q-Opers — what they are and what are they good for?

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Zoom Talk at University of Queensland 07/29/2022

Literature

[arXiv:2108.04184]
q-Opers, QQ-systems, and Bethe Ansatz II: Generalized Minors
P. Koroteev, A. M. Zeitlin

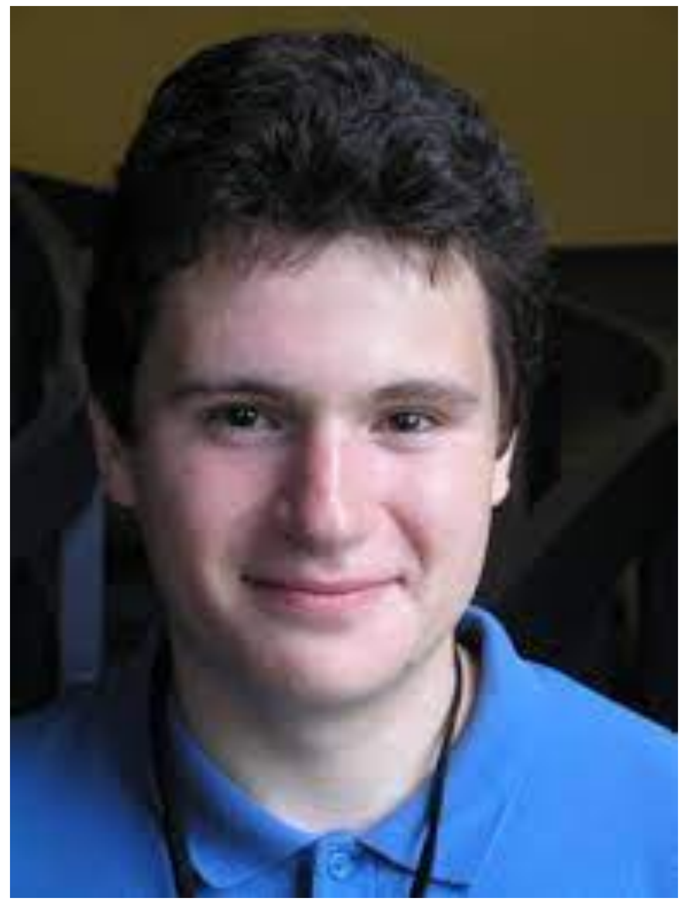
[arXiv:2105.00588]
3d Mirror Symmetry for Instanton Moduli Spaces
P. Koroteev, A. M. Zeitlin

[arXiv:2007.11786] J. Inst. Math. Jussieu
Toroidal q-Opers
P. Koroteev, A. M. Zeitlin

[arXiv:2002.07344] J. Europ. Math. Soc.
q-Opers, QQ-Systems, and Bethe Ansatz
E. Frenkel, P. Koroteev, D. S. Sage, A. M. Zeitlin

[arXiv:1811.09937] Commun.Math.Phys. **381** (2021) 641
(SL(N),q)-opers, the q-Langlands correspondence, and quantum/classical duality
P. Koroteev, D. S. Sage, A. M. Zeitlin

[arXiv:1705.10419] Selecta Math. **27** (2021) 87
Quantum K-theory of Quiver Varieties and Many-Body Systems
P. Koroteev, P. P. Pushkar, A. V. Smirnov, A. M. Zeitlin



Motivation

Quantum Geometry and Integrable Systems [Okounkov et al] [Pushkar, Zeitlin, Smirnov] [PK, Pushkar, Smirnov, Zeitlin]

BPS/CFT Correspondence [Nekrasov Shatashvili]

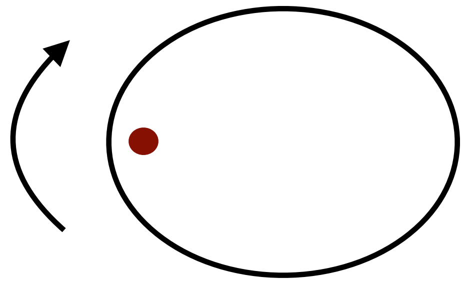
Geometric q-Langlands Correspondence [Frenkel] [Aganagic, Frenkel, Okounkov]

ODE/IM Correspondence [Bazhanov, Lukyanov, Zamolodchikov] [Dorey, Tateo]

I. q-Opers — SL(2) Example

Consider vector bundle *E* over $\mathbb{P}^1$

$$M_q : \mathbb{P}^1 \to \mathbb{P}^1$$
$$z \mapsto qz$$

q 

Map of vector bundles	$A : E \longrightarrow E^q$
Upon trivialization	$A(z) \in \mathfrak{gl}(N, \mathbb{C}(z))$
q-gauge transformation	$A(z) \mapsto g(qz)A(z)g^{-1}(z)$
Difference equation	$D_q(s) = As$

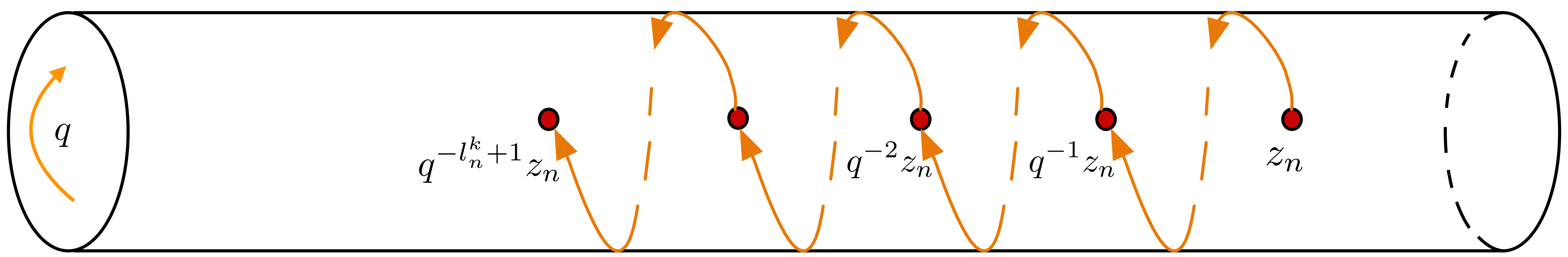
Definition: A meromorphic $(\mathrm{GL}(N), q)$-*connection* over $\mathbb{P}^1$ is a pair (E, A), where E is a (trivializable) vector bundle of rank N over $\mathbb{P}^1$ and A is a meromorphic section of the sheaf $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(E, E^q)$ for which $A(z)$ is invertible, i.e. lies in $\mathrm{GL}(N, \mathbb{C}(z))$. The pair (E, A) is called an $(\mathrm{SL}(N), q)$-connection if there exists a trivialization for which $A(z)$ has determinant 1.

q-Opers

Definition: A $(\mathrm{GL}(2), q)$-*oper* on $\mathbb{P}^1$ is a triple $(E, A, \mathcal{L})$, where (E, A) is a $(\mathrm{GL}(2), q)$-connection and $\mathcal{L}$ is a line subbundle such that the induced map $\bar{A} : \mathcal{L} \longrightarrow (E/\mathcal{L})^q$ is an isomorphism. The triple is called an $(\mathrm{SL}(2), q)$-*oper* if (E, A) is an $(\mathrm{SL}(2), q)$-connection.

in a trivialization $s(z) \wedge A(z)s(qz) \neq 0$

Definition: A $(\mathrm{SL}(2), q)$-*oper with regular singularities* at the points $z_1, \ldots, z_L \neq 0, \infty$ with weights $k_1, \ldots k_L$ is a meromorphic $(\mathrm{SL}(2), q)$-oper $(E, A, \mathcal{L})$ for which $\bar{A}$ is an isomorphism everywhere on $\mathbb{P}^1 \setminus \{0, \infty\}$ except at the points z_m, $q^{-1}z_m$, $q^{-2}z_m$, $\ldots$, $q^{-k_m+1}z_m$ for $m \in \{1, \ldots, L\}$, where it has simple zeros.



Finally, (SL(2),q)-oper is ***Z-twisted*** in A(z) is gauge equivalent to a diagonal matrix Z

$$Z = g(qz)A(z)g(z)^{-1}$$

Miura q-Opers

Miura (SL(2),q)-oper is a quadruple $(E, A, \mathcal{L}, \hat{\mathcal{L}})$ where $(E, A, \mathcal{L})$ is an (SL(2),q)-oper and $\hat{\mathcal{L}}$ is preserved by the q-connection *A*

Chose trivialization of $\mathcal{L}$

$$s(z) = \begin{pmatrix} Q_+(z) \\ Q_-(z) \end{pmatrix}$$

Twist element $Z = \mathrm{diag}(\zeta, \zeta^{-1})$

q-Oper condition — SL(2) **QQ-system**

$$s(z) \wedge A(z) s(qz) = \Lambda(z)$$

$$\det \begin{pmatrix} Q_+(z) & \zeta Q_+(qz) \\ Q_-(z) & \zeta^{-1} Q_-(qz) \end{pmatrix} = \Lambda(z)$$

$$\zeta Q_-(z) Q_+(zq) - \zeta^{-1} Q_-(zq) Q_+(z) = \Lambda(z)$$

One of the polynomials can be made monic

$$Q_+(z) = \prod_{k=1}^{m} (z - w_k)$$

singularities

$$\Lambda(z) = \prod_{p=1}^{L} \prod_{j_p=0}^{r_p-1} (z - q^{-j_p} z_p)$$

From QQ-system to Bethe equations

$$\frac{\Lambda(w_k)}{\Lambda(q^{-1}w_k)} = -\zeta^2 \frac{Q_+(qw_k)}{Q_+(q^{-1}w_k)}, \qquad k = 1, \dots, m.$$

$$q^r \prod_{p=1}^{L} \frac{w_k - q^{1-r_p} z_p}{w_k - q z_p} = -\zeta^2 q^m \prod_{j=1}^{m} \frac{q w_k - w_j}{w_k - q w_j}, \qquad k = 1, \dots, m$$

q-Miura Transformation

$$A(z) = \begin{pmatrix} \zeta \frac{Q_+(qz)}{Q_+(z)} & \Lambda(z) \\ 0 & \zeta^{-1} \frac{Q_+(z)}{Q_+(qz)} \end{pmatrix}$$

