

q-Operators — what they are and what are they good for?

Peter Koroteev

Talk at M-Seminar at Kansas State University 03/31/2022

Literature

[arXiv:2108.04184]

q-Operators, QQ-systems, and Bethe Ansatz II: Generalized Minors

[P. Koroteev](#), [A. M. Zeitlin](#)

[arXiv:2105.00588]

3d Mirror Symmetry for Instanton Moduli Spaces

[P. Koroteev](#), [A. M. Zeitlin](#)

[arXiv:2007.11786] J. Inst. Math. Jussieu

Toroidal q-Operators

[P. Koroteev](#), [A. M. Zeitlin](#)

[arXiv:2002.07344] J. Europ. Math. Soc.

q-Operators, QQ-Systems, and Bethe Ansatz

[E. Frenkel](#), [P. Koroteev](#), [D. S. Sage](#), [A. M. Zeitlin](#)

[arXiv:1811.09937] Commun.Math.Phys. **381** (2021) 641

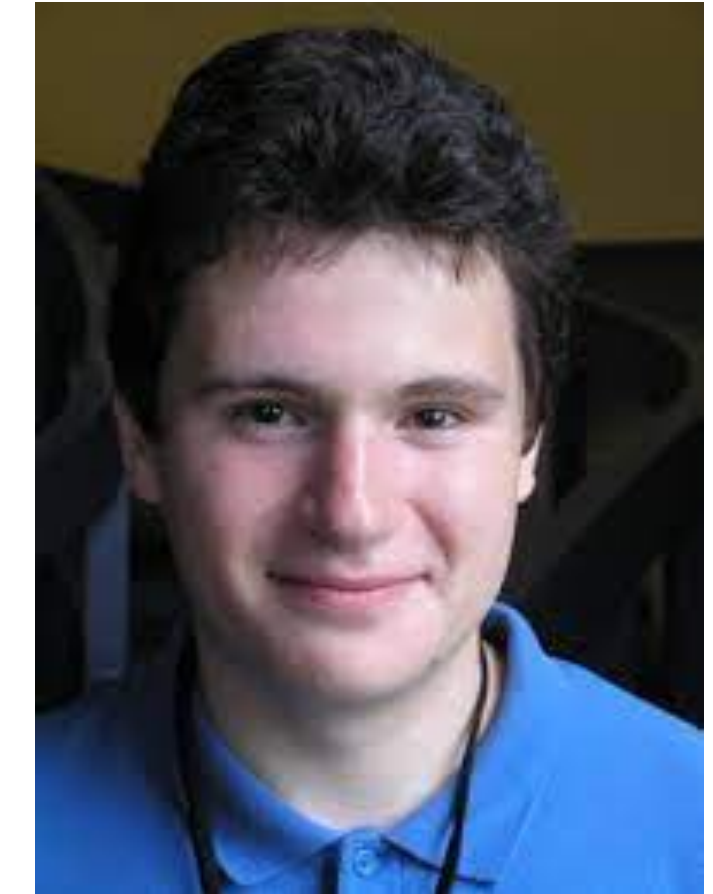
(SL(N),q)-opers, the q-Langlands correspondence, and quantum/classical duality

[P. Koroteev](#), [D. S. Sage](#), [A. M. Zeitlin](#)

[arXiv:1705.10419] Selecta Math. **27** (2021) 87

Quantum K-theory of Quiver Varieties and Many-Body Systems

[P. Koroteev](#), [P. P. Pushkar](#), [A. V. Smirnov](#), [A. M. Zeitlin](#)



Motivation

Quantum Geometry and Integrable Systems

[Okounkov et al]

[Pushkar, Zeitlin, Smirnov]

[PK, Pushkar, Smirnov, Zeitlin]

BPS/CFT Correspondence

[Nekrasov Shatashvili]

Geometric q-Langlands Correspondence

[Frenkel] [Aganagic, Frenkel, Okounkov]

ODE/IM Correspondence

[Bazhanov, Lukyanov, Zamolodchikov]

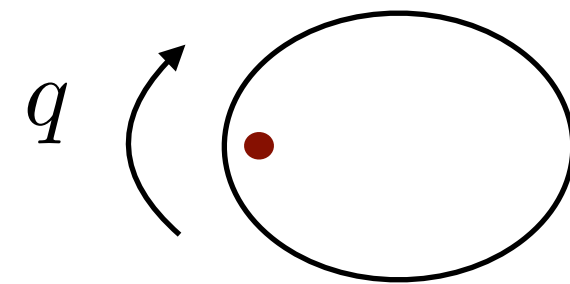
[Dorey, Tateo]

I. q-Oper — SL(2) Example

Consider vector bundle E over \mathbb{P}^1

$$M_q : \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

$$z \mapsto qz$$



Map of vector bundles

$$A : E \rightarrow E^q$$

Upon trivialization

$$A(z) \in \mathfrak{gl}(N, \mathbb{C}(z))$$

q-gauge transformation

$$A(z) \mapsto g(qz)A(z)g^{-1}(z)$$

Difference equation

$$D_q(s) = As.$$

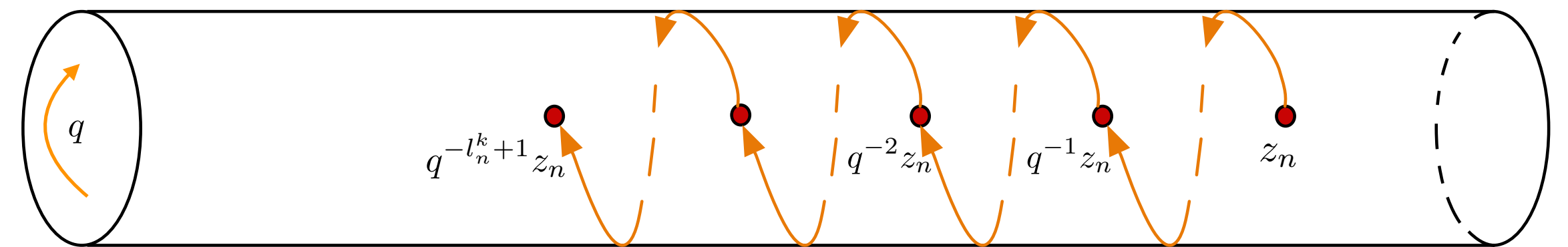
Definition: A meromorphic $(GL(N), q)$ -connection over \mathbb{P}^1 is a pair (E, A) , where E is a (trivializable) vector bundle of rank N over \mathbb{P}^1 and A is a meromorphic section of the sheaf $\text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(E, E^q)$ for which $A(z)$ is invertible, i.e. lies in $GL(N, \mathbb{C}(z))$. The pair (E, A) is called an $(SL(N), q)$ -connection if there exists a trivialization for which $A(z)$ has determinant 1.

q-Operators

Definition: A $(GL(2), q)$ -oper on \mathbb{P}^1 is a triple (E, A, \mathcal{L}) , where (E, A) is a $(GL(2), q)$ -connection and \mathcal{L} is a line subbundle such that the induced map $\bar{A} : \mathcal{L} \rightarrow (E/\mathcal{L})^q$ is an isomorphism. The triple is called an $(SL(2), q)$ -oper if (E, A) is an $(SL(2), q)$ -connection.

$$\text{in a trivialization } s(qz) \wedge A(z)s(z) \neq 0$$

Definition: A $(SL(2), q)$ -oper with regular singularities at the points $z_1, \dots, z_L \neq 0, \infty$ with weights k_1, \dots, k_L is a meromorphic $(SL(2), q)$ -oper (E, A, \mathcal{L}) for which \bar{A} is an isomorphism everywhere on $\mathbb{P}^1 \setminus \{0, \infty\}$ except at the points $z_m, q^{-1}z_m, q^{-2}z_m, \dots, q^{-k_m+1}z_m$ for $m \in \{1, \dots, L\}$, where it has simple zeros.



Finally, $(SL(2), q)$ -oper is **Z-twisted** in $A(z)$ is gauge equivalent to a diagonal matrix Z

$$Z = g(qz)A(z)g(z)^{-1}$$

Miura q-Operators

Miura (SL(2),q)-oper is a quadruple $(E, A, \mathcal{L}, \hat{\mathcal{L}})$ where (E, A, \mathcal{L}) is an (SL(2),q)-oper and $\hat{\mathcal{L}}$ is preserved by the q-connection A

Chose trivialization of \mathcal{L}

$$s(z) = \begin{pmatrix} Q_+(z) \\ Q_-(z) \end{pmatrix} \quad \text{Twist element} \quad Z = \text{diag}(\zeta, \zeta^{-1})$$

q-Oper condition — SL(2) **QQ-system**

$$\zeta Q_-(z)Q_+(zq) - \zeta^{-1}Q_-(zq)Q_+(z) = \Lambda(z)$$

singularities

$$\Lambda(z) = \prod_{p=1}^L \prod_{j_p=0}^{r_p-1} (z - q^{-j_p} z_p)$$

One of the polynomials can be made monic

$$Q_+(z) = \prod_{k=1}^m (z - w_k)$$

From QQ-system to Bethe equations

$$\frac{\Lambda(w_k)}{\Lambda(q^{-1}w_k)} = -\zeta^2 \frac{Q_+(qw_k)}{Q_+(q^{-1}w_k)}, \quad k = 1, \dots, m.$$

$$q^r \prod_{p=1}^L \frac{w_k - q^{1-r_p} z_p}{w_k - q z_p} = -\zeta^2 q^m \prod_{j=1}^m \frac{qw_k - w_j}{w_k - qw_j}, \quad k = 1, \dots, m$$

q-Miura Transformation

$$A(z) = \begin{pmatrix} g(z) & \Lambda(z) \\ 0 & g(z)^{-1} \end{pmatrix}$$

Z-twisted q-oper condition

$$A(z) = v(zq)Zv(z)^{-1}, \quad Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

Gauge transformation reads

$$v(z) = \begin{pmatrix} y(z) & 0 \\ 0 & y(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\frac{Q_-(z)}{Q_+(z)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y(z) & -y(z)\frac{Q_-(z)}{Q_+(z)} \\ 0 & y(z)^{-1} \end{pmatrix}$$

We find

$$g(z) = \zeta_i y(zq)y(z)^{-1}$$

$$\Lambda(z) = y(z)y(zq) \left(\zeta \frac{Q_-(z)}{Q_+(z)} - \zeta^{-1} \frac{Q_-(zq)}{Q_+(zq)} \right)$$

The q-oper condition becomes the **SL(2) QQ-system**

$$\zeta Q_-(z)Q_+(zq) - \zeta^{-1} Q_-(zq)Q_+(z) = \Lambda(z)$$

Difference Equation

$$D_q(s) = As.$$

$$D_q(s_1) = \Lambda(z)s_2$$

after elimination

$$\left(D_q^2 - T(qz)D_q - \frac{\Lambda(qz)}{\Lambda(z)} \right) s_1 = 0$$

Trigonometric Ruijsenaars Hamiltonians

Recover 2-body tRS Hamiltonian from a q-Oper

Let $Q_- = z - p_-$ and $Q_+ = c(z - p_+)$

$$z^2 - \frac{z}{q} \left[\frac{\zeta - q\zeta^{-1}}{\zeta - \zeta^{-1}} p_+ + \frac{q\zeta - \zeta^{-1}}{\zeta - \zeta^{-1}} p_- \right] + \frac{p_+ p_-}{q} = (z - z_+)(z - z_-)$$

qOper condition yields
tRS Hamiltonians!

