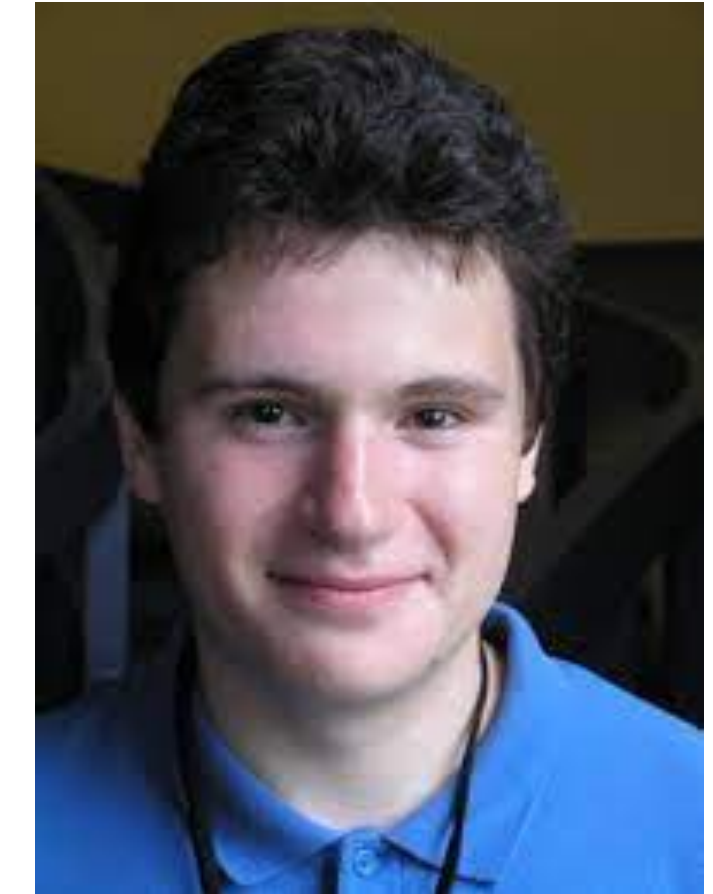
[arXiv:1811.09937] Commun.Math.Phys. **381** (2021) 641
(SL(N),q)-opers, the q-Langlands correspondence, and quantum/classical duality
P. Koroteev, D. S. Sage, A. M. Zeitlin

[arXiv:1705.10419] Selecta Math. **27** (2021) 87
Quantum K-theory of Quiver Varieties and Many-Body Systems
P. Koroteev, P. P. Pushkar, A. V. Smirnov, A. M. Zeitlin

Motivation

Quantum Geometry and Integrable Systems   [Okounkov et al]   [Pushkar, Zeitlin, Smirnov]

[PK, Pushkar, Smirnov, Zeitlin]

BPS/CFT Correspondence   [Nekrasov Shatashvili]

Geometric q-Langlands Correspondence   [Frenkel] [Aganagic, Frenkel, Okounkov]

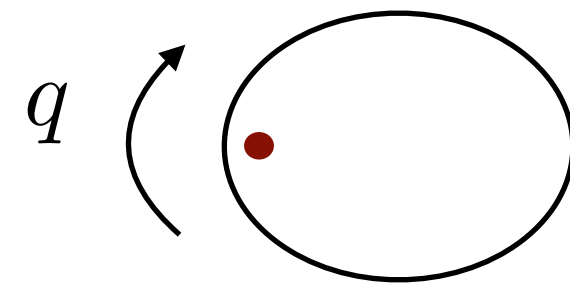ODE/IM Correspondence   [Bazhanov, Lukyanov, Zamolodchikov]

[Dorey, Tateo]

Consider vector bundle E over $\mathbb{P}^1$

$$M_q : \mathbb{P}^1 \to \mathbb{P}^1$$
$$z \mapsto qz$$

q

Map of vector bundles $\quad A : E \longrightarrow E^q$

Upon trivialization $\quad A(z) \in \mathfrak{gl}(N, \mathbb{C}(z))$

q-gauge transformation $\quad A(z) \mapsto g(qz)A(z)g^{-1}(z)$

Difference equation $\quad D_q(s) = As$

Definition: A meromorphic $(\mathrm{GL}(N), q)$-*connection* over $\mathbb{P}^1$ is a pair (E, A), where E is a (trivializable) vector bundle of rank N over $\mathbb{P}^1$ and A is a meromorphic section of the sheaf $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(E, E^q)$ for which $A(z)$ is invertible, i.e. lies in $\mathrm{GL}(N, \mathbb{C}(z))$. The pair (E, A) is called an $(\mathrm{SL}(N), q)$-connection if there exists a trivialization for which $A(z)$ has determinant 1.

q-Opers

Definition: A $(\mathrm{GL}(2), q)$-*oper* on $\mathbb{P}^1$ is a triple $(E, A, \mathcal{L})$, where (E, A) is a $(\mathrm{GL}(2), q)$-connection and $\mathcal{L}$ is a line subbundle such that the induced map $\bar{A} : \mathcal{L} \longrightarrow (E/\mathcal{L})^q$ is an isomorphism. The triple is called an $(\mathrm{SL}(2), q)$-*oper* if (E, A) is an $(\mathrm{SL}(2), q)$-connection.

in a trivialization $s(qz) \wedge A(z)s(z) \neq 0$

Definition: A $(\mathrm{SL}(2), q)$-*oper with regular singularities* at the points $z_1, \ldots, z_L \neq 0, \infty$ with weights $k_1, \ldots k_L$ is a meromorphic $(\mathrm{SL}(2), q)$-oper $(E, A, \mathcal{L})$ for which $\bar{A}$ is an isomorphism everywhere on $\mathbb{P}^1 \setminus \{0, \infty\}$ except at the points $z_m, q^{-1}z_m, q^{-2}z_m, \ldots, q^{-k_m+1}z_m$ for $m \in \{1, \ldots, L\}$, where it has simple zeros.
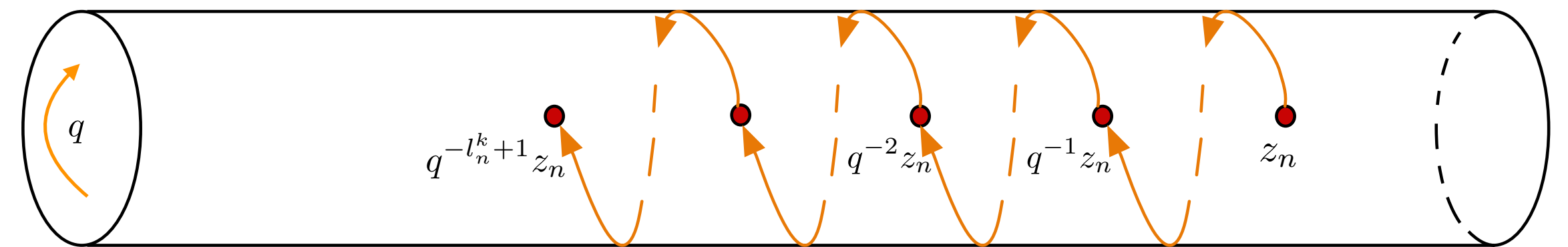
Finally, (SL(2),q)-oper is **Z-twisted** in A(z) is gauge equivalent to a diagonal matrix Z

$$Z = g(qz)A(z)g(z)^{-1}$$

Miura q-Opers

Miura (SL(2),q)-oper is a quadruple $(E, A, \mathcal{L}, \hat{\mathcal{L}})$ where $(E, A, \mathcal{L})$ is an (SL(2),q)-oper and $\hat{\mathcal{L}}$ is preserved by the q-connection A

Chose trivialization of $\mathcal{L}$

$$s(z) = \begin{pmatrix} Q_+(z) \\ Q_-(z) \end{pmatrix}$$

Twist element $\quad Z = \mathrm{diag}(\zeta, \zeta^{-1})$

q-Oper condition — SL(2) **QQ-system**

$$\zeta Q_-(z)Q_+(zq) - \zeta^{-1}Q_-(zq)Q_+(z) = \Lambda(z)$$

singularities

One of the polynomials can be made monic

$$Q_+(z) = \prod_{k=1}^{m}(z - w_k)$$

$$\Lambda(z) = \prod_{p=1}^{L}\prod_{j_p=0}^{r_p-1}(z - q^{-j_p}z_p)$$

From QQ-system to Bethe equations

$$\frac{\Lambda(w_k)}{\Lambda(q^{-1}w_k)} = -\zeta^2 \frac{Q_+(qw_k)}{Q_+(q^{-1}w_k)}, \qquad k = 1, \ldots, m.$$

$$q^r \prod_{p=1}^{L} \frac{w_k - q^{1-r_p}z_p}{w_k - qz_p} = -\zeta^2 q^m \prod_{j=1}^{m} \frac{qw_k - w_j}{w_k - qw_j}, \qquad k = 1, \ldots, m$$

q-Miura Transformation

$$A(z) = \begin{pmatrix} g(z) & \Lambda(z) \\ 0 & g(z)^{-1} \end{pmatrix}$$

Z-twisted q-oper condition

$$A(z) = v(zq)Zv(z)^{-1}, \qquad Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

Gauge transformation reads

$$v(z) = \begin{pmatrix} y(z) & 0 \\ 0 & y(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\frac{Q_-(z)}{Q_+(z)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y(z) & -y(z)\frac{Q_-(z)}{Q_+(z)} \\ 0 & y(z)^{-1} \end{pmatrix}$$