Z-twisted q-oper condition $A(z) = v(zq) Z v(z)^{-1}, \qquad Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$

q-gauge transformation $v(z) = Q_+^{\check{\alpha}} \exp\left(-\frac{Q_-}{Q_+}(z) e \right) = \begin{pmatrix} Q_+(z) & \frac{Q_-}{Q_+}(z) \\ 0 & \frac{1}{Q_+(z)} \end{pmatrix}$

The q-oper condition becomes the **SL(2) QQ-system** $\zeta Q_-(z) Q_+(zq) - \zeta^{-1} Q_-(zq) Q_+(z) = \Lambda(z)$

Difference Equation $D_q(s) = As.$ $\qquad D_q(s_1) = \Lambda(z) s_2$

after elimination $\left(D_q^2 - T(qz) D_q - \frac{\Lambda(qz)}{\Lambda(z)} \right) s_1 = 0$

Trigonometric Ruijsenaars Hamiltonians

Nondegenerate (SL(2),q)-oper condition

$$Q_- = z - p_-$$
$$Q_+ = c(z - p_+)$$

$$\begin{vmatrix} z - p_+ & \zeta(qz - p_+) \\ z - p_- & \zeta^{-1}(qz - p_-) \end{vmatrix} = (\zeta - \zeta^{-1})(z - z_+)(z - z_-)$$

$$z^2 - \frac{z}{q}\left[\frac{\zeta - q\zeta^{-1}}{\zeta - \zeta^{-1}} p_+ + \underbrace{\frac{q\zeta - \zeta^{-1}}{\zeta - \zeta^{-1}} p_-}_{T_1}\right] + \underbrace{\frac{p_+ p_-}{q}}_{T_2} = (z - z_+)(z - z_-)$$

qOper condition yields
tRS Hamiltonians!

$$\det(z - L_{tRS}) = (z - z_+)(z - z_-)$$

Calogero-Moser Space

Let V be an N-dimensional vector space over $\mathbb{C}$. Let $\mathscr{M}'$ be the subset of $GL(V) \times GL(V) \times V \times V^*$ consisting of elements (M, T, u, v) such that

$$qMT - TM = u \otimes v^T$$

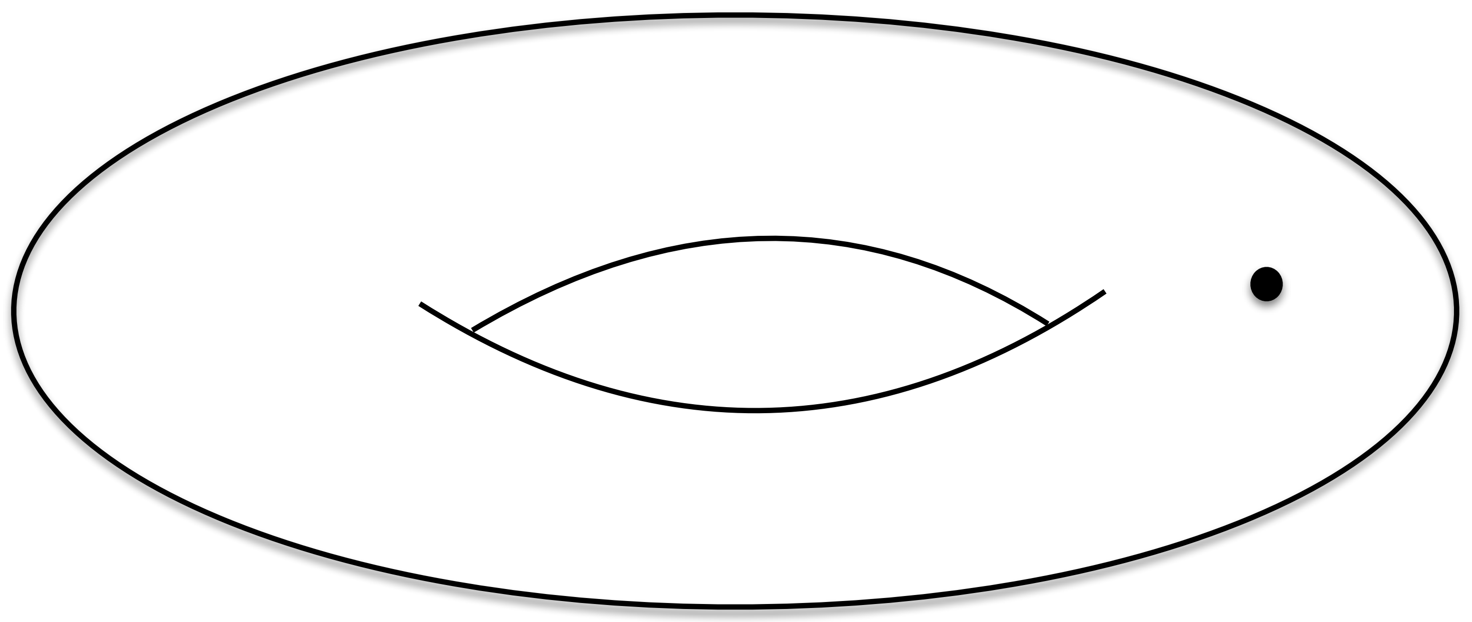
The group $GL(N; \mathbb{C}) = GL(V)$ acts on $\mathscr{M}'$ by conjugation

$$(M, T, u, v) \mapsto (gMg^{-1}, gTg^{-1}, gu, vg^{-1})$$

The quotient of $\mathscr{M}'$ by the action of $GL(V)$ is called **Calogero-Moser space $\mathscr{M}$**

Also can be understood as moduli space of flat connections on punctured torus

Integrable Hamiltonians are ~TrT^k



$$\mathcal{M}_n = \{A, B, C\}/GL(n; \mathbb{C})$$

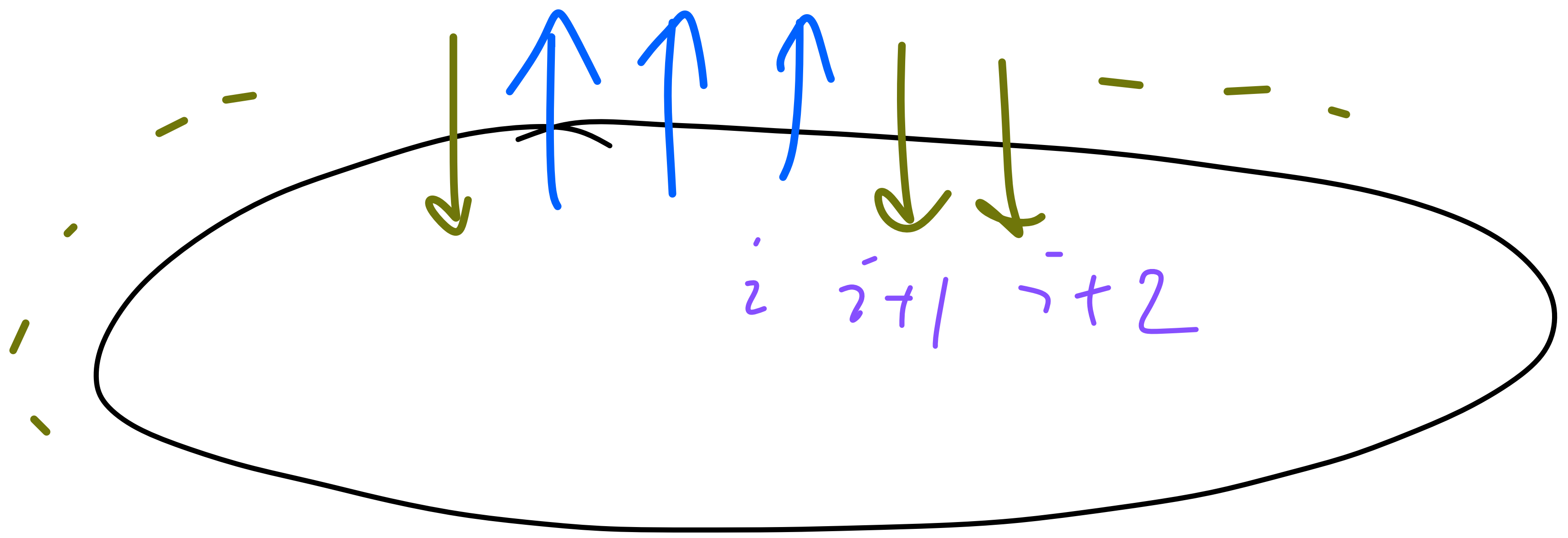
$$ABA^{-1}B^{-1} = C$$

$$C = \text{diag}(q, \ldots, q, q^{n-1})$$

[PK Gaiotto]

Quantum

QQ-Systems



SU(**n**) XXZ spin chain on n sites w/ ***anisotropies***
and ***twisted periodic boundary conditions***