T_1

T_2

$$\det(z - L_{tRS}) = (z - z_+)(z - z_-)$$

Calogero-Moser Space

Let V be an N -dimensional vector space over \mathbb{C} . Let \mathcal{M}' be the subset of $GL(V) \times GL(V) \times V \times V^*$ consisting of elements (M, T, u, v) such that

$$qMT - TM = u \otimes v^T$$

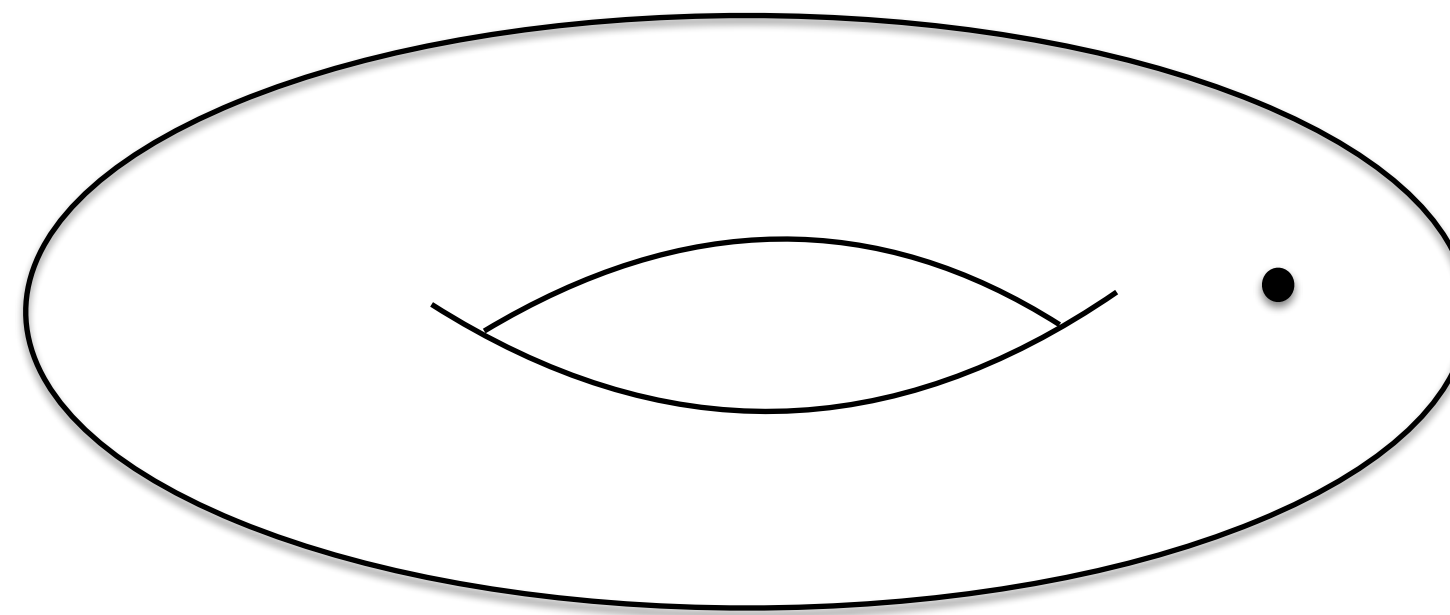
The group $GL(N; \mathbb{C}) = GL(V)$ acts on \mathcal{M}' by conjugation

$$(M, T, u, v) \mapsto (gMg^{-1}, gTg^{-1}, gu, vg^{-1})$$

The quotient of \mathcal{M}' by the action of $GL(V)$ is called **Calogero-Moser space** \mathcal{M}

Also can be understood as moduli space of flat connections on punctured torus

Integrable Hamiltonians are $\sim \text{Tr} T^k$



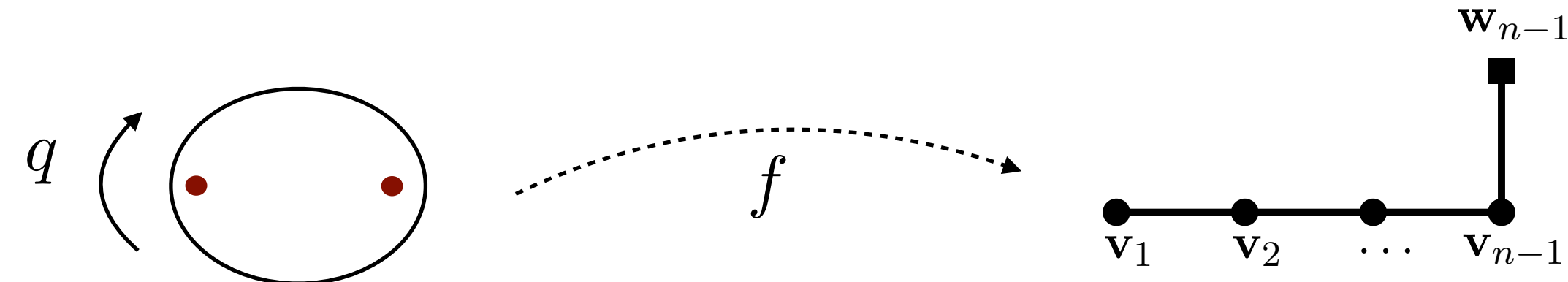
$$\mathcal{M}_n = \{A, B, C\} / GL(n; \mathbb{C})$$

$$ABA^{-1}B^{-1} = C$$

$$C = \text{diag}(q, \dots, q, q^{n-1})$$

Enumerative AG/Integrable Systems

Quantum equivariant K-theory of Nakajima quiver varieties



$$A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$$

$$V^{(\tau)}(\mathbf{z}) = \sum_d \text{ev}_{p_2, *} (\widehat{\mathcal{O}}_{\text{vir}}^d \otimes \tau|_{p_1}, \text{QM}_{\text{nonsing } p_2}^d) \mathbf{z}^d \in K_{\mathbb{T} \times \mathbb{C}_q^\times}(X)_{\text{loc}}[[\mathbf{z}]]$$

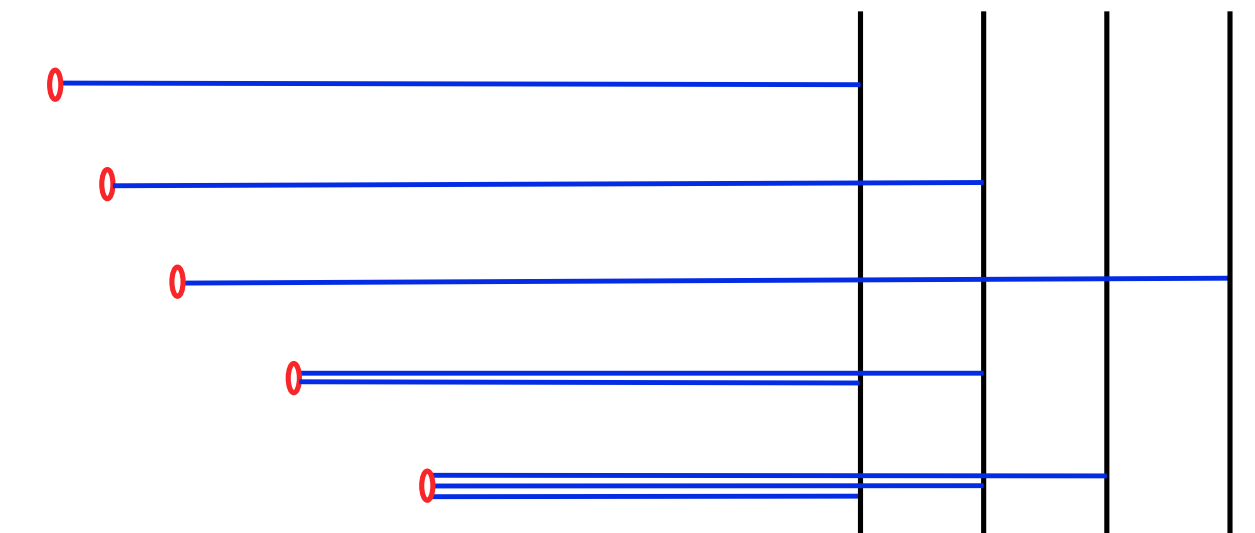
Saddle point limit yields Bethe equations for XXZ

Quantum classes satisfy interesting difference equations in equivariant parameters and Kahler parameters **qKZ, Dynamical equation** [Okounkov, Smirnov]