We find

$$g(z) = \zeta_i y(zq)y(z)^{-1} \qquad\qquad \Lambda(z) = y(z)y(zq)\left(\zeta\frac{Q_-(z)}{Q_+(z)} - \zeta^{-1}\frac{Q_-(zq)}{Q_+(zq)} \right)$$

The q-oper condition becomes the **SL(2) QQ-system**

$$\zeta Q_-(z)Q_+(zq) - \zeta^{-1}Q_-(zq)Q_+(z) = \Lambda(z)$$

Difference Equation

$$D_q(s) = As. \qquad\qquad D_q(s_1) = \Lambda(z)s_2$$

after elimination

$$\left(D_q^2 - T(qz)D_q - \frac{\Lambda(qz)}{\Lambda(z)} \right) s_1 = 0$$

Trigonometric Ruijsenaars Hamiltonians

Recover 2-body tRS Hamiltonian from a q-Oper

Let $\quad Q_- = z - p_- \quad$ **and** $\quad Q_+ = c(z - p_+)$

$$z^2 - \frac{z}{q}\left[\frac{\zeta - q\zeta^{-1}}{\zeta - \zeta^{-1}}p_+ + \frac{q\zeta - \zeta^{-1}}{\zeta - \zeta^{-1}}p_-\right] + \frac{p_+ p_-}{q} = (z - z_+)(z - z_-)$$

qOper condition yields
tRS Hamiltonians!

$$T_1 \qquad T_2$$

$$\det(z - L_{tRS}) = (z - z_+)(z - z_-)$$

Calogero-Moser Space

Let V be an N-dimensional vector space over $\mathbb{C}$. Let $\mathscr{M}'$ be the subset of $GL(V) \times GL(V) \times V \times V^*$ consisting of elements (M, T, u, v) such that
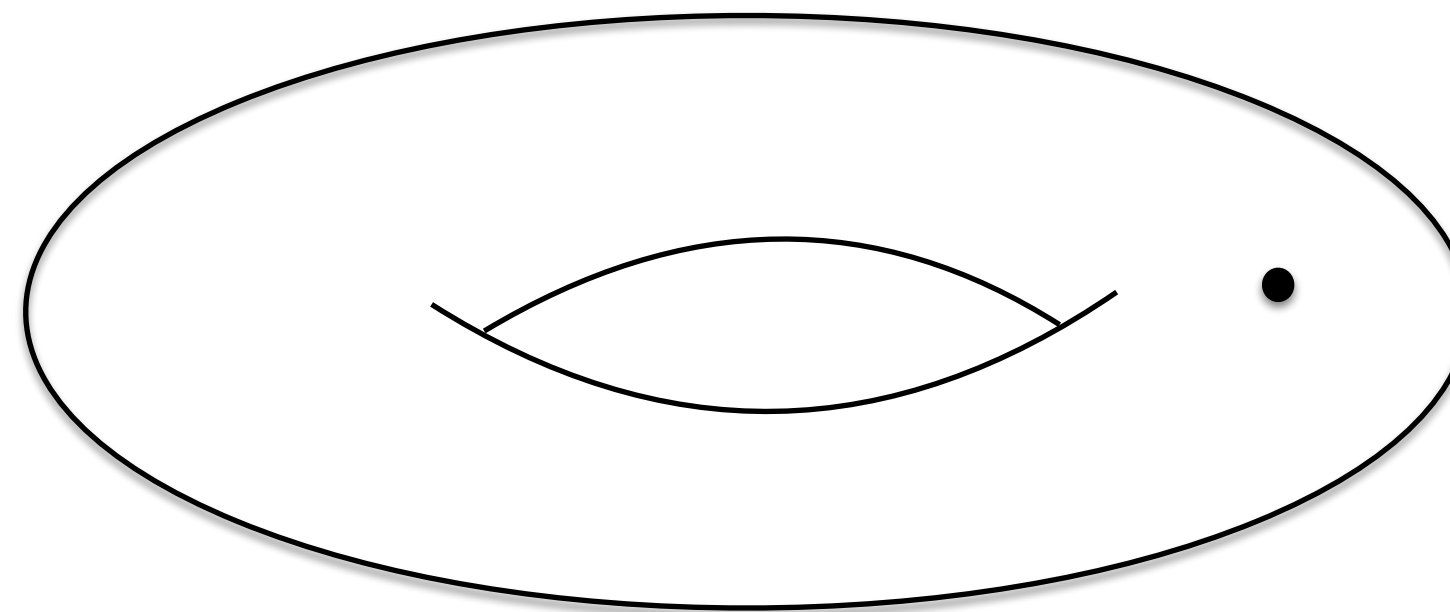
$$qMT - TM = u \otimes v^T$$

The group $GL(N; \mathbb{C}) = GL(V)$ acts on $\mathscr{M}'$ by conjugation

$$(M, T, u, v) \mapsto (gMg^{-1}, gTg^{-1}, gu, vg^{-1})$$

The quotient of $\mathscr{M}'$ by the action of $GL(V)$ is called **Calogero-Moser space** $\mathscr{M}$

Also can be understood as moduli space of flat connections on punctured torus

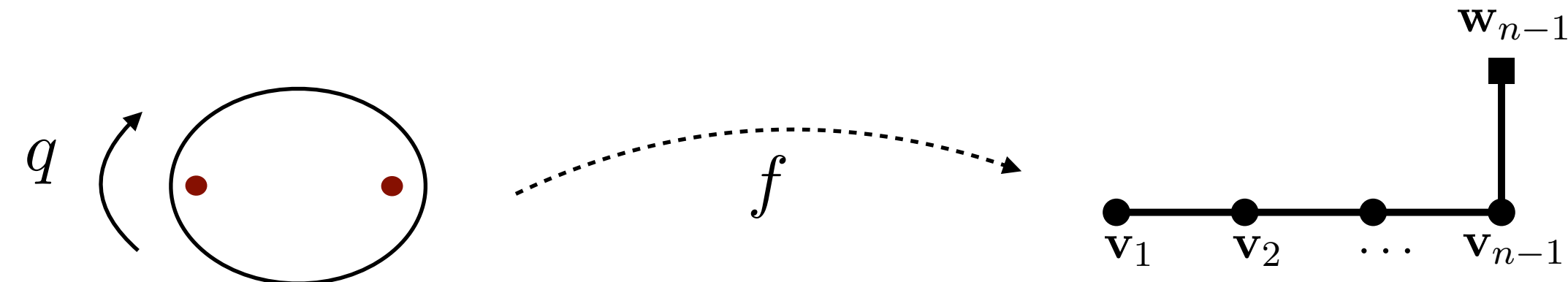Integrable Hamiltonians are $\sim \mathrm{Tr} T^k$

$$\mathcal{M}_n = \{A, B, C\}/GL(n; \mathbb{C})$$

$$ABA^{-1}B^{-1} = C$$

$$C = \mathrm{diag}(q, \ldots, q, q^{n-1})$$

Enumerative AG/Integrable Systems

Quantum equivariant K-theory of Nakajima quiver varieties

$$A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$$

$$\mathbf{V}^{(\tau)}(\boldsymbol{z}) = \sum_{\boldsymbol{d}} \mathrm{ev}_{p_2,*}(\widehat{\mathcal{O}}_{\mathrm{vir}}^{\boldsymbol{d}} \otimes \tau|_{p_1}, \mathrm{QM}_{\mathrm{nonsing}\,p_2}^{\boldsymbol{d}})\boldsymbol{z}^{\boldsymbol{d}} \in K_{\mathsf{T} \times \mathbb{C}_q^{\times}}(X)_{loc}[[\boldsymbol{z}]]$$
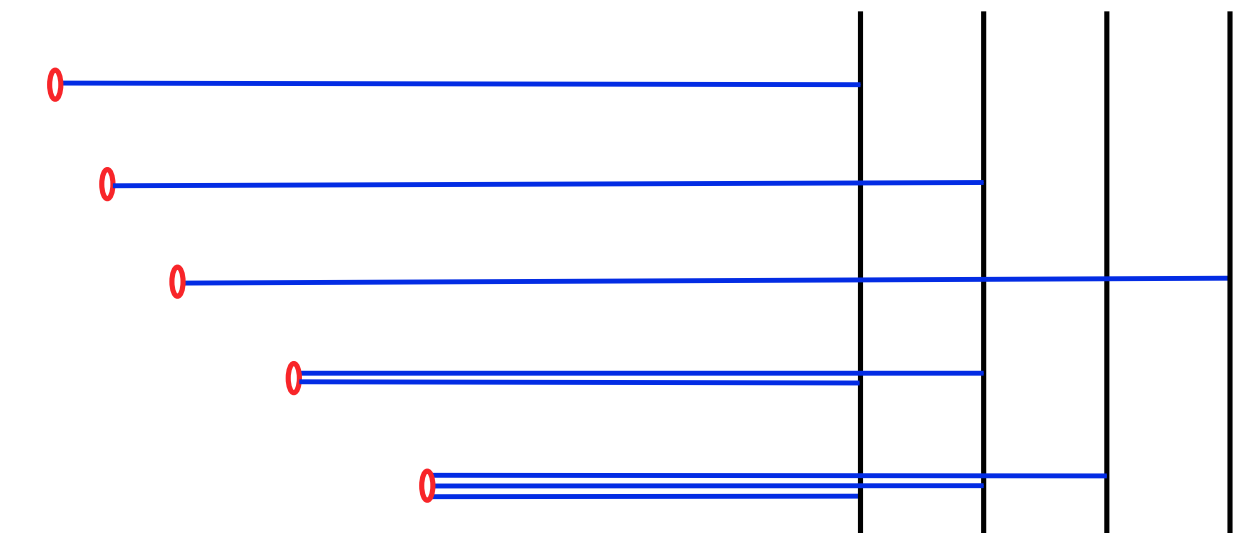
Saddle point limit yields Bethe equations for XXZ