Planck's constant $\hbar$

twist eigenvalues z_i

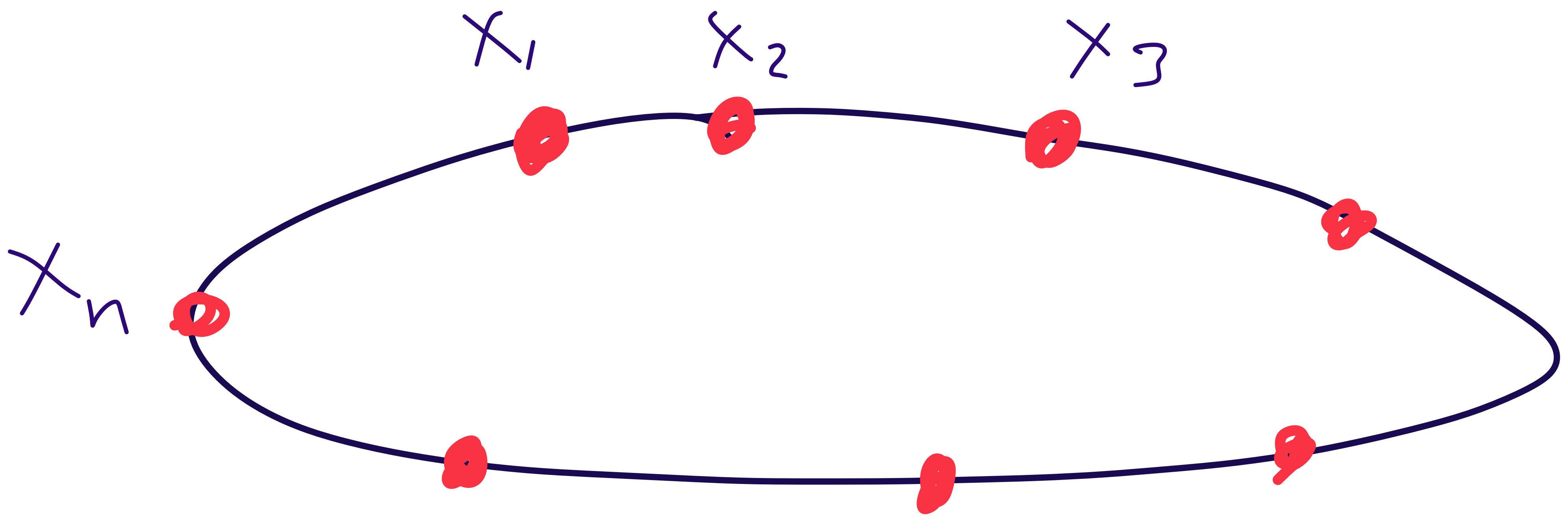
equivariant parameters (anisotropies) a_i

Bethe Ansatz Equations: $\dfrac{\partial Y}{\partial \sigma_i} = 0$

$$\frac{\zeta_i}{\zeta_{i+1}} \prod_{\beta=1}^{\mathbf{v}_{i-1}} \frac{\sigma_{i,\alpha} - \hbar^{1/2}\sigma_{i-1,\beta}}{\sigma_{i-1,\beta} - \hbar^{1/2}\sigma_{i,\alpha}} \cdot \prod_{\beta\neq\alpha}^{\mathbf{v}_i} \frac{\hbar\sigma_{i,\alpha} - \sigma_{i,\beta}}{\hbar\sigma_{i,\beta} - \sigma_{i,\alpha}} \cdot \prod_{\beta=1}^{\mathbf{v}_{i+1}} \frac{\sigma_{i,\alpha} - \hbar^{1/2}\sigma_{i+1,\beta}}{\sigma_{i+1,\beta} - \hbar^{1/2}\sigma_{i,\alpha}} = (-1)^{\delta_i}$$

Classical

q-Opers



$$\Omega = \sum_i \frac{dp_i}{p_i} \wedge \frac{dz_i}{z_i}$$

n-particle trigonometric
Ruijsenaars-Schneider model

$$[T_i, T_j] = 0$$

Coupling constant $\hbar$

$$T_1 = \sum_{i=1}^{n} \prod_{j\neq i} \frac{\hbar z_i - z_j}{z_i - z_j} p_i$$

coordinates z_i

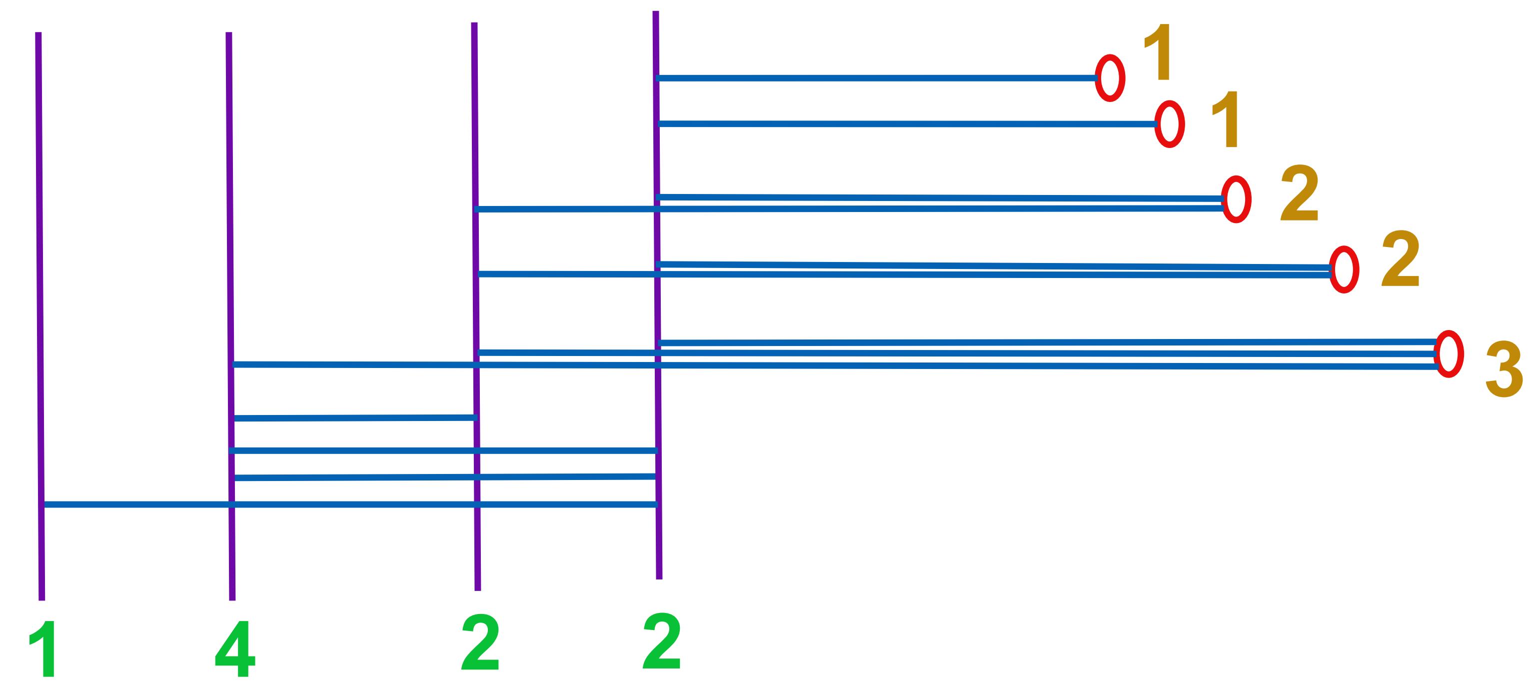
energy (eigenvalues of Hamiltonians) $e_i(a_i)$

Energy level equations

$$T_i(\mathbf{z}, \hbar) = e_i(\mathbf{a}), \qquad i = 1, \ldots, n$$

Quantum/Classical Duality

[PK Gaiotto]
[PK Zeitlin]

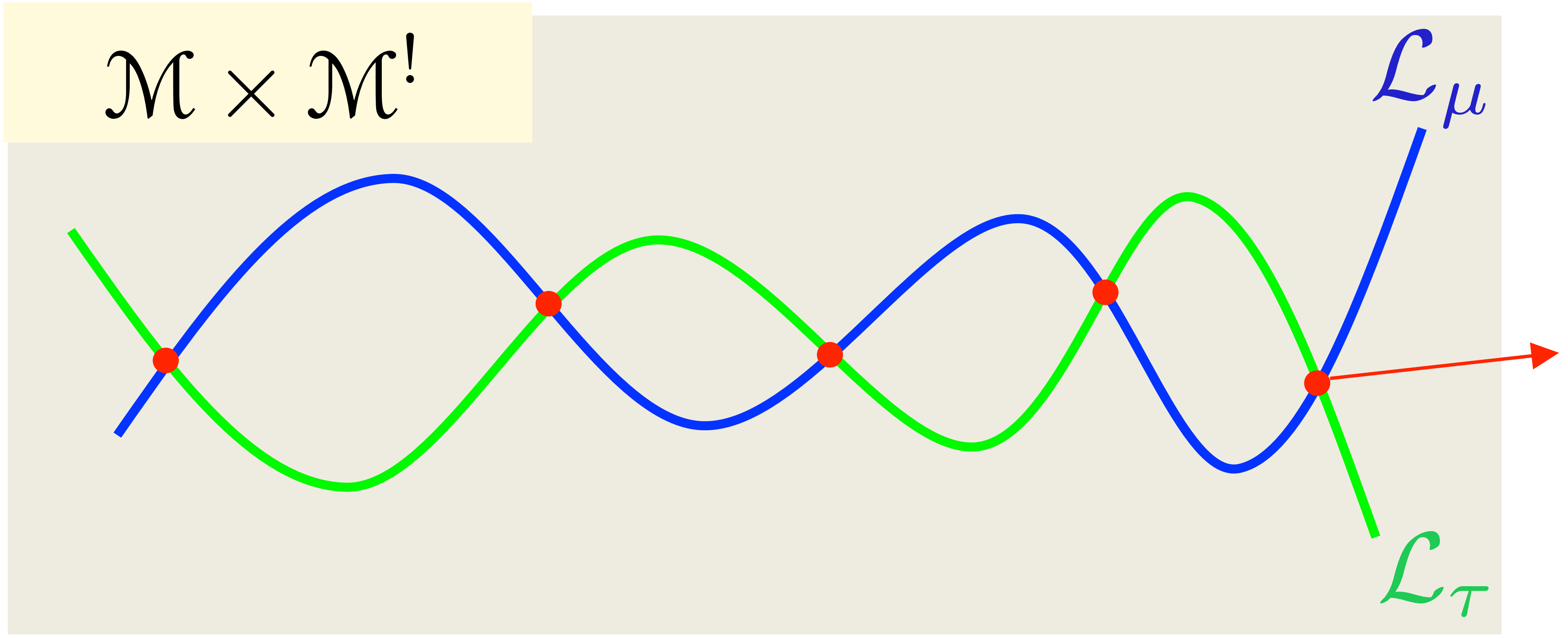


Symplectic form $\Omega = \sum_{i=1}^{N} \frac{dp_i^{\xi}}{p_i^{\xi}} \wedge \frac{d\xi_i}{\xi_i} - \frac{dp_i^{a}}{p_i^{a}} \wedge \frac{da_i}{a_i}$

tRS momenta $p_i^{\xi} = \exp \frac{\partial Y}{\partial \xi_i}\,, \qquad p_i^{a} = \exp \frac{\partial Y}{\partial a_i}$

tRS energy relations

$$\det(u - T) = \prod_{i=1}^{N} (u - a_i)\,, \qquad \det(u - M) = \prod_{i=1}^{N} (u - \xi_i)$$