After symmetrization they can be rewritten as eigenvalue equations for **trigonometric Ruijsenaars-Schneider (tRS)** system [PK, Zeitlin] [PK]

$$T_r(\mathbf{a}) = \sum_{\substack{\mathcal{J} \subset \{1, \dots, n\} \\ |\mathcal{J}|=r}} \prod_{\substack{i \in \mathcal{J} \\ j \notin \mathcal{J}}} \frac{t a_i - a_j}{a_i - a_j} \prod_{i \in \mathcal{J}} p_i$$

$$T_r(\mathbf{a})V(\mathbf{a}, \vec{\zeta}) = S_r(\vec{\zeta}, t)V(\mathbf{a}, \vec{\zeta})$$

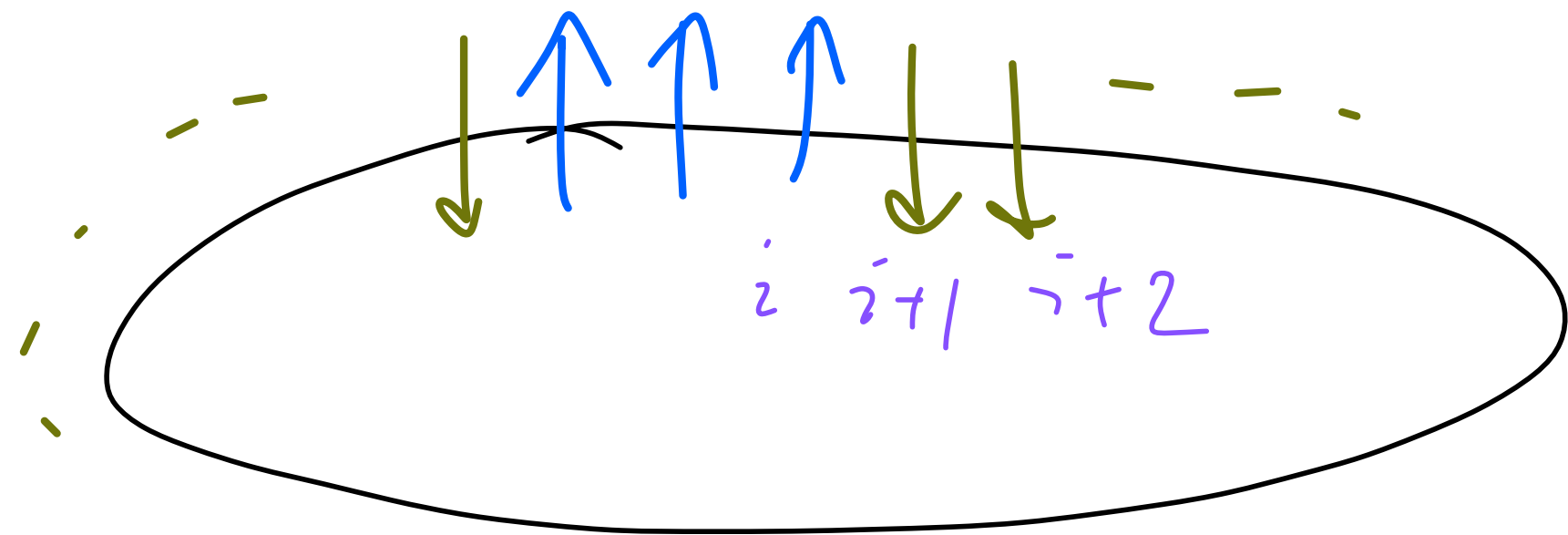


In terms of string/gauge theory tRS eigenproblem is Ward identity

[Gaiotto, PK] [Bullimore, Kim, PK]

Quantum

QQ-Systems



SU(**n**) XXZ spin chain on n sites w/ **anisotropies** and **twisted periodic boundary conditions**

Planck's constant \hbar

twist eigenvalues z_i

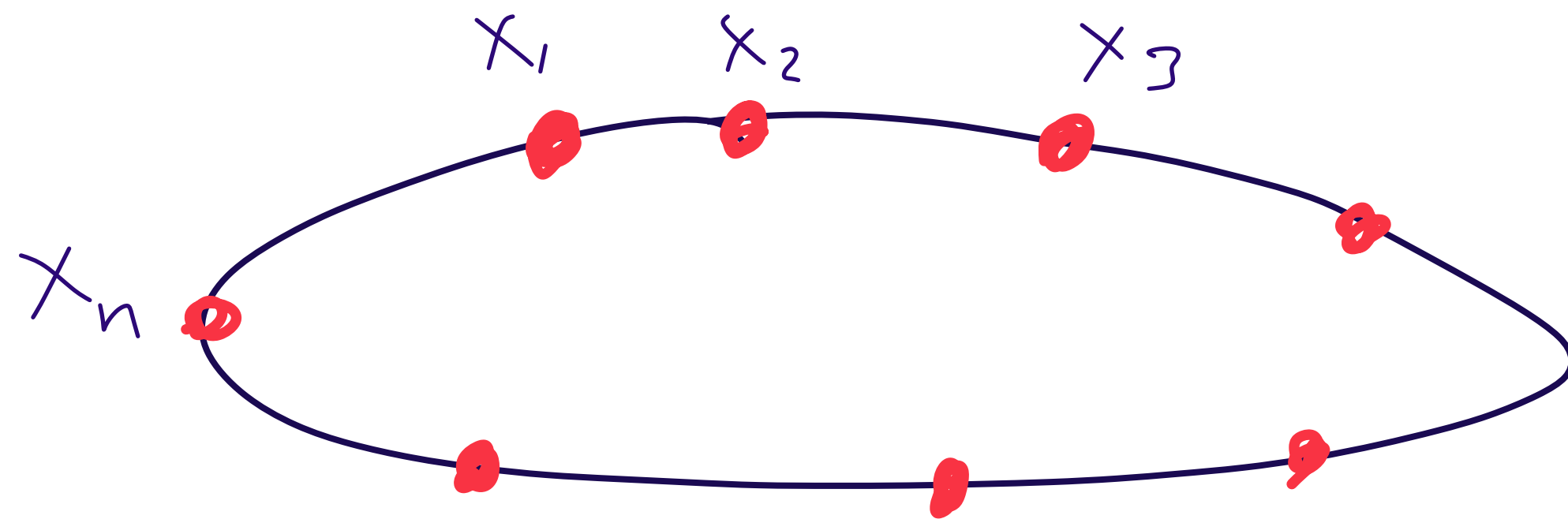
equivariant parameters (anisotropies) a_i

Bethe Ansatz Equations: $\frac{\partial Y}{\partial \sigma_i} = 0$

$$\frac{\zeta_i}{\zeta_{i+1}} \prod_{\beta=1}^{v_{i-1}} \frac{\sigma_{i,\alpha} - \hbar^{1/2} \sigma_{i-1,\beta}}{\sigma_{i-1,\beta} - \hbar^{1/2} \sigma_{i,\alpha}} \cdot \prod_{\beta \neq \alpha}^{v_i} \frac{\hbar \sigma_{i,\alpha} - \sigma_{i,\beta}}{\hbar \sigma_{i,\beta} - \sigma_{i,\alpha}} \cdot \prod_{\beta=1}^{v_{i+1}} \frac{\sigma_{i,\alpha} - \hbar^{1/2} \sigma_{i+1,\beta}}{\sigma_{i+1,\beta} - \hbar^{1/2} \sigma_{i,\alpha}} = (-1)^{\delta_i}$$

Classical

q-Operators



n-particle trigonometric Ruijsenaars-Schneider model

$$\Omega = \sum_i \frac{dp_i}{p_i} \wedge \frac{dz_i}{z_i}$$

$$[T_i, T_j] = 0$$

Coupling constant \hbar

$$T_1 = \sum_{i=1}^n \prod_{j \neq i} \frac{\hbar z_i - z_j}{z_i - z_j} p_i$$

coordinates z_i

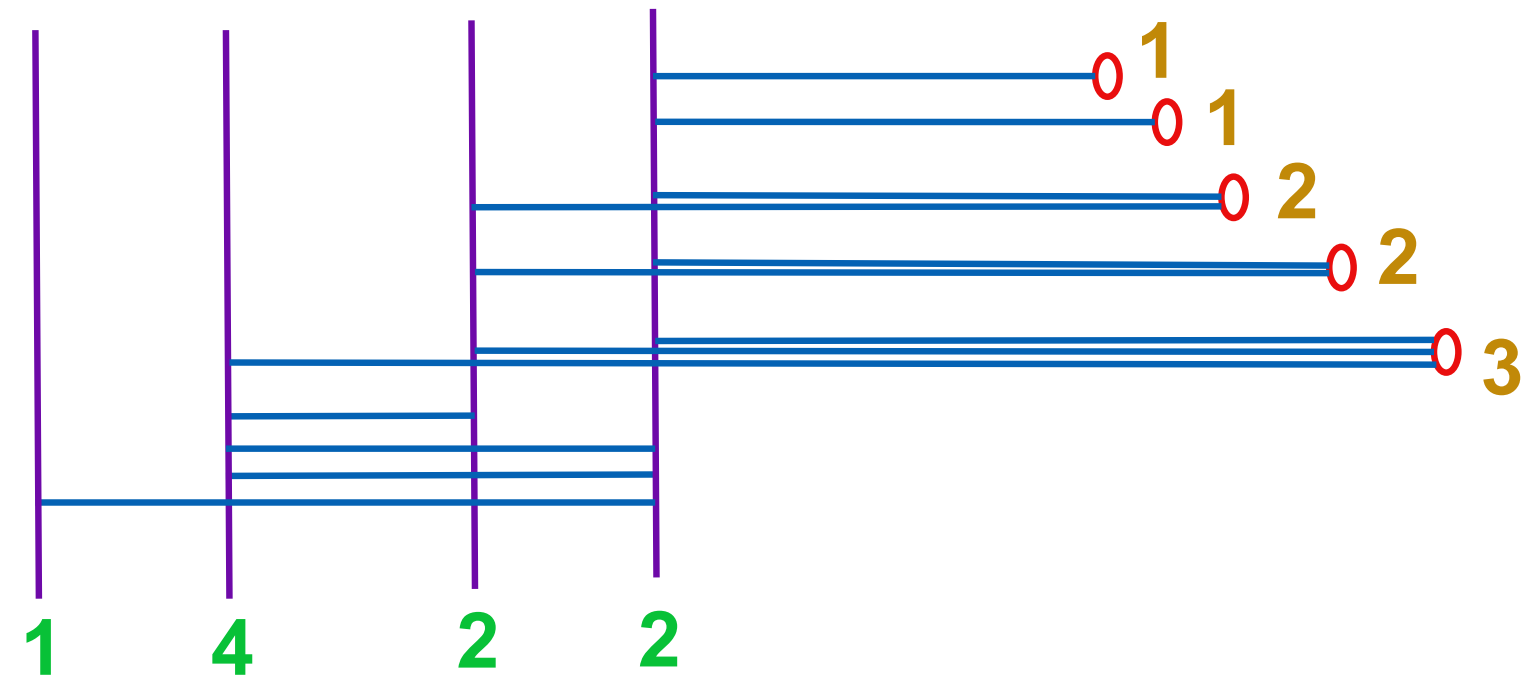
energy (eigenvalues of Hamiltonians) $e_i(a_i)$