Quantum classes satisfy interesting difference equations in equivariant parameters and Kahler parameters **qKZ, Dynamical equation** [Okounkov, Smirnov]

After symmetrization they can be rewritten as eigenvalue equations for **trigonometric Ruijsenaars-Schneider (tRS)** system [PK, Zeitlin] [PK]

$$T_r(\mathbf{a}) = \sum_{\substack{\mathcal{I} \subset \{1,\dots,n\} \\ |\mathcal{I}|=r}} \prod_{\substack{i \in \mathcal{I} \\ j \notin \mathcal{I}}} \frac{t\, a_i - a_j}{a_i - a_j} \prod_{i \in \mathcal{I}} p_i$$

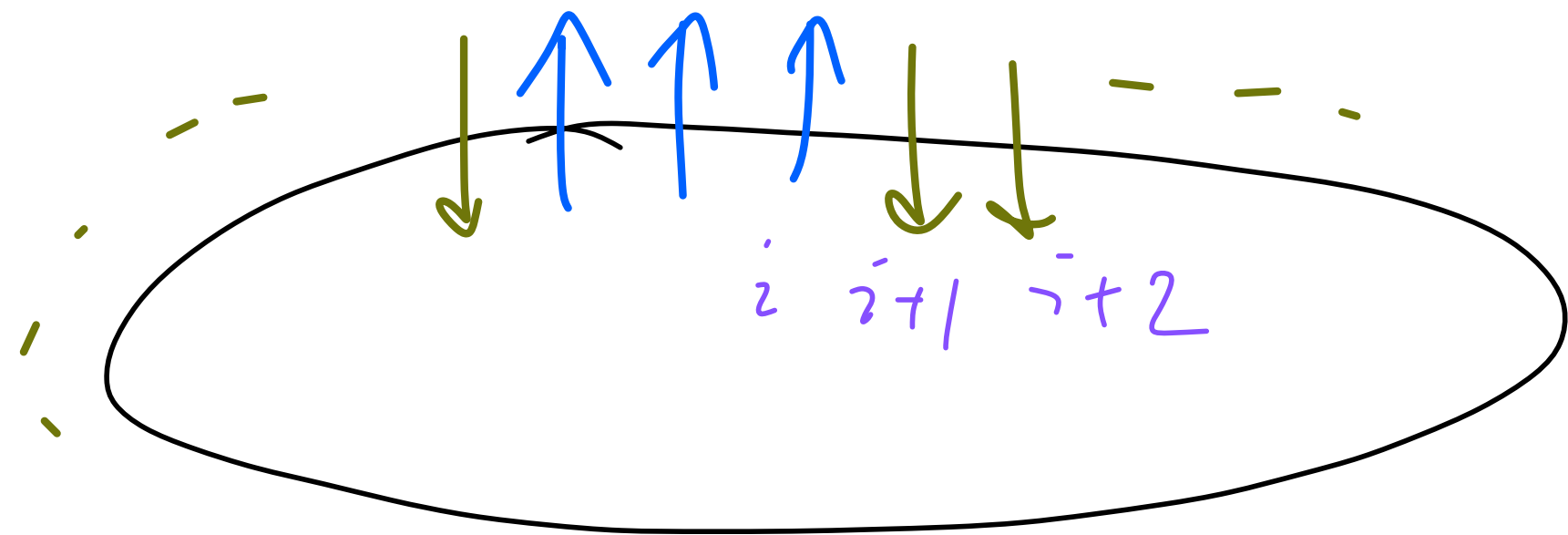$$T_r(\boldsymbol{a})\mathrm{V}(\boldsymbol{a}, \vec{\zeta}) = S_r(\vec{\zeta}, t)\mathrm{V}(\boldsymbol{a}, \vec{\zeta})$$

In terms of string/gauge theory tRS eigenproblem is Ward identity [Gaiotto, PK] [Bullimore, Kim, PK]

Quantum

Classical

$x_1 \quad x_2 \quad x_3$

x_n

$$\Omega = \sum_i \frac{dp_i}{p_i} \wedge \frac{dz_i}{z_i}$$

SU(**n**) XXZ spin chain on n sites w/ **anisotropies** and **twisted periodic boundary conditions**

n-particle trigonometric Ruijsenaars-Schneider model

$$[T_i, T_j] = 0$$

Planck's constant $\quad \hbar$

Coupling constant $\quad \hbar \qquad T_1 = \sum_{i=1}^{n} \prod_{j \neq i} \frac{\hbar z_i - z_j}{z_i - z_j} p_i$

twist eigenvalues $\quad z_i$

coordinates $\quad z_i$

equivariant parameters (anisotropies) $\quad a_i$

energy (eigenvalues of Hamiltonians) $\quad e_i(a_i)$

Bethe Ansatz Equations: $\dfrac{\partial Y}{\partial \sigma_i} = 0$

Energy level equations

$$T_i(\mathbf{z}, \hbar) = e_i(\mathbf{a}), \qquad i = 1, \ldots, n$$

$$\frac{\zeta_i}{\zeta_{i+1}} \prod_{\beta=1}^{\mathbf{v}_{i-1}} \frac{\sigma_{i,\alpha} - \hbar^{1/2}\sigma_{i-1,\beta}}{\sigma_{i-1,\beta} - \hbar^{1/2}\sigma_{i,\alpha}} \cdot \prod_{\beta \neq \alpha}^{\mathbf{v}_i} \frac{\hbar\sigma_{i,\alpha} - \sigma_{i,\beta}}{\hbar\sigma_{i,\beta} - \sigma_{i,\alpha}} \cdot \prod_{\beta=1}^{\mathbf{v}_{i+1}} \frac{\sigma_{i,\alpha} - \hbar^{1/2}\sigma_{i+1,\beta}}{\sigma_{i+1,\beta} - \hbar^{1/2}\sigma_{i,\alpha}} = (-1)^{\delta_i}$$

Quantum/Classical Duality

[PK Gaiotto]
[PK Zeitlin]
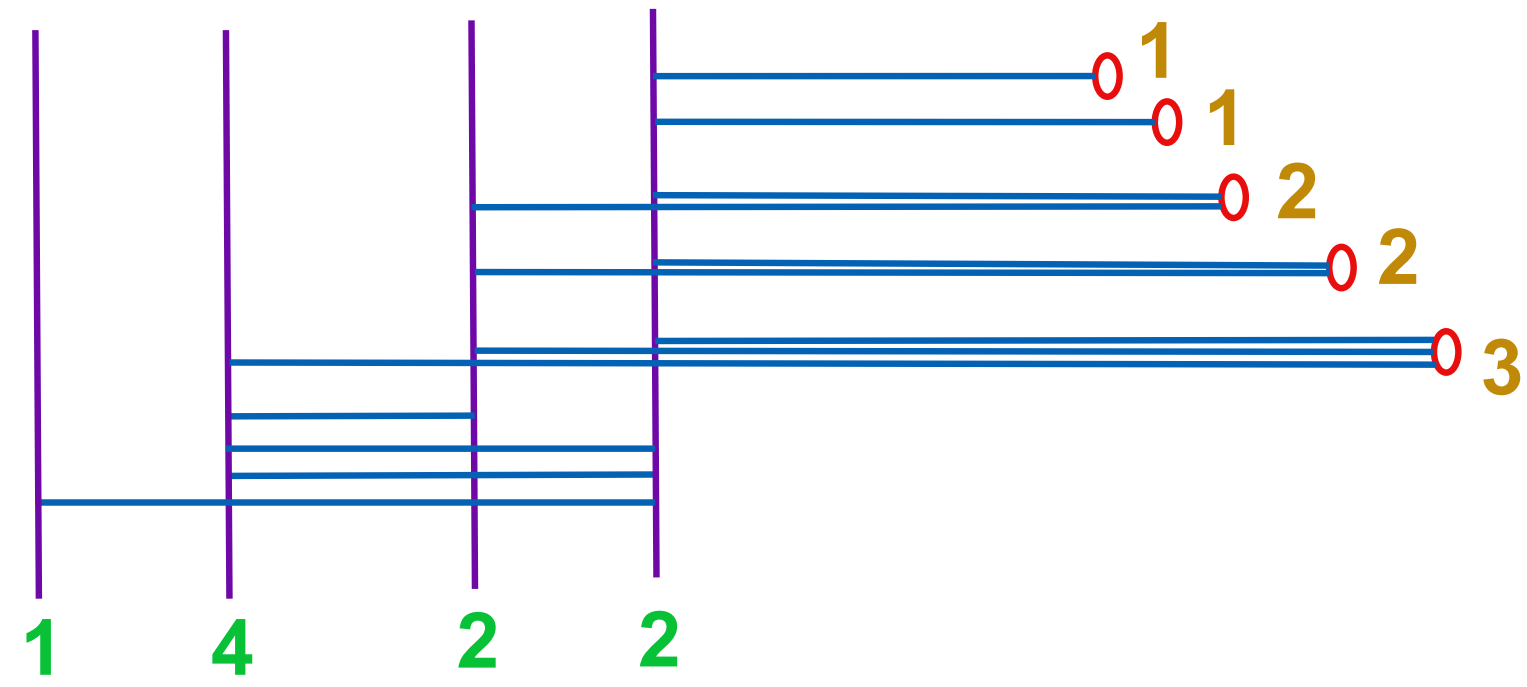
Symplectic form