$Y = Y^!$

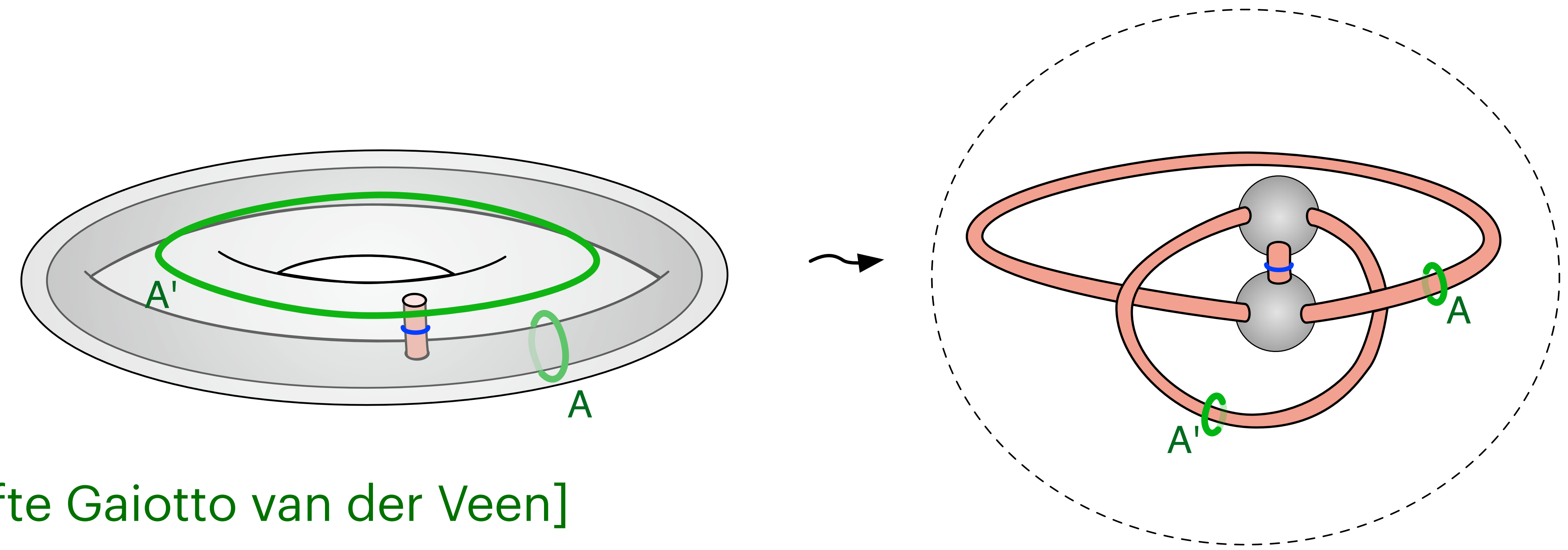
3d mirror symmetry

$$\sum_{\substack{\mathcal{I} \subset \{1,\dots,L\} \\ |\mathcal{I}| = k}} \prod_{\substack{i \in \mathcal{I} \\ j \notin \mathcal{I}}} \frac{a_i - \hbar\, a_j}{a_i - a_j} \prod_{m \in \mathcal{I}} p_m = \ell_k(\xi_i)$$

$\mathcal{L}_\mu$ Eigenvalues of M and Slodowy form on T

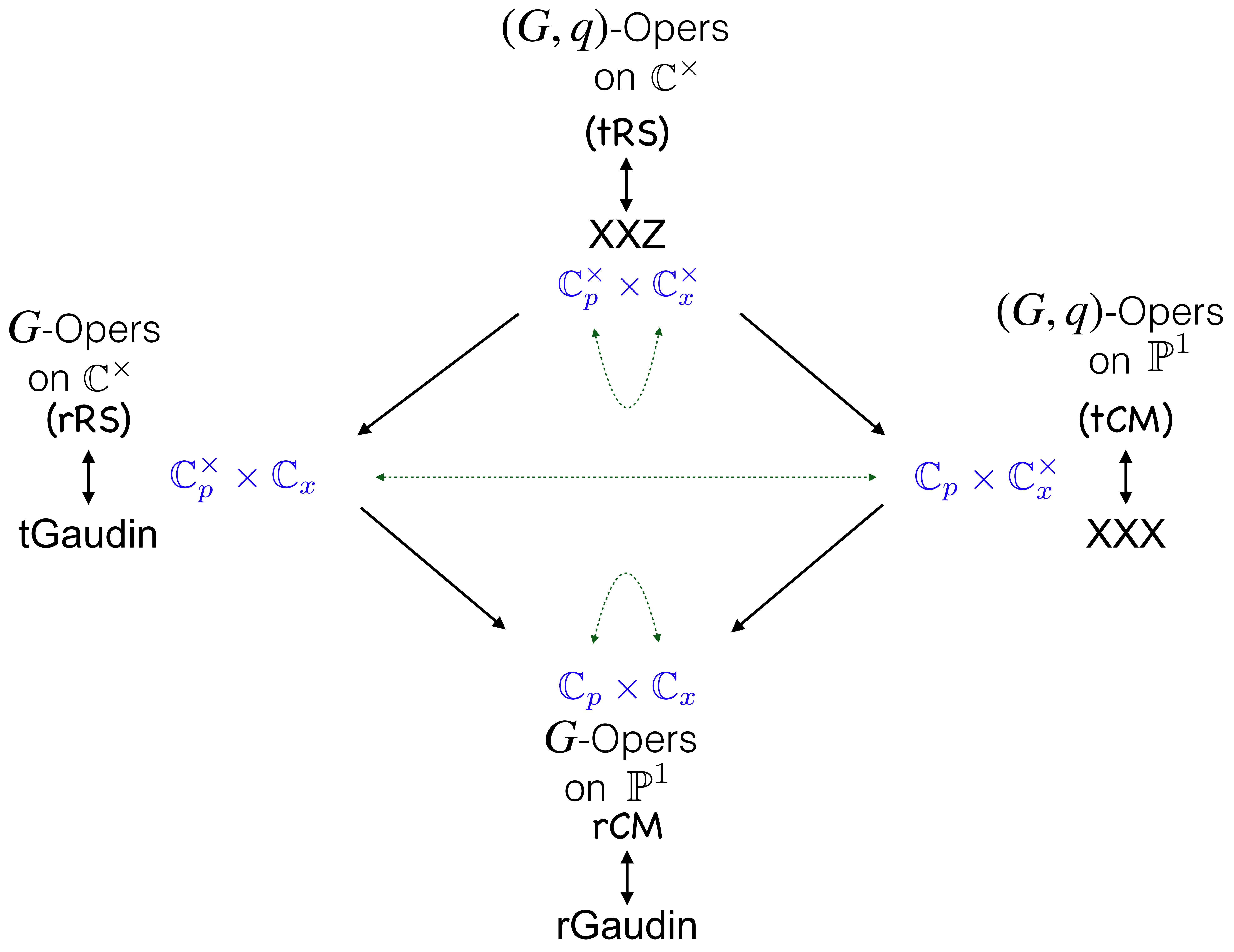
$\mathcal{L}_\tau$ Eigenvalues of T and Slodowy form on M

Solutions of Bethe equations — intersection points



[Dimofte Gaiotto van der Veen]