Energy level equations

$$T_i(\mathbf{z}, \hbar) = e_i(\mathbf{a}), \quad i = 1, \dots, n$$

Quantum/Classical Duality

[PK Gaiotto]
[PK Zeitlin]



Symplectic form

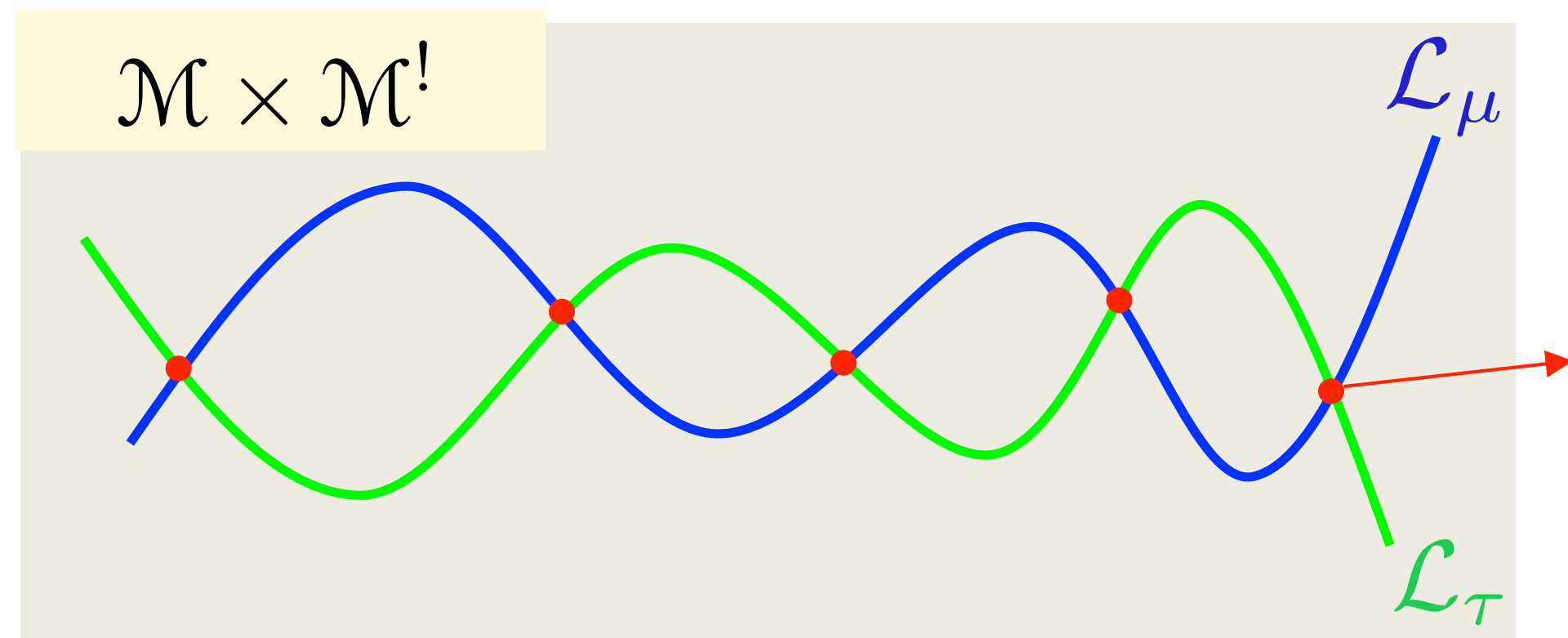
$$\Omega = \sum_{i=1}^N \frac{dp_i^\xi}{p_i^\xi} \wedge \frac{d\xi_i}{\xi_i} - \frac{dp_i^a}{p_i^a} \wedge \frac{da_i}{a_i}$$

tRS momenta

$$p_i^\xi = \exp \frac{\partial Y}{\partial \xi_i}, \quad p_i^a = \exp \frac{\partial Y}{\partial a_i}$$

tRS energy relations

$$\det(u - T) = \prod_{i=1}^N (u - a_i), \quad \det(u - M) = \prod_{i=1}^N (u - \xi_i)$$



$$Y = Y'$$

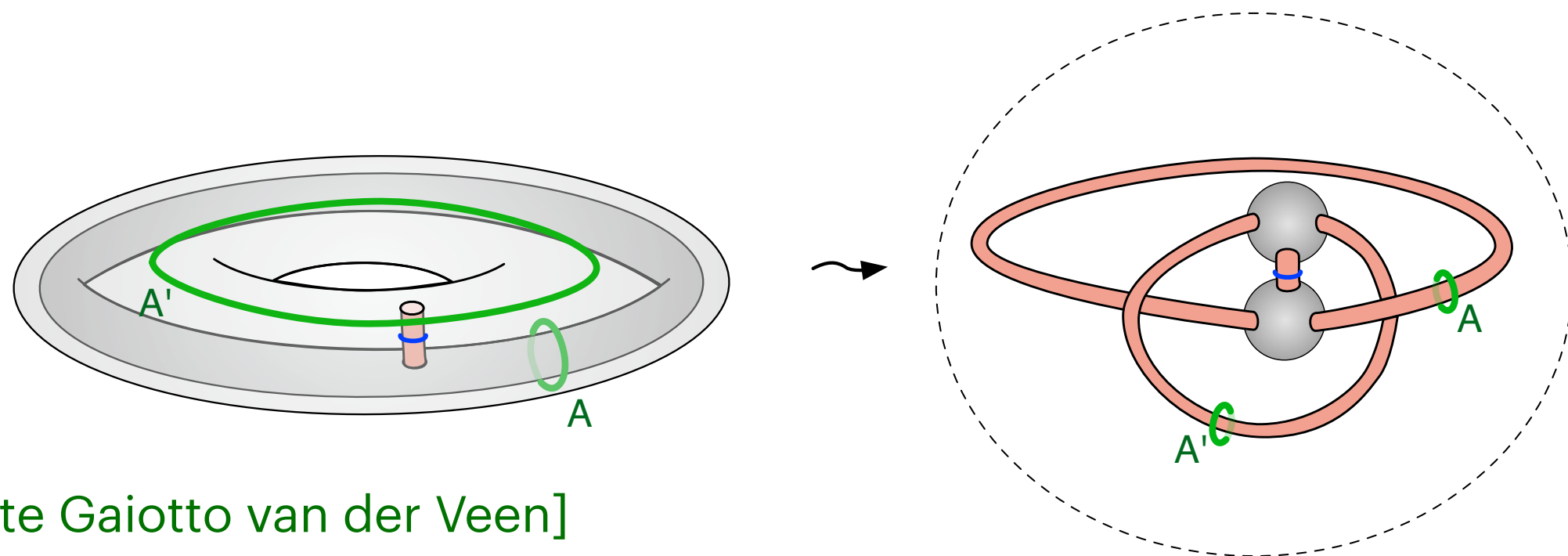
3d mirror symmetry

$$\sum_{\substack{\mathcal{J} \subset \{1, \dots, L\} \\ |\mathcal{J}|=k}} \prod_{\substack{i \in \mathcal{J} \\ j \notin \mathcal{J}}} \frac{a_i - \hbar a_j}{a_i - a_j} \prod_{m \in \mathcal{J}} p_m = \ell_k(\xi_i)$$

\mathcal{L}_μ Eigenvalues of M and Slodowy form on T

\mathcal{L}_τ Eigenvalues of T and Slodowy form on M

Solutions of Bethe equations — intersection points



[Dimofte Gaiotto van der Veen]

II. (\mathbf{G}, \mathbf{q}) -Connection

G -simple simply-connected complex Lie group

Principal G -bundle \mathcal{F}_G over \mathbb{P}^1

$$M_q : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \\ z \mapsto qz$$

A meromorphic **(\mathbf{G}, \mathbf{q}) -connection** on \mathcal{F}_G is a section A of $\text{Hom}_{\mathcal{O}_U}(\mathcal{F}_G, \mathcal{F}_G^q)$

U -Zariski open dense set

Choose U so that the restriction $\mathcal{F}_G|_U$ of \mathcal{F}_G to U is isomorphic to a trivial G -bundle

$$A(z) \in G(\mathbb{C}(z)) \quad \text{on} \quad U \cap M_q^{-1}(U)$$

Change of trivialization $A(z) \mapsto g(qz)A(z)g(z)^{-1}$

(G,q)-Oper

A meromorphic (G,q)-oper on \mathbb{P}^1 is a triple $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$

A is a meromorphic (G, q) -connection

\mathcal{F}_{B_-} is a reduction of \mathcal{F}_G to B_-

Oper condition: Restriction of the connection on some Zariski open dense set U

$$A : \mathcal{F}_G \longrightarrow \mathcal{F}_G^q \text{ to } U \cap M_q^{-1}(U)$$

takes values in the double Bruhat cell

$$B_-(\mathbb{C}[U \cap M_q^{-1}(U)])cB_-(\mathbb{C}[U \cap M_q^{-1}(U)])$$

Coxeter element: $c = \prod_i s_i$

Locally

$$A(z) = n'(z) \prod_i (\phi_i(z)^{\check{\alpha}_i} s_i) n(z)$$

$\phi_i(z) \in \mathbb{C}(z)$ and $n(z), n'(z) \in N_-(z)$

Miura (G, q) -Operers

Definition: A *Miura (G, q) -oper* on \mathbb{P}^1 is a quadruple $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$, where $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ is a meromorphic (G, q) -oper on \mathbb{P}^1 and \mathcal{F}_{B_+} is a reduction of the G -bundle \mathcal{F}_G to B_+ that is preserved by the q -connection A .