$$\Omega = \sum_{i=1}^{N} \frac{dp_i^\xi}{p_i^\xi} \wedge \frac{d\xi_i}{\xi_i} - \frac{dp_i^a}{p_i^a} \wedge \frac{da_i}{a_i}$$
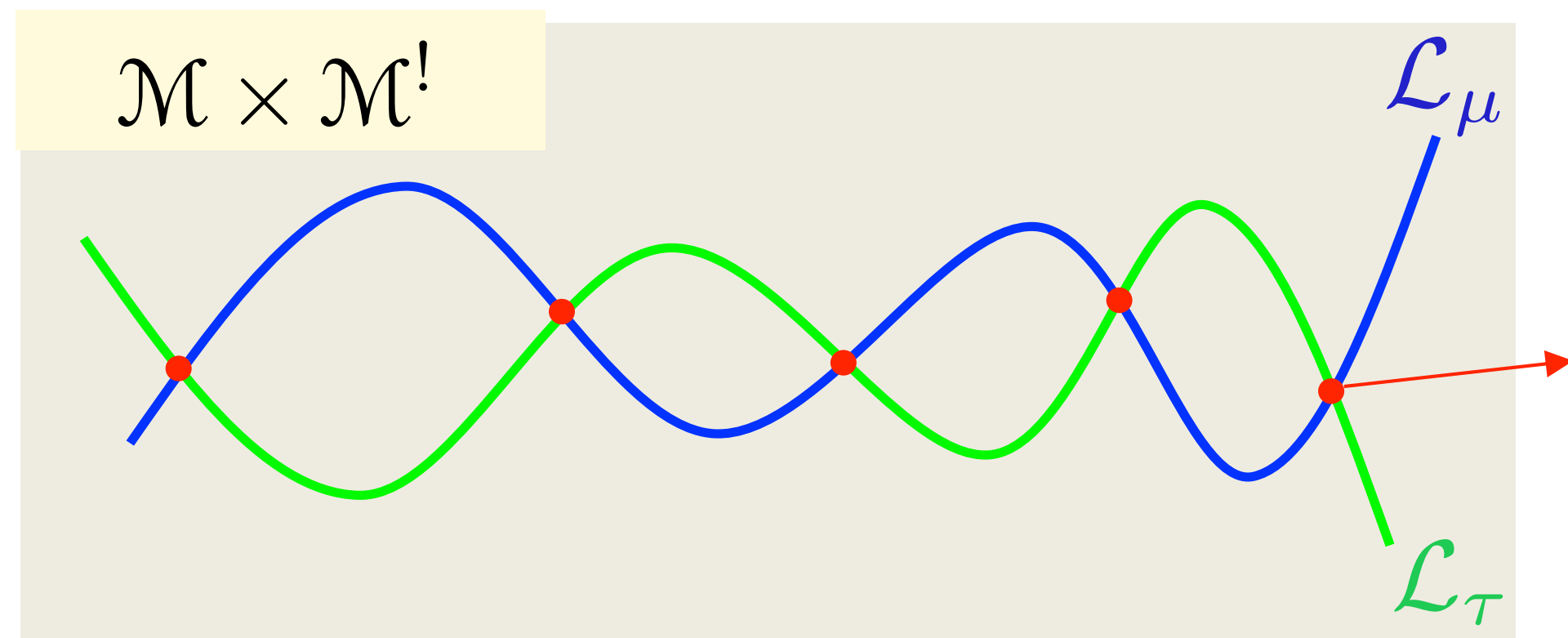
tRS momenta

$$p_i^\xi = \exp \frac{\partial Y}{\partial \xi_i}, \qquad p_i^a = \exp \frac{\partial Y}{\partial a_i}$$

tRS energy relations

$$\det(u - T) = \prod_{i=1}^{N}(u - a_i), \qquad \det(u - M) = \prod_{i=1}^{N}(u - \xi_i)$$

$$\sum_{\substack{\mathcal{I} \subset \{1,\ldots,L\} \\ |\mathcal{I}| = k}} \prod_{\substack{i \in \mathcal{I} \\ j \notin \mathcal{I}}} \frac{a_i - \hbar\, a_j}{a_i - a_j} \prod_{m \in \mathcal{I}} p_m = \ell_k(\xi_i)$$

$\mathcal{M} \times \mathcal{M}^!$

$$Y = Y^!$$

3d mirror symmetry

$\mathcal{L}_\mu$ Eigenvalues of M and Slodowy form on T

$\mathcal{L}_\tau$ Eigenvalues of T and Slodowy form on M

Solutions of Bethe equations — intersection points
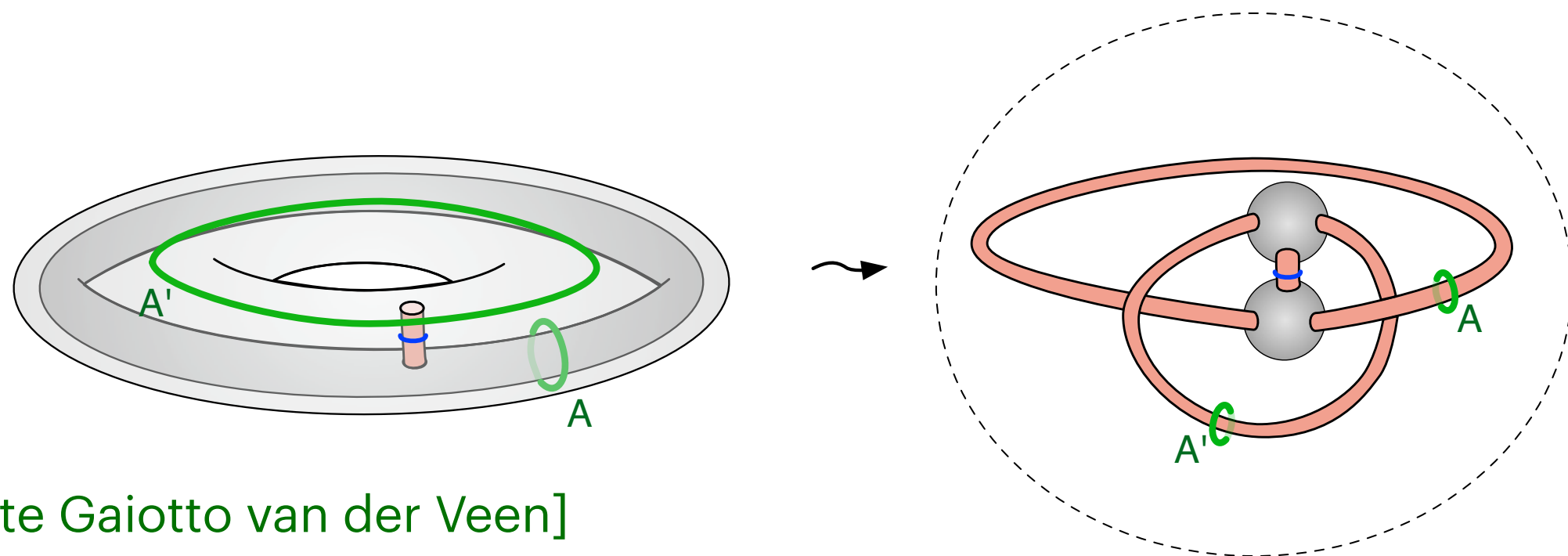
[Dimofte Gaiotto van der Veen]

II. (G,q)-Connection

G-simple simply-connected complex Lie group

Principal G-bundle $\mathcal{F}_G$ over $\mathbb{P}^1$

$$M_q : \mathbb{P}^1 \to \mathbb{P}^1$$
$$z \mapsto qz$$

A meromorphic **(G,q)-connection** on $\mathcal{F}_G$ is a section A of $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_G, \mathcal{F}_G^q)$ U-Zariski open dense set

Choose U so that the restriction $\mathcal{F}_G|_U$ of $\mathcal{F}_G$ to U is isomorphic to a trivial G-bundle

$$A(z) \in G(\mathbb{C}(z)) \quad \text{on} \quad U \cap M_q^{-1}(U)$$

Change of trivialization $A(z) \mapsto g(qz)A(z)g(z)^{-1}$

(G,q)-Opers

A meromorphic (G,q)-oper on $\mathbb{P}^1$ is a triple $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$

$\quad\quad A$ is a meromorphic (G, q)-connection

$\quad\quad \mathcal{F}_{B_-}$ is a reduction of $\mathcal{F}_G$ to B_-

Oper condition: Restriction of the connection on some Zariski open dense set U

$$A : \mathcal{F}_G \longrightarrow \mathcal{F}_G^q \text{ to } U \cap M_q^{-1}(U)$$

takes values in the double Bruhat cell

$$B_-(\mathbb{C}[U \cap M_q^{-1}(U)])cB_-(\mathbb{C}[U \cap M_q^{-1}(U)]) \quad\quad\quad \text{Coxeter element: } c = \prod_i s_i$$

Locally

$$A(z) = n'(z) \prod_i (\phi_i(z)^{\check{\alpha}_i} s_i) n(z)$$

$$\phi_i(z) \in \mathbb{C}(z) \text{ and } n(z), n'(z) \in N_-(z)$$

Miura (G,q)-Opers

Definition: A *Miura (G,q)-oper* on $\mathbb{P}^1$ is a quadruple $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$, where $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ is a meromorphic (G,q)-oper on $\mathbb{P}^1$ and $\mathcal{F}_{B_+}$ is a reduction of the G-bundle $\mathcal{F}_G$ to B_+ that is preserved by the q-connection A.