Network of Dualities



(SL(N),q)-Opers

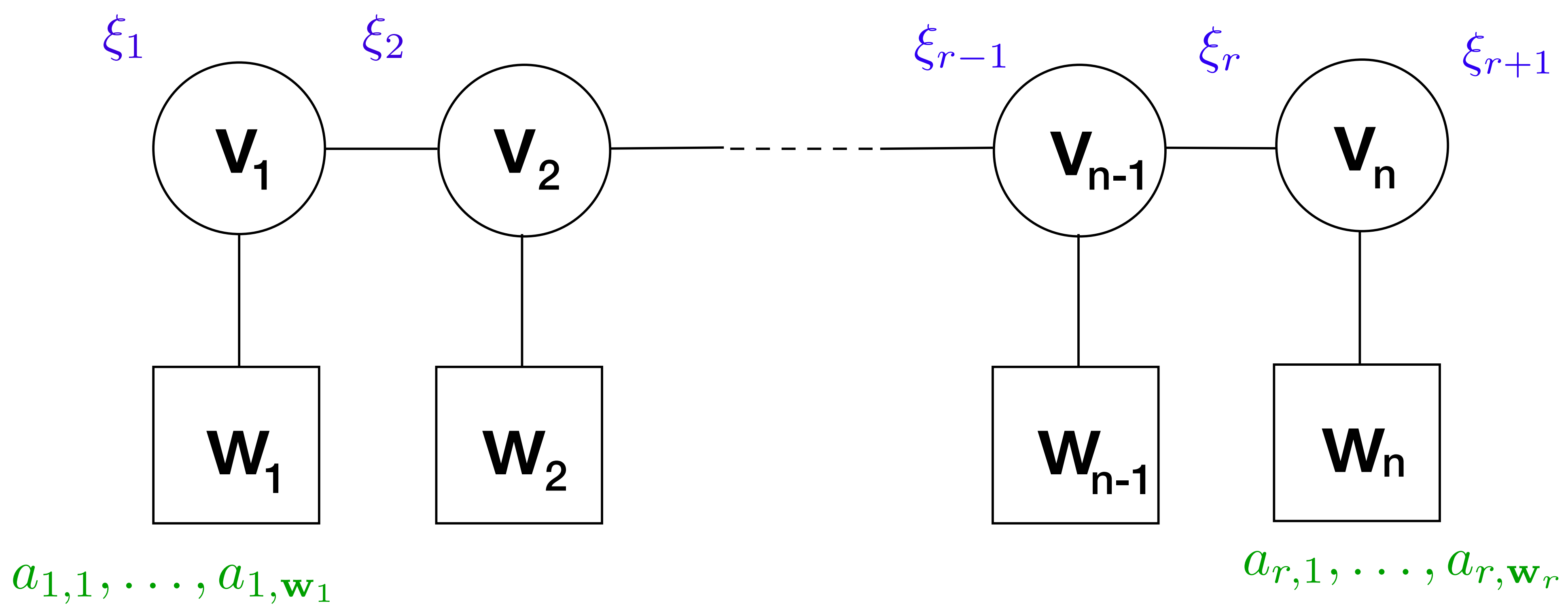
(SL(N),q)-oper can also be constructed from flag of subbundles $(E, A, \mathcal{L}_\bullet)$ such that the induced maps $\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i-1} \longrightarrow \mathcal{L}^q_{i+1}/\mathcal{L}^q_i$ are isomorphisms

The quantum determinants $\mathcal{D}_k(s) = e_1 \wedge \cdots \wedge e_{r+1-k} \wedge Z^{k-1}s(z) \wedge Z^{k-2}s(qz) \wedge \cdots \wedge Zs(q^{k-2}) \wedge s(q^{k-1}z)$

vanish at q-oper singularities $W_k(s) = P_1(z) \cdot P_2(q^2 z) \cdots P_k(q^{k-1}z), \qquad P_i(z) = \Lambda_r \Lambda_{r-1} \cdots \Lambda_{r-i+1}(z)$

Diagonalizing condition

$$\det_{i,j}\left[\xi^{k-j}_{r+1-k+i} s_{r+1-k+i}(q^{j-1}z)\right] = \alpha_k W_k \mathcal{V}_k$$



Components of the section of the
line subbundle are the Q-polynomials!

$$s_{r+1}(z) = Q^+_r(z)\,, \qquad s_r(z) = Q^-_r(z)\,, \qquad s_k(z) = Q^-_{k,\ldots,r}(z)$$

(SL(N),q)-Opers

The extended QQ-system

$$\xi_i Q_i^+(qz)Q_i^-(z) - \xi_{i+1}Q_i^+(z)Q_i^-(qz) = \Lambda_i(z)Q_{i-1}^+(qz)Q_{i+1}^+(z)\,,$$
$$\xi_i Q_{i+1}^+(qz)Q_{i,i+1}^-(z) - \xi_{i+2}Q_{i+1}^+(z)Q_{i,i+1}^-(qz) = \Lambda_{i+1}(z)Q_i^-(qz)Q_{i+2}^+(z)$$
$$\cdots\cdots$$

q-Oper condition $v(qz)^{-1}A(z) = Zv(z)^{-1}$

Diagonalizing element

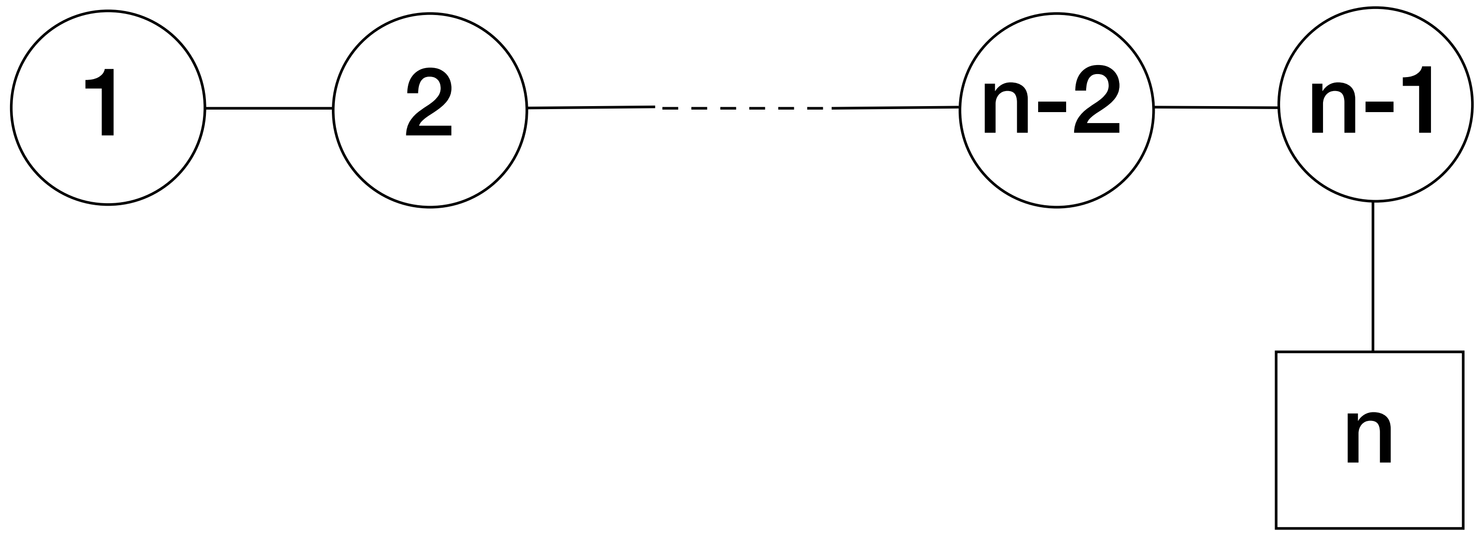
Polynomials $Q^-_{i,\ldots,j}(z)$

form extended QQ-system

$$v(z)^{-1} = \begin{pmatrix} \frac{1}{Q_1^+(z)} & \frac{Q_1^-(z)}{Q_2^+(z)} & \frac{Q_{12}^-(z)}{Q_3^+(z)} & \ldots & \frac{Q^-_{1,\ldots,r-1}(z)}{Q_r^+(z)} & Q^-_{1,\ldots,r}(z) \\ 0 & \frac{Q_1^+(z)}{Q_2^+(z)} & \frac{Q_2^-(z)}{Q_3^+(z)} & \ldots & \frac{Q^-_{2,\ldots,r-1}(z)}{Q_r^+(z)} & Q^-_{2,\ldots,r}(z) \\ 0 & 0 & \frac{Q_2^+(z)}{Q_3^+(z)} & \ldots & \frac{Q^-_{3,\ldots,r-1}(z)}{Q_r^+(z)} & Q^-_{3,\ldots,r}(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \ldots & \ldots & \frac{Q^+_{r-1}(z)}{Q_r^+(z)} & Q_r^-(z) \\ 0 & \ldots & \ldots & \ldots & 0 & Q_r^+(z) \end{pmatrix}$$

Quantum/Classical Duality

Consider T*G/B



Construct the corresponding space of (SL(N),q)-opers

Nondegeneracy defines the section of $\mathscr{L}_1$

$$s_1(z) = z - p_1, \quad , \ldots , \quad s_{k+r}(z) = z - p_{k+l}$$

$$p_{k+l+1-p} = -\frac{Q^+_p(0)}{Q^+_{p-1}(0)}$$

The space of functions on the space of q-opers

$$\mathrm{Fun}(\hbar\mathrm{Op})(F\mathbb{F}l_L)) \cong \frac{\mathbb{C}(\{\xi_i\},\{a_i\},\{p_i\},\hbar)}{\{H_i(\{p_j\},\{\xi_j\},\hbar) = e_i(a_1,\ldots,a_L\})_{i=1,\ldots,L}}$$

is described by trigonometric Ruijsenaars-Schneider model with n particles

$$H_k = \sum_{\substack{\mathcal{I}\subset\{1,\ldots,L\}\\ |\mathcal{I}|=k}} \prod_{\substack{i\in\mathcal{I}\\ j\notin\mathcal{I}}} \frac{\xi_i - \hbar\,\xi_j}{\xi_i - \xi_j} \prod_{m\in\mathcal{I}} p_m$$

n-particle tCM from q-opers

The QQ-system $$\xi_{i+1}Q_i^+(z+\epsilon)Q_i^-(z)-\xi_iQ_i^+(z)Q_i^-(z+\epsilon)=(\xi_{i+1}-\xi_i)\Lambda_i(z)Q_{i-1}(z)Q_{i+1}(z)$$

Theorem: Qs can be represented using *twisted* Wronskians

$$Q_j^+(z)=\frac{det\Big(M_{1,\dots,j}\Big)}{det\Big(V_{1,\dots,j}\Big)}\,,\qquad Q_j^-(z)=\frac{det\Big(M_{1,\dots,j-1,j+1}\Big)}{det\Big(V_{1,\dots,j-1,j+1}\Big)}$$

$$M_{i_1,\dots,i_j}(z)=\begin{bmatrix} s_{i_1}(z) & \xi_{i_1}s_{i_1}(z+\epsilon) & \cdots & \xi_{i_1}^{j-1}s_{i_1}(z+\epsilon(j-1)) \\ \vdots & \vdots & \ddots & \vdots \\ s_{i_j}(z) & \xi_{i_j}s_{i_j}(z+\epsilon) & \cdots & \xi_{i_j}^{j-1}s_{i_j}(z+\epsilon(j-1)) \end{bmatrix}$$

$$V_{i_1,\dots,i_j}=\begin{bmatrix} 1 & \xi_{i_1} & \cdots & \xi_{i_1}^{j-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \xi_{i_j} & \cdots & \xi_{i_j}^{j-1} \end{bmatrix}$$