It can be shown that the two flags \mathcal{F}_{B_-} and \mathcal{F}_{B_+} are in *generic relative position* for some dense set V

The fiber $\mathcal{F}_{G,x}$ of \mathcal{F}_G at x is a G -torsor with reductions $\mathcal{F}_{B_-,x}$ and $\mathcal{F}_{B_+,x}$ to B_- and B_+ , respectively. Choose any trivialization of $\mathcal{F}_{G,x}$, i.e. an isomorphism of G -torsors $\mathcal{F}_{G,x} \simeq G$. Under this isomorphism, $\mathcal{F}_{B_-,x}$ gets identified with $aB_- \subset G$ and $\mathcal{F}_{B_+,x}$ with bB_+ .

Then $a^{-1}b$ is a well-defined element of the double quotient $B_- \backslash G / B_+$, which is in bijection with W_G .

We will say that \mathcal{F}_{B_-} and \mathcal{F}_{B_+} have a *generic relative position* at $x \in X$ if the element of W_G assigned to them at x is equal to 1 (this means that the corresponding element $a^{-1}b$ belongs to the open dense Bruhat cell $B_- \cdot B_+ \subset G$).

Structure Theorems

Theorem 1: For any Miura (G, q) -oper on \mathbb{P}^1 , there exists a trivialization of the underlying G -bundle \mathcal{F}_G on an open dense subset of \mathbb{P}^1 for which the oper q -connection has the form

$$A(z) \in N_-(z) \prod_i ((\phi_i(z)^{\check{\alpha}_i} s_i) N_-(z) \cap B_+(z)).$$

Theorem 2: Let F be any field, and fix $\lambda_i \in F^\times, i = 1, \dots, r$. Then every element of the set $N_- \prod_i \lambda_i^{\check{\alpha}_i} s_i N_- \cap B_+$ can be written in the form

$$\prod_i g_i^{\check{\alpha}_i} e^{\frac{\lambda_i t_i}{g_i} e_i}, \quad g_i \in F^\times,$$

where each $t_i \in F^\times$ is determined by the lifting s_i .

Adding Singularities and Twists

Consider family of polynomials $\{\Lambda_i(z)\}_{i=1,\dots,r}$

(G,q)-oper with regular singularities can be written as

$$A(z) = n'(z) \prod_i (\Lambda_i(z)^{\check{\alpha}_i} s_i) n(z), \quad n(z), n'(z) \in N_-(z)$$

Using structure theorem every Miura (G,q)-oper with singularities reads

$$A(z) = \prod_i g_i(z)^{\check{\alpha}_i} e^{\frac{\Lambda_i(z)}{g_i(z)} e_i}, \quad g_i(z) \in \mathbb{C}(z)^\times$$

(G,q)-oper is Z-twisted if it is equivalent to a constant element of G $Z \in H \subset H(z)$ Z is regular semisimple. There are W_G

$$A(z) = g(qz) Z g(z)^{-1}$$

Miura (G,q)-opers for each (G,q)-opers

Z-twisted Miura (G,q)-oper if gauge transform is from Borel

$$A(z) = v(qz) Z v(z)^{-1}, \quad v(z) \in B_+(z)$$

Plucker Relations

V_i^+ irrep of G with highest weight ω_i Line $L_i \subset V_i$ stable under B_+

Plucker relations: for two integral dominant weights $L_{\lambda+\mu} \subset V_{\lambda+\mu}$ is the image of $L_\lambda \otimes L_\mu \subset V_\lambda \otimes V_\mu$
 under canonical projection $V_\lambda \otimes V_\mu \longrightarrow V_{\lambda+\mu}$

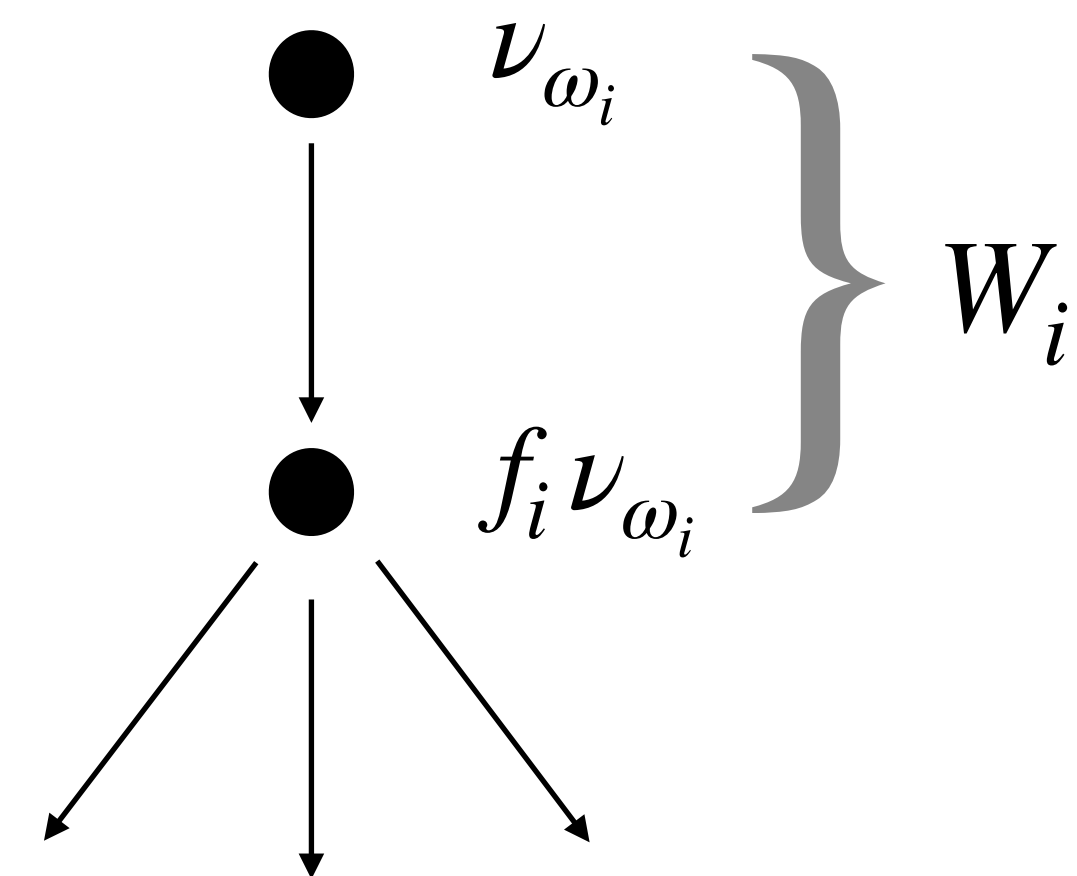
Conversely, for a collection of lines $L_\lambda \subset V_\lambda$ satisfying Plucker relations $\exists B \subset G$ such that L_λ is stabilized by B for all λ

A choice of B is equivalent to a choice of B_+ -torsor in G

Let ν_{ω_i} be a generator of the line $L_i \subset V_i$. This is a vector of weight ω_i wrt $H \subset B_+$

The subspace of V_i of weight $\omega_i - \alpha_i$ is one-dimensional and spanned $f_i \cdot \nu_{\omega_i}$

Thus the 2d subspace spanned by $\{\nu_{\omega_i}, f_i \cdot \nu_{\omega_i}\}$ is a B_+ -invariant subspace of V_i



Miura-Pucker (G,q)-Operators

let $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$ be a Miura (G, q) -oper with regular singularities $\{\Lambda_i(z)\}_{i=1, \dots, r}$

Associated vector bundle $\mathcal{V}_i = \mathcal{F}_{B_+} \times_{B_+} V_i = \mathcal{F}_G \times_G V_i$ contains rank-two subbundle $\mathcal{W}_i = \mathcal{F}_{B_+} \times_{B_+} W_i$

associated to $W_i \subset V_i$, and \mathcal{W}_i in turn contains a line subbundle $\mathcal{L}_i = \mathcal{F}_{B_+} \times_{B_+} L_i$

Using structure theorems we obtain r Miura $(GL(2), q)$ -operators

$$A_i(z) = \begin{pmatrix} g_i(z) & \Lambda_i(z) \prod_{j>i} g_j(z)^{-a_{ji}} \\ 0 & g_i^{-1}(z) \prod_{j \neq i} g_j(z)^{-a_{ji}} \end{pmatrix}$$

Z-twisted Miura-Plucker (G,q)-oper is meromorphic Miura (G, q) -oper on P^1 such that for each Miura $(GL(2), q)$ -oper

$$A_i(z) = v(zq) Z v(z)^{-1}|_{W_i} = v_i(zq) Z_i v_i(z)^{-1}$$

where $v_i(z) = v(z)|_{W_i}$ and $Z_i = Z|_{W_i}$

QQ-System

Theorem: *There is a one-to-one correspondence between the set of nondegenerate Z -twisted Miura-Plücker (G, q) -opers and the set of nondegenerate polynomial solutions of the QQ-system*

$$\tilde{\xi}_i Q_-^i(z) Q_+^i(qz) - \xi_i Q_-^i(qz) Q_+^i(z) = \Lambda_i(z) \prod_{j>i} [Q_+^j(qz)]^{-a_{ji}} \prod_{j<i} [Q_+^j(z)]^{-a_{ji}}, \quad i = 1, \dots, r,$$

$$\tilde{\xi}_i = \zeta_i \prod_{j>i} \zeta_j^{a_{ji}}, \quad \xi_i = \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}}$$

Proof uses

$$v(z) = \prod_{i=1}^r y_i(z)^{\check{\alpha}_i} \prod_{i=1}^r e^{-\frac{Q_-^i(z)}{Q_+^i(z)} e_i} \dots,$$

$$g_i(z) = \zeta_i \frac{Q_+^i(qz)}{Q_+^i(z)}.$$

XXZ Bethe Ansatz Equations for G

roots of Q+

$$\frac{Q_+^i(qw_i^k)}{Q_+^i(q^{-1}w_i^k)} \prod_j \zeta_j^{a_{ji}} = - \frac{\Lambda_i(w_k^i) \prod_{j>i} [Q_+^j(qw_k^i)]^{-a_{ji}} \prod_{j<i} [Q_+^j(w_k^i)]^{-a_{ji}}}{\Lambda_i(q^{-1}w_k^i) \prod_{j>i} [Q_+^j(w_k^i)]^{-a_{ji}} \prod_{j<i} [Q_+^j(q^{-1}w_k^i)]^{-a_{ji}}}$$

Space of nondegenerate solutions of
QQ-system for G

Nondegenerate **Z-twisted Miura-Plucker** (G,q)-opers
with regular singularities



Space of nondegenerate solutions of
XXZ for G

?