It can be shown that the two flags $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ are in *generic relative position* for some dense set V

The fiber $\mathcal{F}_{G,x}$ of $\mathcal{F}_G$ at x is a G-torsor with reductions $\mathcal{F}_{B_-,x}$ and $\mathcal{F}_{B_+,x}$ to B_- and B_+, respectively. Choose any trivialization of $\mathcal{F}_{G,x}$, i.e. an isomorphism of G-torsors $\mathcal{F}_{G,x} \simeq G$. Under this isomorphism, $\mathcal{F}_{B_-,x}$ gets identified with $aB_- \subset G$ and $\mathcal{F}_{B_+,x}$ with bB_+.

Then $a^{-1}b$ is a well-defined element of the double quotient $B_- \backslash G / B_+$, which is in bijection with W_G.

We will say that $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ have a generic relative position at $x \in X$ if the element of W_G assigned to them at x is equal to 1 (this means that the corresponding element $a^{-1}b$ belongs to the open dense Bruhat cell $B_- \cdot B_+ \subset G$).

Structure Theorems

Theorem 1: *For any Miura (G, q)-oper on $\mathbb{P}^1$, there exists a trivialization of the underlying G-bundle $\mathcal{F}_G$ on an open dense subset of $\mathbb{P}^1$ for which the oper q-connection has the form*

$$A(z) \in N_-(z) \prod_i ((\phi_i(z)^{\check{\alpha}_i} s_i) N_-(z) \cap B_+(z).$$

Theorem 2: *Let F be any field, and fix $\lambda_i \in F^\times, i = 1, \ldots, r$. Then every element of the set $N_- \prod_i \lambda_i^{\check{\alpha}_i} s_i N_- \cap B_+$ can be written in the form*

$$\prod_i g_i^{\check{\alpha}_i} e^{\frac{\lambda_i t_i}{g_i} e_i}, \qquad g_i \in F^\times,$$

where each $t_i \in F^\times$ is determined by the lifting s_i.

Adding Singularities and Twists

Consider family of polynomials $\quad \{\Lambda_i(z)\}_{i=1,\ldots,r}$

(G,q)-oper with regular singularities can be written as

$$A(z) = n'(z) \prod_i (\Lambda_i(z)^{\check{\alpha}_i} \, s_i) n(z), \qquad n(z), n'(z) \in N_-(z)$$

Using structure theorem every Miura (G,q)-oper with singularities reads

$$A(z) = \prod_i g_i(z)^{\check{\alpha}_i} \, e^{\frac{\Lambda_i(z)}{g_i(z)} e_i}, \qquad g_i(z) \in \mathbb{C}(z)^{\times}$$

(G,q)-oper is **Z-twisted** if it is equivalent to a constant element of G $\quad Z \in H \subset H(z) \quad$ Z is regular semisimple. There are W_G

Miura (G,q)-opers for each (G,q)-opers

$$A(z) = g(qz)Zg(z)^{-1}$$

Z-twisted Miura (G,q)-oper if gauge transform is from Borel

$$A(z) = v(qz)Zv(z)^{-1}, \qquad v(z) \in B_+(z)$$

Plucker Relations

V_i^+ irrep of G with highest weight ω_i Line $L_i \subset V_i$ stable under B_+

Plucker relations: for two integral dominant weights $\quad L_{\lambda+\mu} \subset V_{\lambda+\mu}$ is the image of $L_\lambda \otimes L_\mu \subset V_\lambda \otimes V_\mu$

under canonical projection $\quad V_\lambda \otimes V_\mu \longrightarrow V_{\lambda+\mu}$

Conversely, for a collection of lines $\quad L_\lambda \subset V_\lambda \quad$ satisfying Plucker relations $\quad \exists B \subset G \quad$ such that $\quad L_\lambda$ is stabilized by B for all λ

A choice of B is equivalent to a choice of B_+-torsor in G

Let ν_{ω_i} be a generator of the line $L_i \subset V_i$. This is a vector of weight ω_i wrt $H \subset B_+$

The subspace of V_i of weight $\omega_i - \alpha_i$ is one-dimensional and spanned $f_i \cdot \nu_{\omega_i}$

Thus the 2d subspace spanned by $\{\nu_{\omega_i}, f_i \cdot \nu_{\omega_i}\}$ is a B_+-invariant subspace of V_i
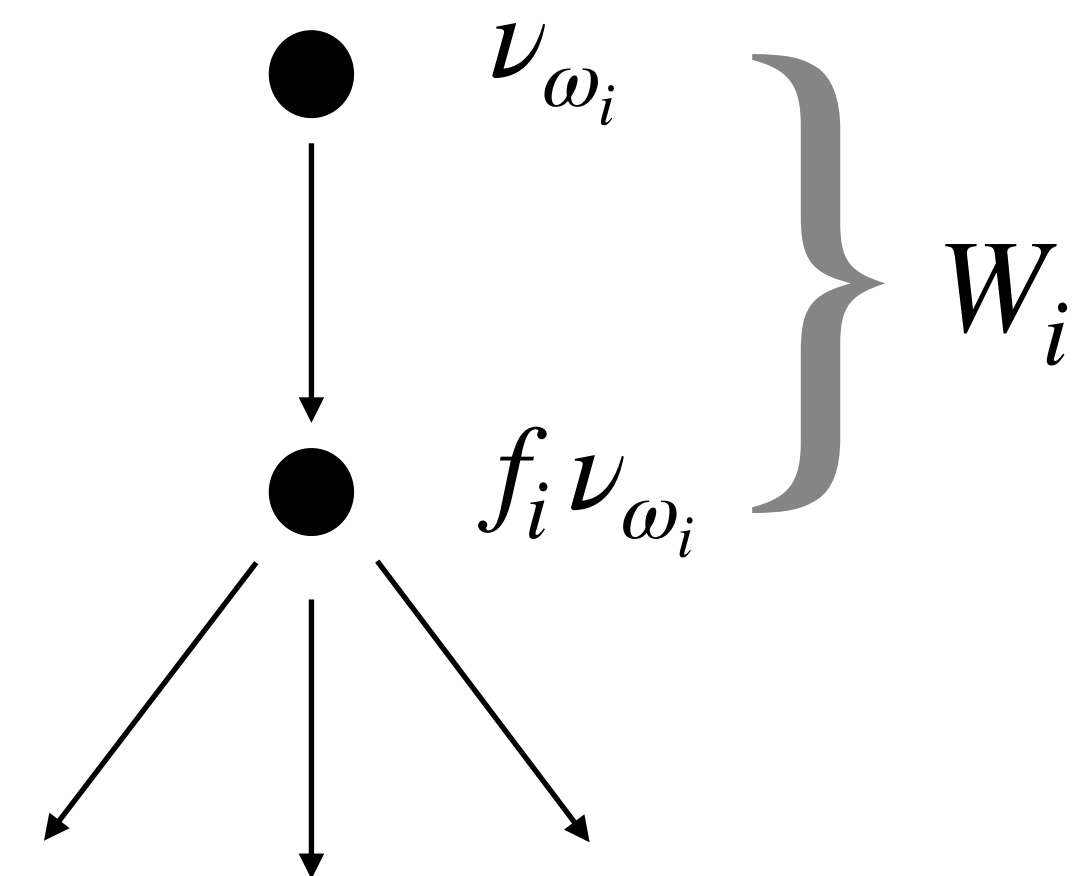
Miura-Pucker (G,q)-Opers

let $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$ be a Miura (G, q)-oper with regular singularities $\quad \{\Lambda_i(z)\}_{i=1,\dots,r}$

Associated vector bundle $\quad \mathcal{V}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} V_i = \mathcal{F}_G \underset{G}{\times} V_i \quad$ contains rank-two subbundle $\quad \mathcal{W}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} W_i$

associated to $W_i \subset V_i$, and $\mathcal{W}_i$ in turn contains a line subbundle $\quad \mathcal{L}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} L_i$

Using structure theorems we obtain **r** Miura (GL(2),q)-opers

$$A_i(z) = \begin{pmatrix} g_i(z) & \Lambda_i(z) \prod_{j>i} g_j(z)^{-a_{ji}} \\ 0 & g_i^{-1}(z) \prod_{j \neq i} g_j(z)^{-a_{ji}} \end{pmatrix}$$

Z-twisted Miura-Plucker (G,q)-oper is meromorphic Miura (G,q)-oper on P1 such that for each Miura (GL(2),q)-oper

$$A_i(z) = v(zq)Zv(z)^{-1}|_{W_i} = v_i(zq)Z_i v_i(z)^{-1}$$

where $v_i(z) = v(z)|_{W_i}$ and $Z_i = Z|_{W_i}$

QQ-System

Theorem: *There is a one-to-one correspondence between the set of nondegenerate Z-twisted Miura-Plücker (G, q)-opers and the set of nondegenerate polynomial solutions of the QQ-system*

$$\widetilde{\xi}_i Q^i_-(z) Q^i_+(qz) - \xi_i Q^i_-(qz) Q^i_+(z) =$$

$$\Lambda_i(z) \prod_{j>i} \left[Q^j_+(qz) \right]^{-a_{ji}} \prod_{j<i} \left[Q^j_+(z) \right]^{-a_{ji}}, \qquad i = 1, \ldots, r,$$

$$\widetilde{\xi}_i = \zeta_i \prod_{j>i} \zeta_j^{a_{ji}}, \qquad \xi_i = \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}}$$

Proof uses

$$v(z) = \prod_{i=1}^r y_i(z)^{\check{\alpha}_i} \prod_{i=1}^r e^{-\frac{Q^i_-(z)}{Q^i_+(z)} e_i} \ldots,$$

$$g_i(z) = \zeta_i \frac{Q^i_+(qz)}{Q^i_+(z)}$$

XXZ Bethe Ansatz Equations for G

roots of Q+

$$\frac{Q_+^i(qw_i^k)}{Q_+^i(q^{-1}w_i^k)} \prod_j \zeta_j^{a_{ji}} = - \frac{\Lambda_i(w_k^i) \prod_{j>i} \left[Q_+^j(qw_k^i)\right]^{-a_{ji}} \prod_{j<i} \left[Q_+^j(w_k^i)\right]^{-a_{ji}}}{\Lambda_i(q^{-1}w_k^i) \prod_{j>i} \left[Q_+^j(w_k^i)\right]^{-a_{ji}} \prod_{j<i} \left[Q_+^j(q^{-1}w_k^i)\right]^{-a_{ji}}}$$

Space of nondegenerate solutions of QQ-system for G ⟷ Nondegenerate **Z-twisted Miura-Plucker** (G,q)-opers with regular singularities

‖ ?