The QQ-system is equivalent to the Desnanot-Jacobi-Lewis Carrol identity

$$\det M_1^1\,\det M_{k+1}^2-\det M_{k+1}^1\,\det M_1^2=\det M_{1,k+1}^{1,2}\,\det M$$

n-particle tCM from q-opers cont'd

Theorem: Let the last $\Lambda_i(z)$ in the QQ-system $P(z) = \prod_{i=1}^{n}(z-a_i)$

Let the (SL(n+1),q)-oper be non degenerate, meaning that all polynomials $s_i(z) = z - p_i$

then $P(z) = det\Big(z - m\Big)$ where m is the tCM Lax matrix

Proof:

$$P(z) = \frac{\det\Big(M_{1,\dots,n}\Big)(z)}{\det\Big(V_{1,\dots,n}\Big)}$$

where

$$\left(M_{1,\dots,n}\right)_{i,j} = \xi_i^{j-1}(z - p_i + (j-1)\epsilon) \qquad \left(V_{1,\dots,n}\right)_{ij} = \xi_i^{j-1}$$

So $$P(z) = \det\Big(z - M_{1,\dots,n}(0)V^{-1}_{1,\dots,n}\Big)$$

tCM Lax matrix is a product

$$-M_{1,\dots,n}(0)V^{-1}_{1,\dots,n} = m$$

Diagonal compts

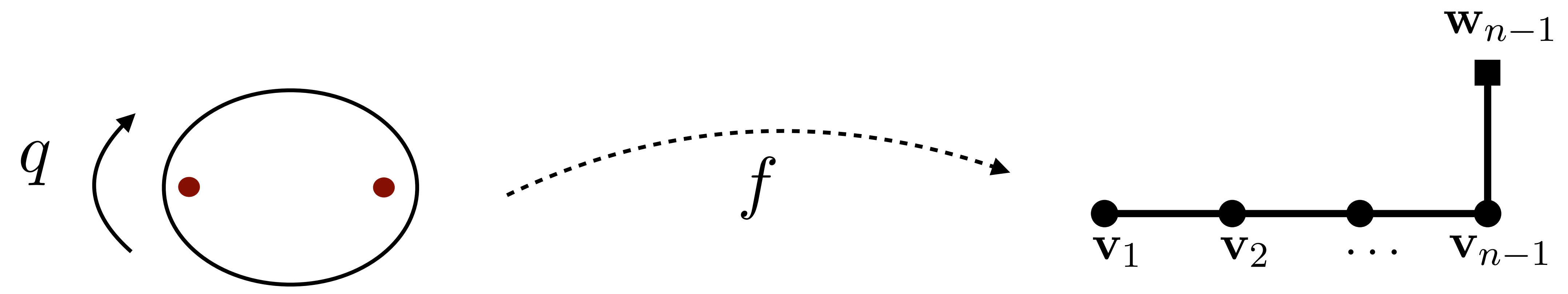
$$m_{ii} = p_i - \epsilon\xi_i \sum_{k\neq i} \frac{1}{\xi_i - \xi_k}$$

Off diagonal compts

$$m_{ij} = \frac{\epsilon\xi_i}{\xi_i - \xi_j} \frac{\prod_{k\neq i}(\xi_i - \xi_k)}{\prod_{k\neq j}(\xi_j - \xi_k)}$$

Enumerative AG/Integrable Systems

Quantum equivariant K-theory of Nakajima quiver varieties



$$A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$$

$$\mathrm{V}^{(\tau)}(\boldsymbol{z}) = \sum_{\boldsymbol{d}} \mathrm{ev}_{p_2,*}(\widehat{\mathcal{O}}^{\boldsymbol{d}}_{\mathrm{vir}} \otimes \tau|_{p_1}, \mathsf{QM}^{\boldsymbol{d}}_{\mathrm{nonsing}\, p_2})\boldsymbol{z}^{\boldsymbol{d}} \in K_{\mathsf{T}\times\mathbb{C}^\times_q}(X)_{loc}[[\boldsymbol{z}]]$$

Saddle point limit yields Bethe equations for XXZ

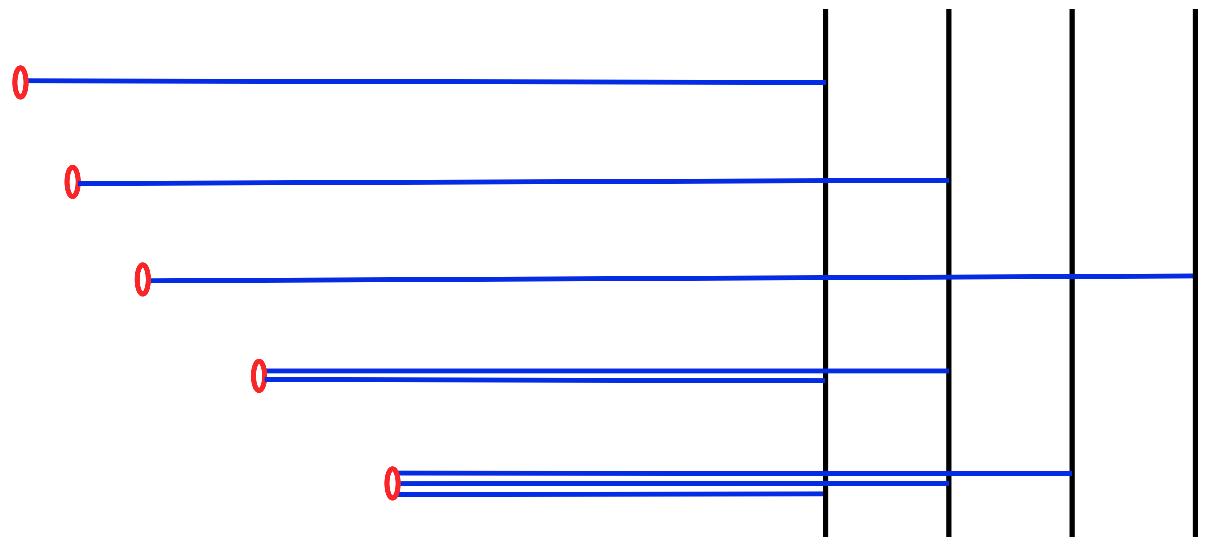
Quantum classes satisfy interesting difference equations in equivariant parameters and Kahler parameters ***qKZ, Dynamical equation*** [Okounkov, Smirnov]

After symmetrization they can be rewritten as eigenvalue equations for ***trigonometric Ruijsenaars-Schneider (tRS)*** system [PK, Zeitlin] [PK]

$$T_r(\mathbf{a}) = \sum_{\substack{\mathcal{I}\subset\{1,\dots,n\}\\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I}\\ j\notin\mathcal{I}}} \frac{t\, a_i - a_j}{a_i - a_j} \prod_{i\in\mathcal{I}} p_i$$

$$T_r(\boldsymbol{a})\mathrm{V}(\boldsymbol{a},\vec{\zeta}) = S_r(\vec{\zeta},t)\mathrm{V}(\boldsymbol{a},\vec{\zeta})$$

In terms of string/gauge theory tRS eigenproblem is Ward identity [Gaiotto, PK] [Bullimore, Kim, PK]



II. (G,q)-Connection

G-simple simply-connected complex Lie group

Principal G-bundle $\mathcal{F}_G$ over $\mathbb{P}^1$

$$M_q : \mathbb{P}^1 \to \mathbb{P}^1$$
$$z \mapsto qz$$

A meromorphic **(G,q)-connection** on $\mathcal{F}_G$ is a section A of $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_G, \mathcal{F}_G^q)$

U-Zariski open dense set

Choose U so that the restriction $\mathcal{F}_G|_U$ of $\mathcal{F}_G$ to U is isomorphic to a trivial G-bundle

$A(z) \in G(\mathbb{C}(z))$ on $U \cap M_q^{-1}(U)$

Change of trivialization $A(z) \mapsto g(qz)A(z)g(z)^{-1}$

(G,q)-Opers

A meromorphic (G,q)-oper on $\mathbb{P}^1$ is a triple $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$

A is a meromorphic (G,q)-connection

$\mathcal{F}_{B_-}$ is a reduction of $\mathcal{F}_G$ to B_-

Oper condition: Restriction of the connection on some Zariski open dense set U

$$A : \mathcal{F}_G \longrightarrow \mathcal{F}_G^q \text{ to } U \cap M_q^{-1}(U)$$

takes values in the double Bruhat cell

$$B_-(\mathbb{C}[U \cap M_q^{-1}(U)])cB_-(\mathbb{C}[U \cap M_q^{-1}(U)])$$

Coxeter element: $c = \prod_i s_i$

Locally

$$A(z) = n'(z)\prod_i(\phi_i(z)^{\check{\alpha}_i}\, s_i)n(z)$$

$\phi_i(z) \in \mathbb{C}(z)$ and $n(z), n'(z) \in N_-(z)$

Miura (G,q)-Opers

Definition: A *Miura* (G,q)-*oper* on $\mathbb{P}^1$ is a quadruple $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$, where $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ is a meromorphic (G,q)-oper on $\mathbb{P}^1$ and $\mathcal{F}_{B_+}$ is a reduction of the G-bundle $\mathcal{F}_G$ to B_+ that is preserved by the q-connection A.