?

Nondegenerate **Z-twisted Miura** (G,q)-opers
with regular singularities

SL(2) Example

$$A(z) = \begin{pmatrix} g(z) & \Lambda(z) \\ 0 & g(z)^{-1} \end{pmatrix}$$

Z-twisted q-oper condition

$$A(z) = v(zq)Zv(z)^{-1}, \quad Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

Gauge transformation reads

$$v(z) = \begin{pmatrix} y(z) & 0 \\ 0 & y(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\frac{Q_-(z)}{Q_+(z)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y(z) & -y(z)\frac{Q_-(z)}{Q_+(z)} \\ 0 & y(z)^{-1} \end{pmatrix}$$

We find

$$g(z) = \zeta_i y(zq)y(z)^{-1}$$

$$\Lambda(z) = y(z)y(zq) \left(\zeta \frac{Q_-(z)}{Q_+(z)} - \zeta^{-1} \frac{Q_-(zq)}{Q_+(zq)} \right)$$

The q-oper condition becomes the **SL(2) QQ-system**

$$\zeta Q_-(z)Q_+(zq) - \zeta^{-1} Q_-(zq)Q_+(z) = \Lambda(z)$$

To get Bethe equations

$$Q_+(z) = \prod_{k=1}^m (z - w_k)$$

evaluate at roots of Q

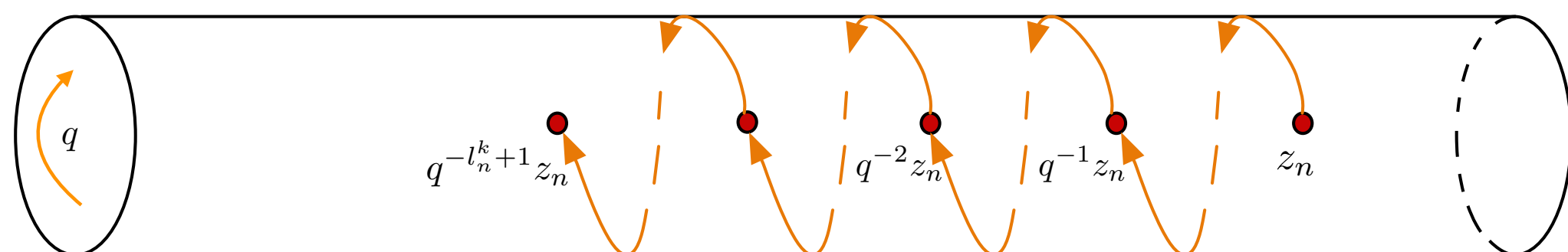
$$\frac{\Lambda(w_k)}{\Lambda(q^{-1}w_k)} = -\zeta^2 \frac{Q_+(qw_k)}{Q_+(q^{-1}w_k)}, \quad k = 1, \dots, m.$$

Singularities

$$\Lambda(z) = \prod_{p=1}^L \prod_{j_p=0}^{r_p-1} (z - q^{-j_p} z_p)$$

XXZ Bethe equations

$$q^r \prod_{p=1}^L \frac{w_k - q^{1-r_p} z_p}{w_k - q z_p} = -\zeta^2 q^m \prod_{j=1}^m \frac{qw_k - w_j}{w_k - qw_j}, \quad k = 1, \dots, m.$$



Quantum Bäcklund Transformation

Theorem: Consider the following q-gauge transformation

$$A \mapsto A^{(i)} = e^{\mu_i(qz)f_i} A(z) e^{-\mu_i(z)f_i}, \quad \text{where} \quad \mu_i(z) = \frac{\prod_{j \neq i} [Q_+^j(z)]^{-a_{ji}}}{Q_+^i(z) Q_-^i(z)}$$

changes the set of Q-functions

$$\begin{aligned} Q_+^j(z) &\mapsto Q_+^j(z), & j \neq i, & & \{\tilde{Q}_+^j\}_{j=1, \dots, r} &= \{Q_+^1, \dots, Q_+^{i-1}, Q_-^i, Q_+^{i+1}, \dots, Q_+^r\} \\ Q_+^i(z) &\mapsto Q_-^i(z), & Z &\mapsto s_i(Z) & \{\tilde{z}_j\}_{j=1, \dots, r} &= \{z_1, \dots, z_{i-1}, z_i^{-1} \prod_{j \neq i} z_j^{-a_{ji}}, \dots, z_r\} \end{aligned}$$

Now the strategy is to successively apply Backlund transformations according to the reduced decomposition of the element of the Weyl group

Consider longest element $w_0 = s_{i_1} \dots s_{i_\ell}$

Theorem: Every Z-twisted Miura-Plucker (G,q)-oper is Z-twisted Miura (G,q)-oper

The proof based on properties of double Bruhat cells addresses existence of the diagonalizing element $v(z)$ (to be constructed later)

(SL(N),q)-Operators

The QQ-system $\xi_i \phi_i(z) - \xi_{i+1} \phi_i(qz) = \rho_i(z)$ $\phi_i(z) = \frac{Q_i^-(z)}{Q_i^+(z)}$, $\rho_i(z) = \Lambda_i(z) \frac{Q_{i-1}^+(qz) Q_{i+1}^+(z)}{Q_i^+(z) Q_i^+(qz)}$

q-Oper condition $v(qz)^{-1} A(z) = Z v(z)^{-1}$

Diagonalizing element

Polynomials $Q_{i,\dots,j}^-(z)$

form extended QQ-system

$$v(z)^{-1} = \begin{pmatrix} \frac{1}{Q_1^+(z)} & \frac{Q_1^-(z)}{Q_2^+(z)} & \frac{Q_{12}^-(z)}{Q_3^+(z)} & \cdots & \frac{Q_{1,\dots,r-1}^-(z)}{Q_r^+(z)} & Q_{1,\dots,r}^-(z) \\ 0 & \frac{Q_1^+(z)}{Q_2^+(z)} & \frac{Q_2^-(z)}{Q_3^+(z)} & \cdots & \frac{Q_{2,\dots,r-1}^-(z)}{Q_r^+(z)} & Q_{2,\dots,r}^-(z) \\ 0 & 0 & \frac{Q_2^+(z)}{Q_3^+(z)} & \cdots & \frac{Q_{3,\dots,r-1}^-(z)}{Q_r^+(z)} & Q_{3,\dots,r}^-(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \frac{Q_{r-1}^+(z)}{Q_r^+(z)} & Q_r^-(z) \\ 0 & \cdots & \cdots & \cdots & 0 & Q_r^+(z) \end{pmatrix}$$

Quantum Wronskians

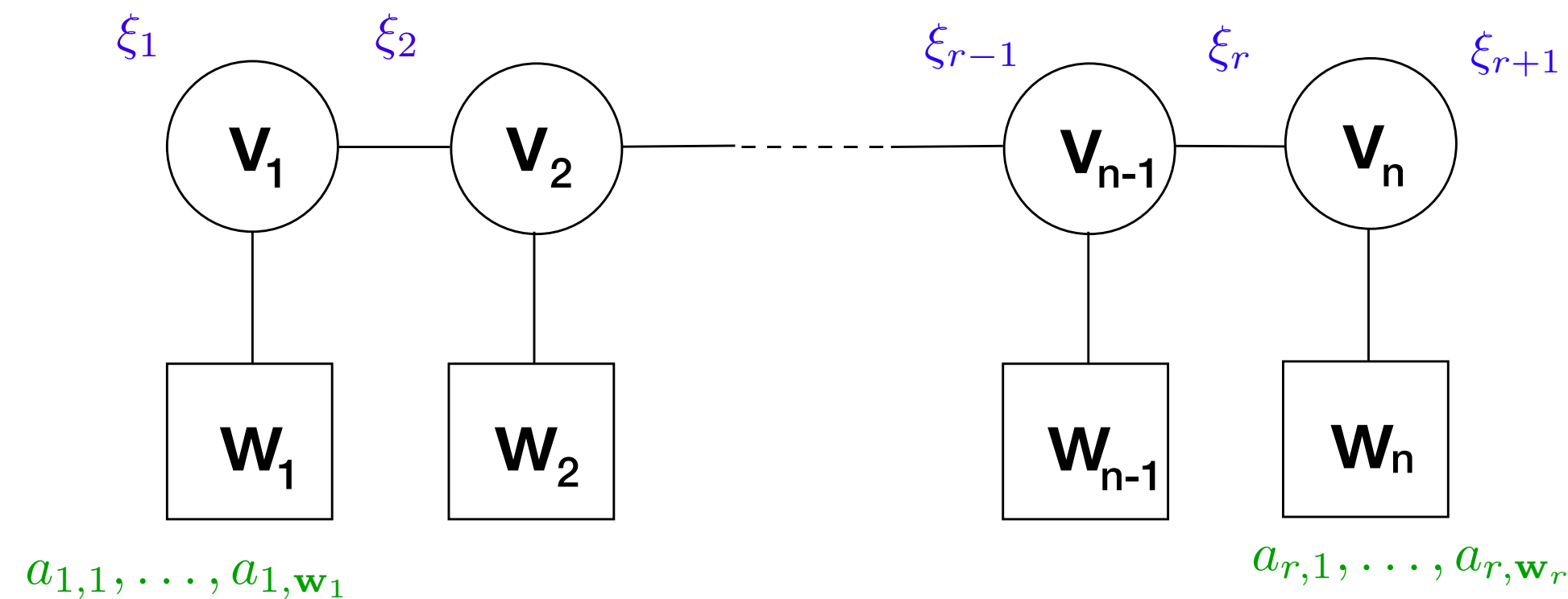
(SL(N),q)-oper can also be constructed from flag of subbundles $(E, A, \mathcal{L}_\bullet)$ such that the induced maps $\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i-1} \longrightarrow \mathcal{L}_{i+1}^q/\mathcal{L}_i^q$ are isomorphisms