Space of nondegenerate solutions of XXZ for G

?

Nondegenerate **Z-twisted Miura** (G,q)-opers with regular singularities

SL(2) Example

$$A(z) = \begin{pmatrix} g(z) & \Lambda(z) \\ 0 & g(z)^{-1} \end{pmatrix}$$

Z-twisted q-oper condition

$$A(z) = v(zq)Zv(z)^{-1}, \qquad Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

Gauge transformation reads

$$v(z) = \begin{pmatrix} y(z) & 0 \\ 0 & y(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\frac{Q_-(z)}{Q_+(z)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y(z) & -y(z)\frac{Q_-(z)}{Q_+(z)} \\ 0 & y(z)^{-1} \end{pmatrix}$$

We find

$$g(z) = \zeta_i y(zq) y(z)^{-1}$$

$$\Lambda(z) = y(z)y(zq)\left(\zeta \frac{Q_-(z)}{Q_+(z)} - \zeta^{-1}\frac{Q_-(zq)}{Q_+(zq)} \right)$$

The q-oper condition becomes the **SL(2) QQ-system**

$$\zeta Q_-(z)Q_+(zq) - \zeta^{-1} Q_-(zq)Q_+(z) = \Lambda(z)$$

To get Bethe equations

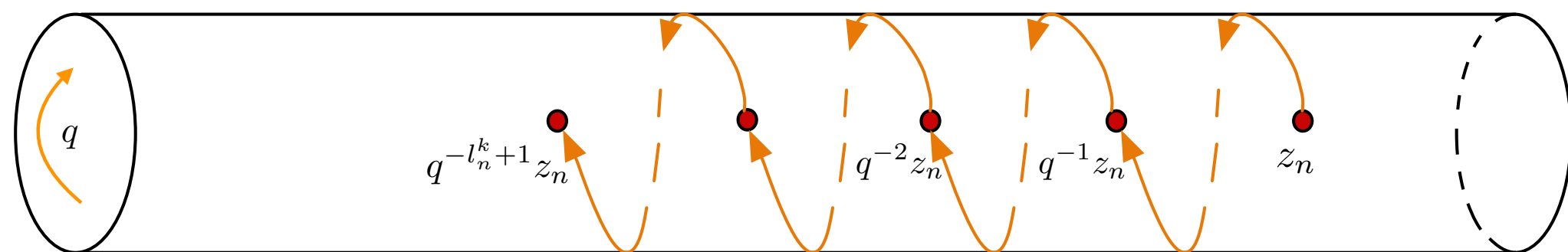$$Q_+(z) = \prod_{k=1}^{m}(z - w_k)$$

evaluate at roots of Q

$$\frac{\Lambda(w_k)}{\Lambda(q^{-1}w_k)} = -\zeta^2 \frac{Q_+(qw_k)}{Q_+(q^{-1}w_k)}, \qquad k = 1,\dots,m.$$

Singularities

$$\Lambda(z) = \prod_{p=1}^{L}\prod_{j_p=0}^{r_p-1}(z - q^{-j_p}z_p)$$

XXZ Bethe equations

$$q^r \prod_{p=1}^{L} \frac{w_k - q^{1-r_p}z_p}{w_k - qz_p} = -\zeta^2 q^m \prod_{j=1}^{m} \frac{qw_k - w_j}{w_k - qw_j}, \qquad k = 1,\dots,m.$$

Quantum Bäcklund Transformation

Theorem: Consider the following q-gauge transformation

$$A \mapsto A^{(i)} = e^{\mu_i(qz)f_i} A(z) e^{-\mu_i(z)f_i}, \quad \text{where} \quad \mu_i(z) = \frac{\prod\limits_{j \neq i} \left[Q_+^j(z)\right]^{-a_{ji}}}{Q_+^i(z) Q_-^i(z)}$$

changes the set of Q-functions
$$Q_+^j(z) \mapsto Q_+^j(z), \qquad j \neq i,$$
$$Q_+^i(z) \mapsto Q_-^i(z), \qquad Z \mapsto s_i(Z)$$

$$\{\widetilde{Q}_+^j\}_{j=1,\ldots,r} = \{Q_+^1, \ldots, Q_+^{i-1}, Q_-^i, Q_+^{i+1} \ldots, Q_+^r\}$$

$$\{\widetilde{z_j}\}_{j=1,\ldots,r} = \{z_1, \ldots, z_{i-1}, z_i^{-1} \prod z_j^{-a_{ji}}, \ldots, z_r\}$$

Now the strategy is to successively apply Backlund transformations according to the reduced decomposition of the element of the Weyl group

Consider longest element $\quad w_0 = s_{i_1} \ldots s_{i_\ell}$

Theorem: Every Z-twisted Miura-Plucker (G,q)-oper is Z-twisted Miura (G,q)-oper

The proof based on properties of double Bruhat cells addresses existence of the diagonalizing element v(z) (to be constructed later)

(SL(N),q)-Opers

The QQ-system

$$\xi_i \phi_i(z) - \xi_{i+1} \phi_i(qz) = \rho_i(z)$$

$$\phi_i(z) = \frac{Q_i^-(z)}{Q_i^+(z)}, \qquad \rho_i(z) = \Lambda_i(z) \frac{Q_{i-1}^+(qz) Q_{i+1}^+(z)}{Q_i^+(z) Q_i^+(qz)}$$

q-Oper condition

$$v(qz)^{-1} A(z) = Z v(z)^{-1}$$

Diagonalizing element

Polynomials $\quad Q_{i,\dots,j}^-(z)$

form extended QQ-system

$$v(z)^{-1} = \begin{pmatrix} \dfrac{1}{Q_1^+(z)} & \dfrac{Q_1^-(z)}{Q_2^+(z)} & \dfrac{Q_{12}^-(z)}{Q_3^+(z)} & \cdots & \dfrac{Q_{1,\dots,r-1}^-(z)}{Q_r^+(z)} & Q_{1,\dots,r}^-(z) \\[2mm] 0 & \dfrac{Q_1^+(z)}{Q_2^+(z)} & \dfrac{Q_2^-(z)}{Q_3^+(z)} & \cdots & \dfrac{Q_{2,\dots,r-1}^-(z)}{Q_r^+(z)} & Q_{2,\dots,r}^-(z) \\[2mm] 0 & 0 & \dfrac{Q_2^+(z)}{Q_3^+(z)} & \cdots & \dfrac{Q_{3,\dots,r-1}^-(z)}{Q_r^+(z)} & Q_{3,\dots,r}^-(z) \\[2mm] \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\[2mm] 0 & \cdots & \cdots & \cdots & \dfrac{Q_{r-1}^+(z)}{Q_r^+(z)} & Q_r^-(z) \\[2mm] 0 & \cdots & \cdots & \cdots & 0 & Q_r^+(z) \end{pmatrix}$$
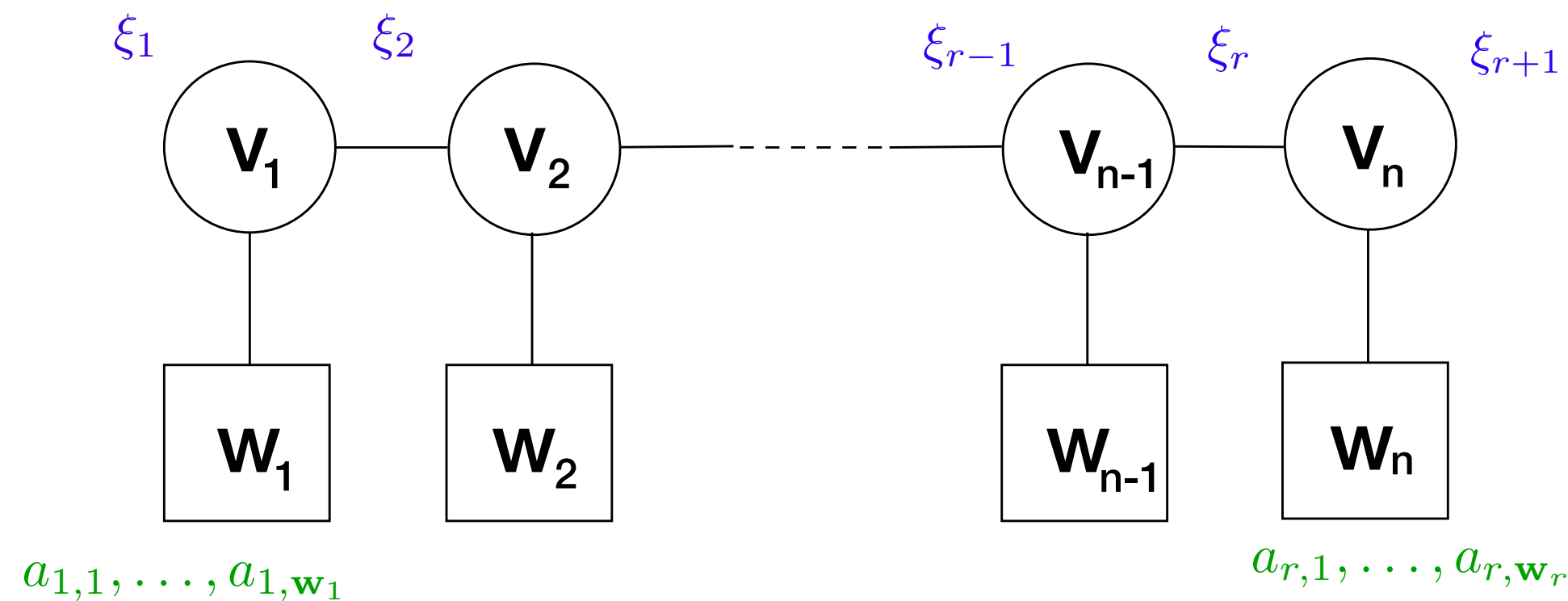
Quantum Wronskians