Choose a trivialization $\mathcal{F}_{G,x} \simeq G$ under this isomorphism
$$\begin{aligned} \mathcal{F}_{B_-,x} &\simeq aB_- \subset G \\ \mathcal{F}_{B_+,x} &\simeq bB_+ \end{aligned}$$

Then $a^{-1}b$ is a well defined element of the double quotient of $B\_\backslash B/B_+ \simeq W_G$

Flags $\mathcal{F}_{B\_}$ and $\mathcal{F}_{B_+}$ are in *generic relative position* at $x \in X$ if the corresponding element of the Weyl group assigned to them at x is equal to 1 or $a^{-1}b \in B\_ \cdot B_+$

Structure Theorems

Theorem 1: *For any Miura (G, q)-oper on $\mathbb{P}^1$, there exists a trivialization of the underlying G-bundle $\mathcal{F}_G$ on an open dense subset of $\mathbb{P}^1$ for which the oper q-connection has the form*

$$A(z) \in N_-(z) \prod_i ((\phi_i(z)^{\check{\alpha}_i} s_i) N_-(z) \ \cap \ B_+(z).$$

Theorem 2: *Let F be any field, and fix $\lambda_i \in F^\times, i = 1, \ldots, r$. Then every element of the set $N_- \prod_i \lambda_i^{\check{\alpha}_i} s_i N_- \ \cap \ B_+$ can be written in the form*

$$\prod_i g_i^{\check{\alpha}_i} e^{\frac{\lambda_i t_i}{g_i} e_i}, \qquad g_i \in F^\times,$$

where each $t_i \in F^\times$ is determined by the lifting s_i.

Adding Singularities and Twists

Consider family of polynomials $\{\Lambda_i(z)\}_{i=1,\dots,r}$

(G,q)-oper with regular singularities can be written as

$$A(z) = n'(z)\prod_i (\Lambda_i(z)^{\check{\alpha}_i}\, s_i)n(z), \qquad n(z), n'(z) \in N_-(z)$$

Using structure theorem every Miura (G,q)-oper with singularities reads

$$A(z) = \prod_i g_i(z)^{\check{\alpha}_i}\, e^{\frac{\Lambda_i(z)}{g_i(z)} e_i}, \qquad g_i(z) \in \mathbb{C}(z)^\times$$

(G,q)-oper is **Z-twisted** if it is equivalent to a constant element of G $Z \in H \subset H(z)$ Z is regular semisimple. There are W_G Miura (G,q)-opers for each (G,q)-opers

$$A(z) = g(qz)Zg(z)^{-1}$$

Z-twisted Miura (G,q)-oper if gauge transform is from Borel

$$A(z) = v(qz)Zv(z)^{-1}, \qquad v(z) \in B_+(z)$$

Plücker Relations

V_i^+ irrep of G with highest weight ω_i Line $L_i \subset V_i$ stable under B_+

Plucker relations: for two integral dominant weights $L_{\lambda+\mu} \subset V_{\lambda+\mu}$ is the image of $L_\lambda \otimes L_\mu \subset V_\lambda \otimes V_\mu$

under canonical projection $V_\lambda \otimes V_\mu \longrightarrow V_{\lambda+\mu}$

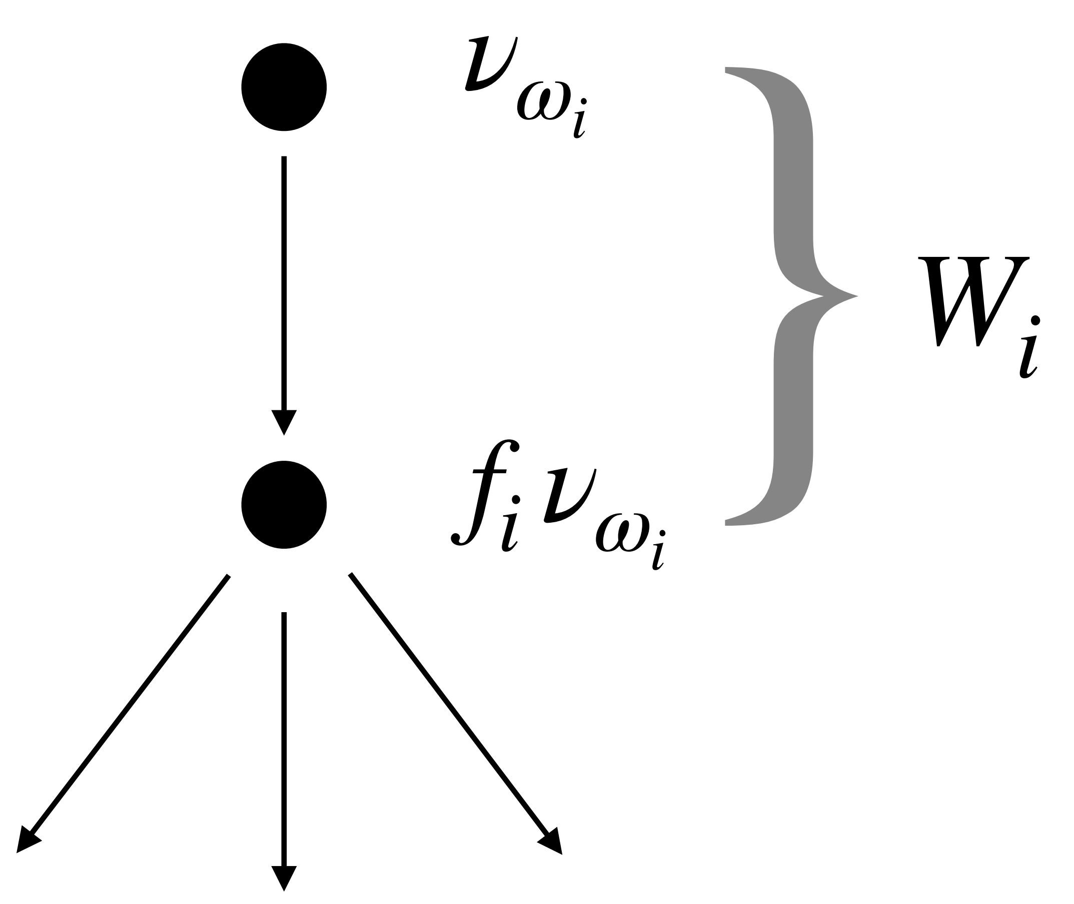
Conversely, for a collection of lines $L_\lambda \subset V_\lambda$ satisfying Plucker relations $\exists B \subset G$ such that L_λ is stabilized by B for all λ

A choice of B is equivalent to a choice of B_+-torsor in G

Let ν_{ω_i} be a generator of the line $L_i \subset V_i$. This is a vector of weight ω_i wrt $H \subset B_+$

The subspace of V_i of weight $\omega_i - \alpha_i$ is one-dimensional and spanned $f_i \cdot \nu_{\omega_i}$

Thus the 2d subspace spanned by $\{\nu_{\omega_i}, f_i \cdot \nu_{\omega_i}\}$ is a B_+-invariant subspace of V_i



Miura-Plücker (G,q)-Opers

let $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$ be a Miura (G,q)-oper with regular singularities $\{\Lambda_i(z)\}_{i=1,\dots,r}$

Associated vector bundle $\mathcal{V}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} V_i = \mathcal{F}_G \underset{G}{\times} V_i$ contains rank-two subbundle $\mathcal{W}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} W_i$

associated to $W_i \subset V_i$, and $\mathcal{W}_i$ in turn contains a line subbundle $\mathcal{L}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} L_i$

Using structure theorems we obtain **r** Miura (GL(2),q)-opers

$$A_i(z) = \begin{pmatrix} g_i(z) & \Lambda_i(z) \prod_{j>i} g_j(z)^{-a_{ji}} \\ 0 & g_i^{-1}(z) \prod_{j\neq i} g_j(z)^{-a_{ji}} \end{pmatrix}$$

Z-twisted Miura-Plücker (G,q)-oper is meromorphic Miura (G,q)-oper on P1 such that for each Miura (GL(2),q)-oper

$$A_i(z) = v(zq) Z v(z)^{-1}|_{W_i} = v_i(zq) Z_i v_i(z)^{-1}$$

$$\text{where } v_i(z) = v(z)|_{W_i} \text{ and } Z_i = Z|_{W_i}$$

QQ-System

Theorem: *There is a one-to-one correspondence between the set of nondegenerate Z-twisted Miura-Plücker (G,q)-opers and the set of nondegenerate polynomial solutions of the QQ-system*

$$\widetilde{\xi}_i Q^i_-(z) Q^i_+(qz) - \xi_i Q^i_-(qz) Q^i_+(z) =$$
$$\Lambda_i(z) \prod_{j>i} \left[Q^j_+(qz)\right]^{-a_{ji}} \prod_{j<i} \left[Q^j_+(z)\right]^{-a_{ji}}, \qquad i = 1, \dots, r,$$

$$\widetilde{\xi}_i = \zeta_i \prod_{j>i} \zeta_j^{a_{ji}}, \qquad \xi_i = \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}}$$

Proof uses $$v(z) = \prod_{i=1}^{r} y_i(z)^{\check{\alpha}_i} \prod_{i=1}^{r} e^{-\frac{Q^i_-(z)}{Q^i_+(z)} e_i} \dots, \qquad g_i(z) = \zeta_i \frac{Q^i_+(qz)}{Q^i_+(z)}$$

XXZ Bethe Ansatz Equations for G

roots of Q+

$$\frac{Q^i_+(qw^k_i)}{Q^i_+(q^{-1}w^k_i)}\prod_j \zeta_j^{a_{ji}} = -\frac{\Lambda_i(w^i_k)\prod_{j>i}\left[Q^j_+(qw^i_k)\right]^{-a_{ji}}\prod_{j<i}\left[Q^j_+(w^i_k)\right]^{-a_{ji}}}{\Lambda_i(q^{-1}w^i_k)\prod_{j>i}\left[Q^j_+(w^i_k)\right]^{-a_{ji}}\prod_{j<i}\left[Q^j_+(q^{-1}w^i_k)\right]^{-a_{ji}}}$$

Space of nondegenerate solutions of QQ-system for G

Nondegenerate ***Z-twisted Miura-Plucker*** (G,q)-opers with regular singularities

?

Space of nondegenerate solutions of XXZ for G

?