The quantum determinants $\mathcal{D}_k(s) = e_1 \wedge \cdots \wedge e_{r+1-k} \wedge Z^{k-1}s(z) \wedge Z^{k-2}s(qz) \wedge \cdots \wedge Zs(q^{k-2}) \wedge s(q^{k-1}z)$

vanish at q-oper singularities $W_k(s) = P_1(z) \cdot P_2(q^2z) \cdots P_k(q^{k-1}z), \quad P_i(z) = \Lambda_r \Lambda_{r-1} \cdots \Lambda_{r-i+1}(z)$

Diagonalizing condition

$$\det_{i,j} \left[\xi^{k-j} \xi_{r+1-k+i} s_{r+1-k+i}(q^{j-1}z) \right] = \alpha_k W_k \mathcal{V}_k$$

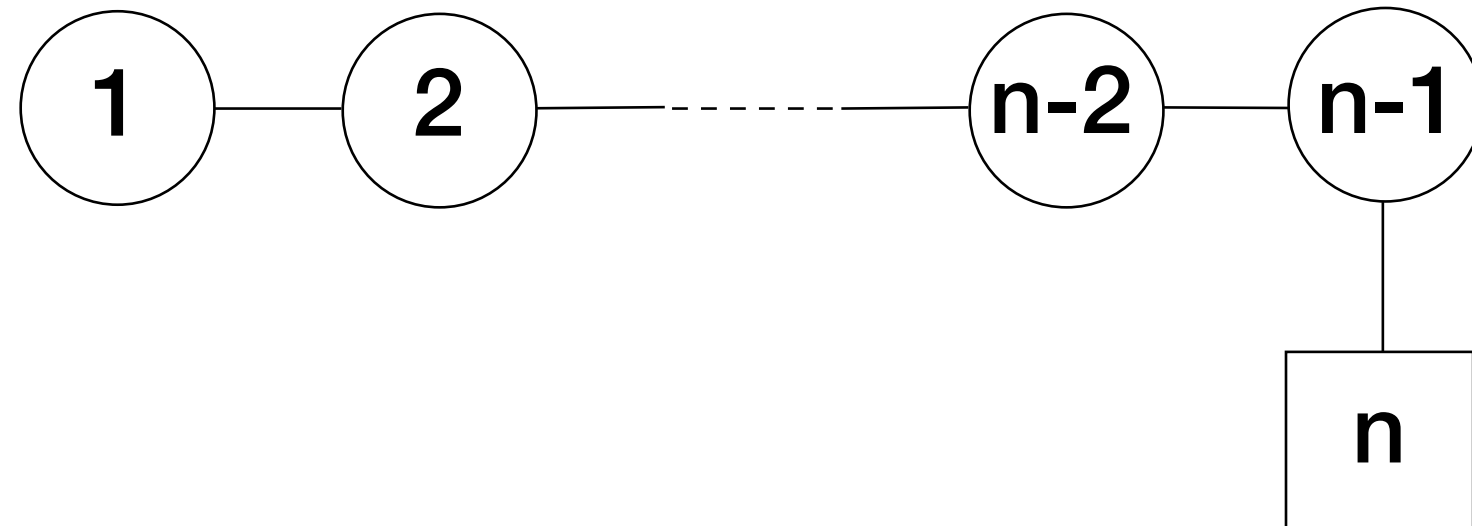


Components of the section of the line subbundle are the Q-polynomials!

$$s_{r+1}(z) = Q_r^+(z), \quad s_r(z) = Q_r^-(z), \quad s_k(z) = Q_{k,\dots,r}^-(z)$$

Quantum/Classical Duality

Consider T^*G/B



Construct the corresponding space of $(SL(N), \hbar)$ -opers

Specify components of the section of L_1

$$s_1(z) = z - p_1, \quad \dots, \quad s_{k+r}(z) = z - p_{k+l}$$

$$p_{k+l+1-p} = -\frac{Q_p^+(0)}{Q_{p-1}^+(0)}$$

Then the space of functions on the space of such \hbar -opers

$$\text{Fun}(\hbar\text{Op})(\text{FFl}_L) \cong \frac{\mathbb{C}(\{\xi_i\}, \{a_i\}, \{p_i\}, \hbar)}{\{H_i(\{p_j\}, \{\xi_j\}, \hbar) = e_i(a_1, \dots, a_L)\}_{i=1, \dots, L}}$$

is described by trigonometric Ruijsenaars-Schneider model with n particles

$$H_k = \sum_{\substack{\mathcal{J} \subset \{1, \dots, L\} \\ |\mathcal{J}|=k}} \prod_{\substack{i \in \mathcal{J} \\ j \notin \mathcal{J}}} \frac{\xi_i - \hbar \xi_j}{\xi_i - \xi_j} \prod_{m \in \mathcal{J}} p_m$$

Quantum Wronskians

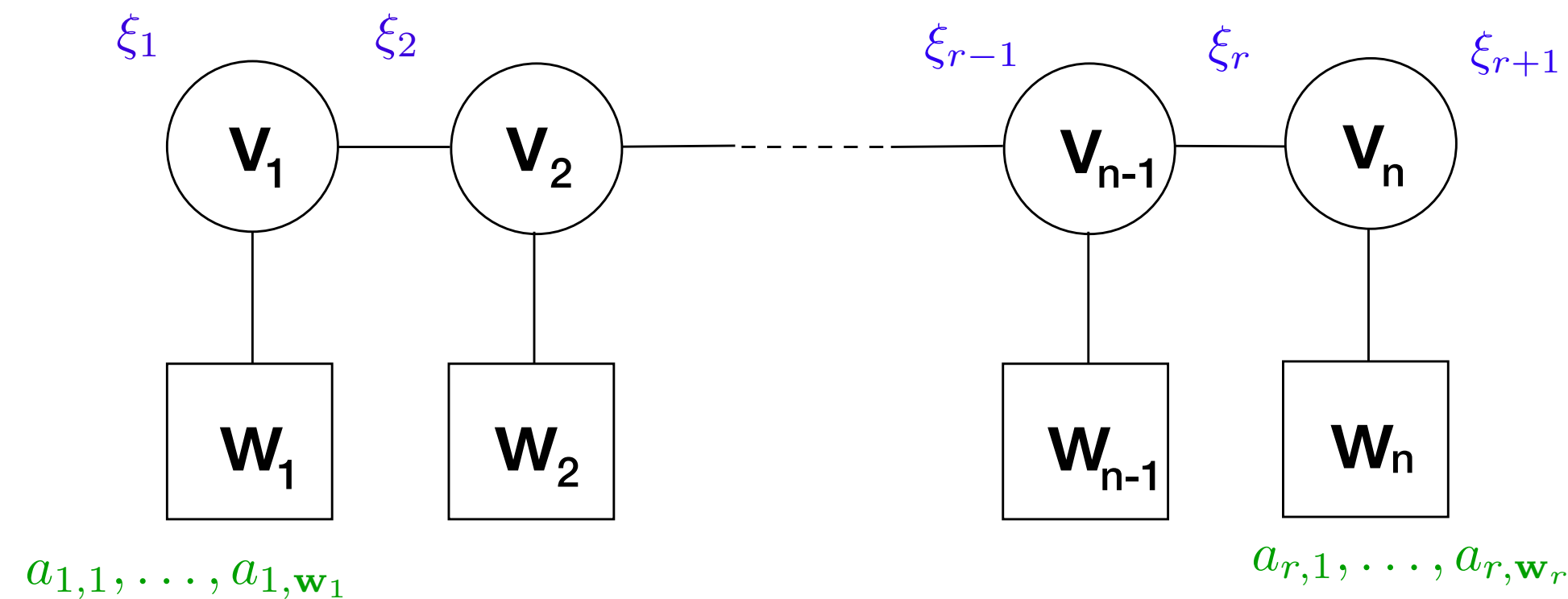
(SL(N),q)-oper can also be constructed from flag of subbundles $(E, A, \mathcal{L}_\bullet)$ such that the induced maps $\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i-1} \longrightarrow \mathcal{L}_{i+1}^q/\mathcal{L}_i^q$ are isomorphisms

The quantum determinants $\mathcal{D}_k(s) = e_1 \wedge \cdots \wedge e_{r+1-k} \wedge Z^{k-1}s(z) \wedge Z^{k-2}s(qz) \wedge \cdots \wedge Zs(q^{k-2}) \wedge s(q^{k-1}z)$

vanish at q-oper singularities $W_k(s) = P_1(z) \cdot P_2(q^2z) \cdots P_k(q^{k-1}z), \quad P_i(z) = \Lambda_r \Lambda_{r-1} \cdots \Lambda_{r-i+1}(z)$

Diagonalizing condition

$$\det_{i,j} \left[\xi^{k-j} \xi_{r+1-k+i} s_{r+1-k+i}(q^{j-1}z) \right] = \alpha_k W_k \mathcal{V}_k$$



Components of the section of the line subbundle are the Q-polynomials!

$$s_{r+1}(z) = Q_r^+(z), \quad s_r(z) = Q_r^-(z), \quad s_k(z) = Q_{k,\dots,r}^-(z)$$

III. Generalized Wronskians

Consider big cell in Bruhat decomposition

$$G_0 = N_- H N_+ \\ g = n_- h n_+$$

$$V_i^+ \text{ irrep of } G \text{ with highest weight } \omega_i \\ h\nu_{\omega_i}^+ = [h]^{\omega_i} \nu_{\omega_i}^+$$

Define **principal minors** for group element g

For $SL(N)$ they are standard minors of matrices

$$\Delta^{\omega_i}(g) = [h]^{\omega_i}, \quad i = 1, \dots, r$$

Then **generalized minors** are regular functions on G

$$\Delta_{u\omega_i, v\omega_i}(g) = \Delta^{\omega_i}(\tilde{u}^{-1} g \tilde{v}) \quad u, v \in W_G.$$

Proposition

Action of the group element on the highest weight vector in

$$g \cdot \nu_{\omega_i}^+ = \sum_{w \in W} \Delta_{w \cdot \omega_i, \omega_i}(g) \tilde{w} \cdot \nu_{\omega_i}^+ + \dots,$$

where dots stand for the vectors, which do not belong to the orbit \mathcal{O}_W .