(SL(N),q)-oper can also be constructed from flag of subbundles $(E, A, \mathcal{L}_\bullet)$ such that the induced maps $\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i-1} \longrightarrow \mathcal{L}^q_{i+1}/\mathcal{L}^q_i$ are isomorphisms

The quantum determinants

$$\mathcal{D}_k(s) = e_1 \wedge \cdots \wedge e_{r+1-k} \wedge Z^{k-1}s(z) \wedge Z^{k-2}s(qz) \wedge \cdots \wedge Zs(q^{k-2}) \wedge s(q^{k-1}z)$$

vanish at q-oper singularities

$$W_k(s) = P_1(z) \cdot P_2(q^2z) \cdots P_k(q^{k-1}z), \qquad P_i(z) = \Lambda_r \Lambda_{r-1} \cdots \Lambda_{r-i+1}(z)$$

Diagonalizing condition

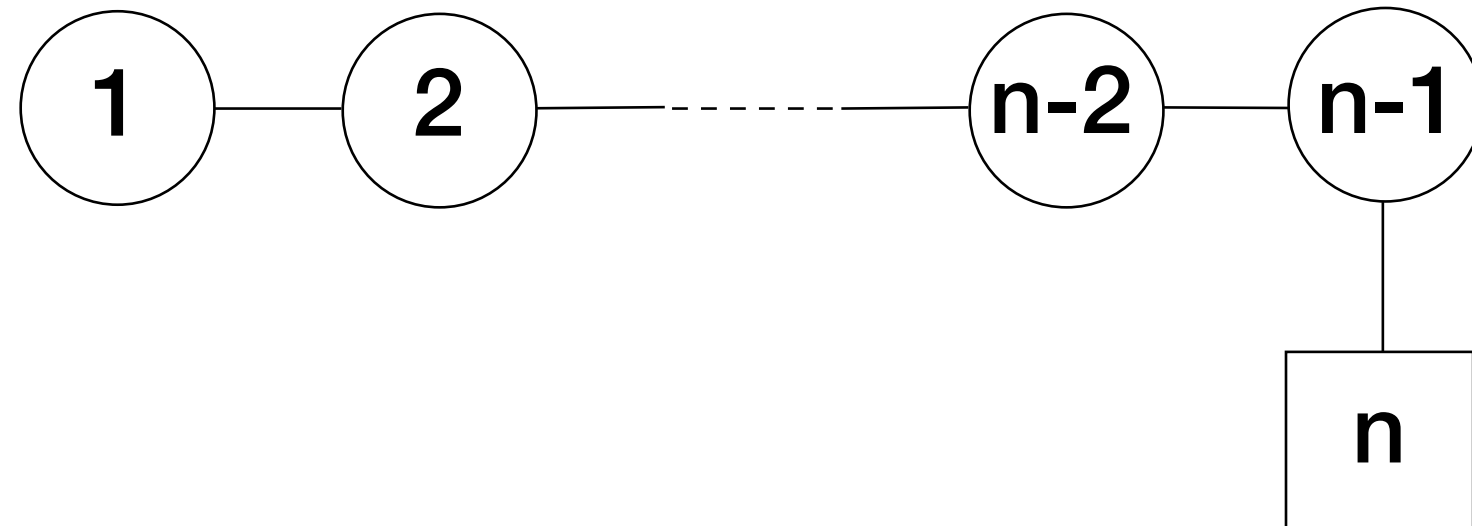$$\det_{i,j} \left[\xi^{k-j}_{r+1-k+i} s_{r+1-k+i}(q^{j-1}z) \right] = \alpha_k W_k \mathcal{V}_k$$

Components of the section of the
line subbundle are the Q-polynomials!

$$s_{r+1}(z) = Q^+_r(z), \qquad s_r(z) = Q^-_r(z), \qquad s_k(z) = Q^-_{k,\ldots,r}(z)$$

Quantum/Classical Duality

Consider T*G/B

Construct the corresponding space of (SL(N),h)-opers

Specify components of the section of L...

Then the space of functions on the space of such h-opers

is described by trigonometric Ruijsenaars-Schneider model with n particles
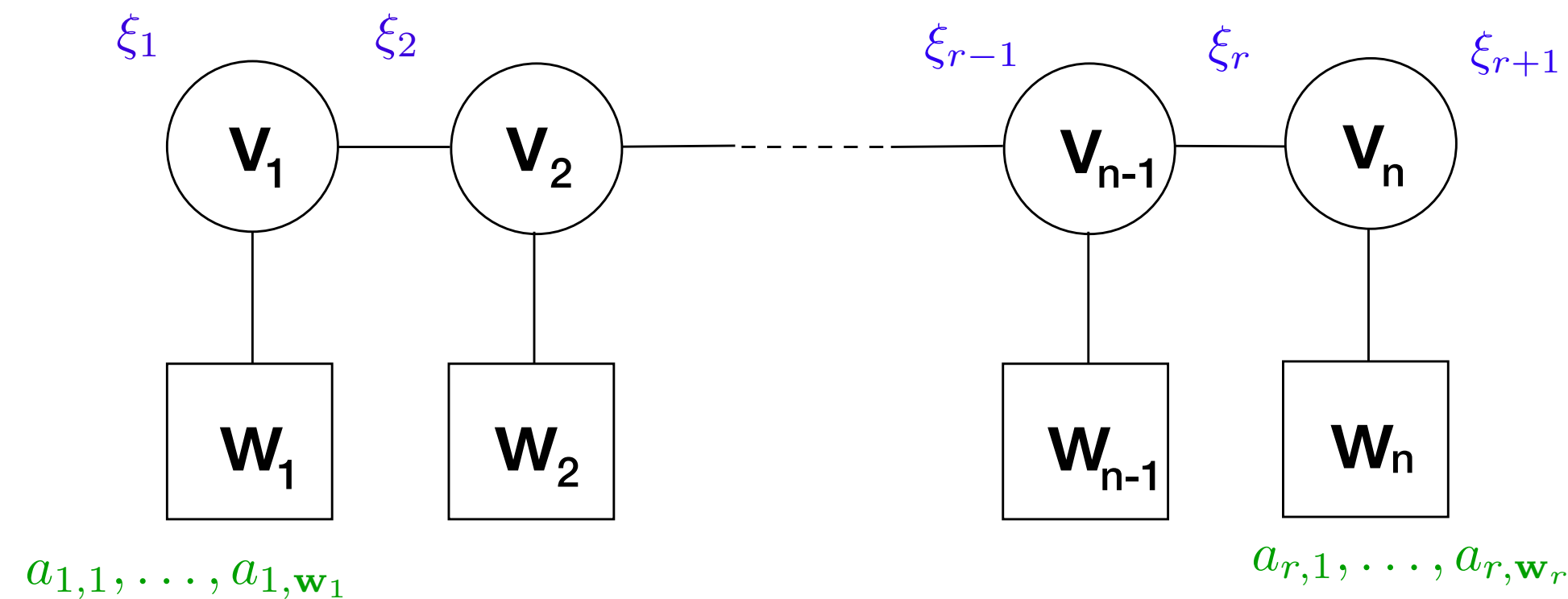
Quantum Wronskians

(SL(N),q)-oper can also be constructed from flag of subbundles $(E, A, \mathcal{L}_\bullet)$ such that the induced maps $\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i-1} \longrightarrow \mathcal{L}_{i+1}^q/\mathcal{L}_i^q$ are isomorphisms

The quantum determinants
$$\mathcal{D}_k(s) = e_1 \wedge \cdots \wedge e_{r+1-k} \wedge Z^{k-1}s(z) \wedge Z^{k-2}s(qz) \wedge \cdots \wedge Zs(q^{k-2}) \wedge s(q^{k-1}z)$$

vanish at q-oper singularities
$$W_k(s) = P_1(z) \cdot P_2(q^2z) \cdots P_k(q^{k-1}z), \qquad P_i(z) = \Lambda_r \Lambda_{r-1} \cdots \Lambda_{r-i+1}(z)$$

Diagonalizing condition
$$\det_{i,j} \left[\xi_{r+1-k+i}^{k-j} s_{r+1-k+i}(q^{j-1}z) \right] = \alpha_k W_k \mathcal{V}_k$$

Components of the section of the line subbundle are the Q-polynomials!

$$s_{r+1}(z) = Q_r^+(z), \qquad s_r(z) = Q_r^-(z), \qquad s_k(z) = Q_{k,\ldots,r}^-(z)$$

III. **Generalized Wronskians**

Consider big cell in Bruhat decomposition

$$G_0 = N_- H N_+$$

$$g = n_- \, h \, n_+$$

V_i^+ irrep of G with highest weight ω_i

$$h\nu_{\omega_i}^+ = [h]^{\omega_i} \nu_{\omega_i}^+$$

Define **principal minors** for group element g

For SL(N) they are standard minors of matrices

$$\Delta^{\omega_i}(g) = [h]^{\omega_i}, \quad i = 1, \ldots, r$$

Then **generalized minors** are regular functions on G

$$\Delta_{u\omega_i, v\omega_i}(g) = \Delta^{\omega_i}(\tilde{u}^{-1} g \tilde{v})$$

$u, v \in W_G$

Proposition *Action of the group element on the highest weight vector in*

$$g \cdot \nu_{\omega_i}^+ = \sum_{w \in W} \Delta_{w \cdot \omega_i, \omega_i}(g) \tilde{w} \cdot \nu_{\omega_i}^+ + \ldots,$$

where dots stand for the vectors, which do not belong to the orbit $\mathcal{O}_W$.