Nondegenerate ***Z-twisted Miura*** (G,q)-opers with regular singularities

Quantum Bäcklund Transformation

Theorem: Consider the following q-gauge transformation

$$A \mapsto A^{(i)} = e^{\mu_i(qz)f_i} A(z) e^{-\mu_i(z)f_i}, \quad \text{where} \quad \mu_i(z) = \frac{\prod_{j\neq i}\left[Q_+^j(z)\right]^{-a_{ji}}}{Q_+^i(z)Q_-^i(z)}$$

changes the set of Q-functions

$$\begin{aligned} Q_+^j(z) &\mapsto Q_+^j(z), & j &\neq i, \\ Q_+^i(z) &\mapsto Q_-^i(z), & Z &\mapsto s_i(Z) \end{aligned}$$

$$\{\widetilde{Q}_+^j\}_{j=1,\dots,r} = \{Q_+^1, \dots, Q_+^{i-1}, Q_-^i, Q_+^{i+1} \dots, Q_+^r\}$$

$$\{\widetilde{z}_j\}_{j=1,\dots,r} = \{z_1, \dots, z_{i-1}, z_i^{-1} \prod_{j\neq i} z_j^{-a_{ji}}, \dots, z_r\}$$

Now the strategy is to successively apply Bäcklund transformations according to the reduced decomposition of the element of the Weyl group

Consider longest element $w_0 = s_{i_1} \dots s_{i_\ell}$

Theorem: Every Z-twisted Miura-Plucker (G,q)-oper is Z-twisted Miura (G,q)-oper

The proof based on properties of double Bruhat cells addresses the existence of the diagonalizing element $v(z)$ (to be constructed later)

III. Generalized Wronskians

Consider big cell in Bruhat decomposition

$$G_0 = N_- H N_+$$

$$g = n_- \; h \; n_+$$

V_i^+ irrep of G with highest weight ω_i

$$h\nu_{\omega_i}^+ = [h]^{\omega_i} \nu_{\omega_i}^+$$

Define **principal minors** for group element g

For SL(N) they are standard minors of matrices

$$\Delta^{\omega_i}(g) = [h]^{\omega_i}, \quad i = 1, \ldots, r$$

Then **generalized minors** are regular functions on G

$$\Delta_{u\omega_i, v\omega_i}(g) = \Delta^{\omega_i}(\tilde{u}^{-1} g \tilde{v}) \qquad u, v \in W_G$$

Proposition 4.5. *For a W-generic Z-twisted Miura-Plücker (G, q)-oper with q-connection $A(z) = v(qz)Zv(z)^{-1}$, where $v(z) \in B_-(z)$ we have the following relation:*

$$\Delta_{w \cdot \omega_i, \omega_i}(v^{-1}(z)) = Q_+^{w,i}(z) \tag{4.5}$$

for any $w \in W$.

Lewis Carroll Identity

In Type A FZ relation reduces to

$$\Delta_{u\omega_i,v\omega_i}\Delta_{us_i\omega_i,vs_i\omega_i} - \Delta_{us_i\omega_i,v\omega_i}\Delta_{u\omega_i,vs_i\omega_i} = \Delta_{u\omega_{i-1},v\omega_{i-1}}\Delta_{u\omega_{i+1},v\omega_{i+1}}$$

$$M_1^1 M_i^2 - M_i^1 M_1^2 = M_{1i}^{12} M$$

IV. The qDE/IM Correspondence

[Bazhanov, Lukyanov, Zamolodchikov]

[Masoero, Raimondo, Valeri]

Consider Schrödinger equation

$$\Psi_1'' + \left(E - \frac{l(l+1)}{x^2} - x^{2M}\right)\Psi_1 = 0\,.$$

Can be presented in the vector form

$$\Psi_1' + \frac{l}{x}\Psi_1 + \Psi_2 = 0$$

$$\Psi_2' - \frac{l}{x}\Psi_2 + p(x,E)\Psi_1 = 0$$

Or as an ***affine*** $\mathfrak{sl}_2$ oper

$$\mathcal{L}\psi = 0, \text{ where } \psi = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} \qquad \mathcal{L} = \partial_x + \frac{l\check{\alpha}}{x} + e + p(x,E)f$$

$$p(x,E) = x^{2M} - E$$

Symmetries

$$\Lambda : x \mapsto x,\ E \mapsto E,\ l \mapsto -l-1$$

$$\Omega : x \mapsto qx,\ E \mapsto q^{-2}E,\ l \mapsto l$$

Ω is realized as shift of Kac-Moody loop parameter t

Solutions of the ODE

Solution of the Schrödinger equation at ∞

$$\chi(x,E,l) \sim x^{-\frac{M}{2}} \exp\left(-\frac{x^{1+M}}{1+M} + \dots\right)$$

Act with Ω to get another solution

$$\chi^{+}(x,E,l) = \chi(x,E,l)$$

$$\chi^{-}(x,E,l) = iq^{-\frac{1}{2}}\chi(qx,q^{-2}E,l)$$

Solutions around $x=0$

$$\psi(x,E,l) \sim ax^{l+1} + O(x^{l+3})$$

Act with Λ to get a different basis

$$\psi^{+}(x,E) = \psi(x,E,l)$$

$$\psi^{-}(x,E) = \psi(x,E,-l-1)$$

Wronskian

$$W[\psi^{+},\psi^{-}] = 2i(q^{l+\frac{1}{2}} - q^{-l-\frac{1}{2}})$$

Discrete values of energy arise when $\psi^{\pm} \to 0$ as $x \to \infty$

Spectral determinants

$$D^{\pm}(E,l) = \prod_{n=1}^{\infty}\left(1 - \frac{E}{E_{\pm}}\right)$$

The ODE/IM Correspondence

Expand ψ in χ basis $\quad \psi^+ = C(E,l)\chi^+ + D(E,l)\chi^-$

Act with Ω, Λ

$$\Lambda\psi^\pm = \psi^\mp \qquad \Lambda\chi^\pm = \chi^\pm$$

$$\Omega\psi^\pm = q^{\frac{1}{2}\pm l \pm \frac{1}{2}}\psi^\pm \qquad \Omega\chi^+ = -iq^{\frac{1}{2}}\chi^- \qquad \Omega\chi^- = -iq^{\frac{1}{2}}\chi^+ + u\chi^-$$

Thus

$$C(E,l) = -iq^{-l-\frac{1}{2}}D(q^{-2}E,l)$$

$$\psi^- = D(E,-l-1)\chi^- - iq^{l+\frac{1}{2}}D(q^{-2}E,-l-1)\chi^+$$

Wronskian yields the **QQ-system** $\quad q^{l+\frac{1}{2}}D(q^E,l)D(E,-l-1) - q^{-l-\frac{1}{2}}D(E,l)D(q^2E,-l-1) = q^{l+\frac{1}{2}} - q^{-l-\frac{1}{2}}$

ODE/IM: Monodromy of solutions around $x = 0$ $\longleftrightarrow$ QQ-system

The qDE/IM Correspondence

[PK, Frenkel, Zeitlin]

Affine $\mathfrak{g}$ oper on formal disk $\longleftrightarrow$ (G, q)-oper on projective line

Theorem 5.6. *1)In case if $M \in \mathbb{Z}_+$ $(q^{M+1} = 1)$, the monodromy matrix is represented by regular semisimple element $Z^{(M+1)h^\vee}$ in the basis of φ_0^{i,v_s}:*

$$\varphi_k^{i,v_s}(e^{2\pi i}x, E) = (-1)^{2\langle\rho^\vee,\omega_i\rangle}\varphi_k^{i,Z^{(M+1)h^\vee}v_s}(x, E), \tag{5.7}$$

2)For $\Psi^{(i)}_{-k+1/2}(x, E)$ solutions, the monodromy operator can be expressed as follows:

$$\Psi^{(i)}_{-k+1/2}(e^{2\pi i}x, E) = (-1)^{2\langle\rho^\vee,\omega_i\rangle}W_e(x, E)Z^{(M+1)h^\vee}W_e^{-1}(x, E)\Psi^{(i)}_{-k+1/2}(x, E) \tag{5.8}$$

3) The Monodromy matrix is conjugated to the following operator

$$\Big[A(q^M E)A(q^{M-1}E)\dots A(E)\Big]^{h^\vee} = v(E)Z^{(M+1)h^\vee}v(E)^{-1}, \tag{5.9}$$

where $A(E) = v(qE)Zv(E)^{-1}$ is the Miura (G, q)-oper connection, defined by the QQ-system.

SL(2) Example

In (5.1) we have $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so

$$\varphi_0^{v_1}(e^{2\pi i}x, E) = \zeta^{M+1}\varphi_0^{v_1}(x, E)\,, \qquad \varphi_0^{v_2}(e^{2\pi i}x, E) = \zeta^{-M-1}\varphi_0^{v_2}(x, E)\,,$$

where $\zeta = \omega^{\frac{l}{2}}$, so the monodromy matrix is Z^{M+1} where $Z = \mathrm{diag}(\zeta\ \zeta^{-1})$.

Let us consider $\Phi^{(i)}$ (5.6). For $G = SL(2)$ then (4.34) reads

$$Z^{-1}\mathscr{W}(qz)v_1 = \mathscr{W}(z)s^{-1}(z)v_1 \tag{5.11}$$

In this case $s^{-1}(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The q-Wronskian in this case reads

$$\mathscr{W}(z) = \begin{pmatrix} Q^+(z) & \zeta^{-1}Q_+(qz) \\ Q^-(z) & \zeta Q^-(qz) \end{pmatrix}\,, \tag{5.12}$$

Let $b(z) = \begin{pmatrix} Q^+(z) \\ Q^-(z) \end{pmatrix}$ then $\mathscr{W}(z)^{-1}b(z) = v_1$. Thus we have from (5.11) that

$$b(qz) = \mathcal{M}(z) \cdot b(z)\,, \tag{5.13}$$

where

$$\mathcal{M}(z) = Z\mathscr{W}(z)s^{-1}(z)\mathscr{W}(z)^{-1} \tag{5.14}$$