Generalized Minors and QQ-system

The set of generalized minors $\{\Delta_{w \cdot \omega_i, \omega_i}\}_{w \in W; i=1, \dots, r}$ creates a set of coordinates on G/B^+ , known as *generalized Plücker coordinates*. In particular, the set of zeroes of each of $\Delta_{w \cdot \omega_i, \omega_i}$ is a uniquely and unambiguously defined hypersurface in G/B .

Proposition For a W -generic Z -twisted Miura-Plücker (G, q) -oper with q -connection $A(z) = v(qz)Zv(z)^{-1}$, where $v(z) \in B_-(z)$ we have the following relation:

$$\Delta_{w \cdot \omega_i, \omega_i}(v^{-1}(z)) = Q_+^{w, i}(z)$$

for any $w \in W$.

Proof: Since $\Delta^{\omega_i}(v^{-1}(z)) = Q_+^i(z)$ Diagonalizing gauge transformation $v^{-1}(z) = \prod_{i=1}^r e^{\frac{Q_-^i(z)}{Q_+^i(z)} f_i} \prod_{i=1}^r [Q_+^i(z)]^{\check{\alpha}_i} \dots$

$$v^{-1}(z)\nu_{\omega_i}^+ = Q_+^i(z)\nu_{\omega_i}^+ + Q_-^i(z)f_i\nu_{\omega_i}^+ + \dots$$

Fundamental Relation for Generalized Minors

[Fomin Zelevinsky]

Proposition 4.8. *Let, $u, v \in W$, such that for $i \in \{1, \dots, r\}$, $\ell(uw_i) = \ell(u) + 1$, $\ell(vw_i) = \ell(v) + 1$. Then*

$$(4.7) \quad \Delta_{u \cdot \omega_i, v \cdot \omega_i} \Delta_{uw_i \cdot \omega_i, vw_i \cdot \omega_i} - \Delta_{uw_i \cdot \omega_i, v \cdot \omega_i} \Delta_{u \cdot \omega_i, vw_i \cdot \omega_i} = \prod_{j \neq i} \Delta_{u \cdot \omega_j, v \cdot \omega_j}^{-a_{ji}}$$

Can we make sense of this relation using our approach of q-Operators?

Generalized Wronskians

The approach is similar to Miura-Plucker q-Operators

Let $\nu_{\omega_i}^+$ be a generator of the line $L_i^+ \subset V_i^+$ V_i^+ irrep of G with highest weight ω_i

The subspace $L_{c,i}^+$ of V_i of weight $c^{-1} \cdot \omega_i$ is one-dimensional and is spanned by $s^{-1}\nu_{\omega_i}^+$

Associated vector bundle $\mathcal{V}_i^+ = \mathcal{F}_{B_+} \times_{B_+} V_i^+ = \mathcal{F}_G \times_G V_i^+$

Contains line subbundles $\mathcal{L}_i^+ = \mathcal{F}_H \times_H L_i^+$, $\mathcal{L}_{c,i}^+ = \mathcal{F}_H \times_H L_{c,i}^+$

Define **generalized Wronskian** on \mathbb{P}^1 as quadruple $(\mathcal{F}_G, \mathcal{F}_{B_+}, \mathcal{G}, Z)$

\mathcal{G} is a meromorphic section of a principle bundle \mathcal{F}_G

s.t. for sections $\{v_i^+, v_{c,i}^+\}_{i=1,\dots,r}$ of line bundles $\{\mathcal{L}_i^+, \mathcal{L}_{c,i}^+\}_{i=1,\dots,r}$ on $U \cap M_q^{-1}(U)$

$$\mathcal{G}^q \cdot v_i^+ = Z \cdot \mathcal{G} \cdot v_{c,i}^+$$

Adding Singularities

Effectively the above definition means that the Wronskian, written as an element of $G(z)$, satisfies

$$Z^{-1} \mathcal{G}(qz) \nu_{\omega_i}^+ = \mathcal{G}(z) \cdot s_{\phi}(z)^{-1} \cdot \nu_{\omega_i}^+ \qquad s_{\phi}(z) = \prod_i \phi_i^{-\check{\alpha}_i} s_i$$

Define **generalized Wronskian with regular singularities** if

$$s_{\Lambda}(z)^{-1} = \prod_i^{\text{inv}} s_i \Lambda_i^{\check{\alpha}_i}$$

Fomin-Zelevinsky relations then read

$$\begin{aligned} \Delta_{\omega_i, \omega_i} \Delta_{\omega_i \cdot \omega_i, c^{-1} \cdot \omega_i} - \Delta_{\omega_i \cdot \omega_i, \omega_i} \Delta_{\omega_i, c^{-1} \cdot \omega_i} \\ = \prod_{j < i = i_l} \Delta_{\omega_j, c^{-1} \cdot \omega_j}^{-a_{ji}} \prod_{j > i = i_l} \Delta_{\omega_j, \omega_j}^{-a_{ji}}, \quad i = 1, \dots, r, \end{aligned}$$

q-Operators and q-Wronskians

Theorem 1:

Nondegenerate generalized q-Wronskians
with regular singularities $\{\Lambda_i\}_{i=1,\dots,r}$



Nondegenerate Z-twisted Miura (G,q)-opers
with regular singularities $\{\Lambda_i\}_{i=1,\dots,r}$

Theorem 2:

For a given Z-twisted (G,q)-Miura oper, there exists a unique generalized q-Wronskian

$$\mathcal{W}(z) \in B_-(z)w_0B_-(z) \cap B_+(z)w_0B_+(z) \subset G(z),$$

satisfying the system of equations

$$(4.32) \quad \begin{aligned} \mathcal{W}(q^{k+1}z)\nu_{\omega_i}^+ &= Z^k \mathcal{W}(z)s^{-1}(z)s^{-1}(qz) \dots s^{-1}(q^k z)\nu_{\omega_i}^+, \\ i &= 1, \dots, r, \quad k = 0, 1, \dots, h-1, \end{aligned}$$

where h is the Coxeter number of G.

Examples: SL(2)

$$\mathcal{W}(qz)\nu_\omega^+ = Z\mathcal{W}(z)s^{-1}(z)\nu_\omega^+$$

$$s^{-1}(z) = \tilde{s}^{-1}\Lambda(z)^{\check{\alpha}} = \begin{pmatrix} 0 & \Lambda(z)^{-1} \\ \Lambda(z) & 0 \end{pmatrix}, \quad \nu_\omega^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

In terms of Q-polynomials

$$\mathcal{W}(z) = \begin{pmatrix} Q^+(z) & \zeta^{-1}\Lambda(z)^{-1}Q_+(qz) \\ Q^-(z) & \zeta\Lambda(z)^{-1}Q^-(qz) \end{pmatrix}$$

$$\zeta Q^+(z)Q^-(qz) - \zeta^{-1}Q^+(qz)Q^-(z) = \Lambda(z)$$

is equivalent to $\det \mathcal{W}(z) = 1$.

Examples SL(N)

$$\mathcal{W}(z) = \left(\Delta_{\mathbf{w}\omega, \omega} \middle| \Delta_{\mathbf{w}\omega, s^{-1}\omega} \middle| \cdots \middle| \Delta_{\mathbf{w}\omega, s^{r+1}\omega} \right) (\mathcal{G}(z))$$

Lift for standard ordering along the Dynkin diagram

$$s_{\Lambda}^{-1}(z) = \tilde{s}^{-1} \prod_i \Lambda_i^{d_i}$$

$$d_i = \sum_{j=1}^i \check{\alpha}_j$$

$$\tilde{s}^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

$$\mathcal{W}(z) = \left(Q^{\mathbf{w}\cdot\omega}(z) \middle| ZF_1(z)Q^{\mathbf{w}\cdot\omega}(qz) \middle| \cdots \middle| Z^{r-1}F_{r-1}(q^{r-1}z)Q^{\mathbf{w}\cdot\omega}(q^{r-1}z) \right)$$

where $F_i(z) = \prod_{j=1}^i \Lambda_j(z)^{-1}$.

Lewis Carroll Identity

In Type A FZ relation reduces to

$$\Delta_{u\omega_i, v\omega_i} \Delta_{u s_i \omega_i, v s_i \omega_i} - \Delta_{u s_i \omega_i, v \omega_i} \Delta_{u \omega_i, v s_i \omega_i} = \Delta_{u\omega_{i-1}, v\omega_{i-1}} \Delta_{u\omega_{i+1}, v\omega_{i+1}}$$

$$M_1^1 M_i^2 - M_i^1 M_1^2 = M_{1i}^{12} M$$