Generalized Minors and QQ-system

The set of generalized minors $\{\Delta_{w \cdot \omega_i, \omega_i}\}_{w \in W; i=1,\dots,r}$ creates a set of coordinates on G/B^+, known as *generalized Plücker coordinates*. In particular, the set of zeroes of each of $\Delta_{w \cdot \omega_i, \omega_i}$ is a uniquely and unambiguously defined hypersurface in G/B.

Proposition *For a W-generic Z-twisted Miura-Plücker (G, q)-oper with q-connection $A(z) = v(qz)Zv(z)^{-1}$, where $v(z) \in B_-(z)$ we have the following relation:*

$$\Delta_{w \cdot \omega_i, \omega_i}(v^{-1}(z)) = Q_+^{w,i}(z)$$

for any $w \in W$.

Proof: Since $\Delta^{\omega_i}(v^{-1}(z)) = Q_+^i(z)$ Diagonalizing gauge transformation $v^{-1}(z) = \prod_{i=1}^{r} e^{\frac{Q_-^i(z)}{Q_+^i(z)} f_i} \prod_{i=1}^{r} \left[Q_+^i(z) \right]^{\check{\alpha}_i} \dots$

$$v^{-1}(z) \nu_{\omega_i}^+ = Q_+^i(z) \nu_{\omega_i}^+ + Q_-^i(z) f_i \nu_{\omega_i}^+ + \dots$$

Fundamental Relation for Generalized Minors

[Fomin Zelevinsky]

Proposition 4.8. *Let, $u, v \in W$, such that for $i \in \{1, \ldots, r\}$, $\ell(uw_i) = \ell(u) + 1$, $\ell(vw_i) = \ell(v) + 1$. Then*

$$(4.7) \qquad \Delta_{u \cdot \omega_i, v \cdot \omega_i} \Delta_{uw_i \cdot \omega_i, vw_i \cdot \omega_i} - \Delta_{uw_i \cdot \omega_i, v \cdot \omega_i} \Delta_{u \cdot \omega_i, vw_i \cdot \omega_i} = \prod_{j \neq i} \Delta_{u \cdot \omega_j, v \cdot \omega_j}^{-a_{ji}},$$

Can we make sense of this relation using our approach of q-Opers?

Generalized Wronskians

The approach is similar to Miura-Plucker q-Opers

Let $\nu_{\omega_i}^+$ be a generator of the line $L_i^+ \subset V_i^+$ $\qquad\qquad V_i^+$ irrep of G with highest weight ω_i

The subspace $L_{c,i}^+$ of V_i of weight $c^{-1} \cdot \omega_i$ is one-dimensional and is spanned by $s^{-1} \nu_{\omega_i}^+$

Associated vector bundle $\qquad \mathcal{V}_i^+ = \mathcal{F}_{B_+} \underset{B_+}{\times} V_i^+ = \mathcal{F}_G \underset{G}{\times} V_i^+$

Contains line subbundles $\qquad \mathcal{L}_i^+ = \mathcal{F}_H \underset{H}{\times} L_i^+, \quad \mathcal{L}_{c,i}^+ = \mathcal{F}_H \underset{H}{\times} L_{c,i}^+$

Define **generalized Wronskian** on $\mathbb{P}^1$ as quadruple $(\mathcal{F}_G, \mathcal{F}_{B_+}, \mathscr{G}, Z)$

$\mathscr{G}$ is a meromorphic section of a principle bundle $\mathcal{F}_G$

s.t. for sections $\{v_i^+, v_{c,i}^+\}_{i=1,\ldots,r}$ of line bundles $\{\mathcal{L}_i^+, \mathcal{L}_{c,i}^+\}_{i=1,\ldots,r}$ on $U \cap M_q^{-1}(U)$

$$\mathscr{G}^q \cdot v_i^+ = Z \cdot \mathscr{G} \cdot v_{c,i}^+$$

Adding Singularities

Effectively the above definition means that the Wronskian, written as an element of G(z), satisfies

$$Z^{-1}\mathscr{G}(qz)\ \nu^+_{\omega_i} = \mathscr{G}(z) \cdot s_\phi(z)^{-1} \cdot \nu^+_{\omega_i} \qquad\qquad s_\phi(z) = \prod_i \phi_i^{-\check{\alpha}_i} s_i$$

Define **generalized Wronskian with regular singularities** if

$$s_\Lambda(z)^{-1} = \prod_i^{\mathrm{inv}} s_i \Lambda_i^{\check{\alpha}_i}$$

Fomin-Zelevinsky relations then read

$$\Delta_{\omega_i,\omega_i}\Delta_{w_i\cdot\omega_i,c^{-1}\cdot\omega_i} - \Delta_{w_i\cdot\omega_i,\omega_i}\Delta_{\omega_i,c^{-1}\cdot\omega_i}$$
$$= \prod_{j<i=i_l} \Delta_{\omega_j,c^{-1}\cdot\omega_j}^{-a_{ji}} \prod_{j>i=i_l} \Delta_{\omega_j,\omega_j}^{-a_{ji}}, \qquad i = 1,\ldots,r,$$

q-Opers and q-Wronskians

Nondegenerate generalized q-Wronskians with regular singularities $\{\acute{\Lambda_i}\}_{i=1,\ldots,r}$ ↔ Nondegenerate Z-twisted Miura (G,q)-opers with regular singularities $\{\acute{\Lambda_i}\}_{i=1,\ldots,r}$

Theorem 2: *For a given Z-twisted (G,q)-Miura oper, there exists a unique generalized q-Wronskian*

$$\mathscr{W}(z) \in B_-(z)w_0 B_-(z) \cap B_+(z)w_0 B_+(z) \subset G(z),$$

satisfying the system of equations

$$\mathscr{W}(q^{k+1}z)\nu^+_{\omega_i} = Z^k \mathscr{W}(z)s^{-1}(z)s^{-1}(qz)\ldots s^{-1}(q^k z)\nu^+_{\omega_i},$$

(4.32)
$$i = 1, \ldots, r, \qquad k = 0, 1, \ldots, h-1,$$

where h is the Coxeter number of G.

Examples: SL(2)

$$\mathscr{W}(qz)\nu_\omega^+ = Z\mathscr{W}(z)s^{-1}(z)\nu_\omega^+$$

$$s^{-1}(z) = \tilde{s}^{-1}\Lambda(z)^{\check{\alpha}} = \begin{pmatrix} 0 & \Lambda(z)^{-1} \\ \Lambda(z) & 0 \end{pmatrix}, \qquad \nu_\omega^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

In terms of Q-polynomials
$$\mathscr{W}(z) = \begin{pmatrix} Q^+(z) & \zeta^{-1}\Lambda(z)^{-1}Q_+(qz) \\ Q^-(z) & \zeta\Lambda(z)^{-1}Q^-(qz) \end{pmatrix}$$

$$\zeta Q^+(z)Q^-(qz) - \zeta^{-1}Q^+(qz)Q^-(z) = \Lambda(z)$$

is equivalent to $\det\mathscr{W}(z) = 1$.

Examples SL(N)

$$\mathscr{W}(z) = \left(\Delta_{\mathbf{w}\omega,\omega} \middle| \Delta_{\mathbf{w}\omega,s^{-1}\omega} \middle| \ldots \middle| \Delta_{\mathbf{w}\omega,s^{r+1}\omega} \right) (\mathscr{G}(z))$$

Lift for standard ordering along the Dynkin diagram

$$s_{\Lambda}^{-1}(z) = \tilde{s}^{-1} \prod_i \Lambda_i^{d_i}$$

$$d_i = \sum_{j=1}^{i} \check{\alpha}_j$$

$$\tilde{s}^{-1} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ldots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

$$\mathscr{W}(z) = \left(Q^{\mathbf{w}\cdot\omega}(z) \middle| Z F_1(z) Q^{\mathbf{w}\cdot\omega}(qz) \middle| \ldots \middle| Z^{r-1} F_{r-1}(q^{r-1}z) Q^{\mathbf{w}\cdot\omega}(q^{r-1}z) \right)$$

where $F_i(z) = \prod_{j=1}^{i} \Lambda_j(z)^{-1}$.

Lewis Carroll Identity

In Type A FZ relation reduces to

$$\Delta_{u\omega_i,v\omega_i}\Delta_{us_i\omega_i,vs_i\omega_i} - \Delta_{us_i\omega_i,v\omega_i}\Delta_{u\omega_i,vs_i\omega_i} = \Delta_{u\omega_{i-1},v\omega_{i-1}}\Delta_{u\omega_{i+1},v\omega_{i+1}}$$

$$M_1^1 M_i^2 - M_i^1 M_1^2 = M_{1i}^{12} M$$