The qDE/IM Correspondence

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with E. Frenkel and A. Zeitlin, to appear

Talk at University of Melbourne

Motivation

Quantum Geometry and Integrable Systems — [Okounkov et al] [Pushkar, Zeitlin, Smirnov] [PK, Pushkar, Smirnov, Zeitlin]

BPS/CFT Correspondence — [Nekrasov Shatashvili]

Geometric q-Langlands Correspondence — [Frenkel] [Aganagic, Frenkel, Okounkov]

CFT as Integrable System — [Bazhanov, Lukyanov, Zamolodchikov] [Dorey, Dunning, Tateo]

I. The ODE/IM Correspondence

[Bazhanov, Lukyanov, Zamolodchikov]
[Dorey, Dunning, Tateo]

Consider Schrödinger equation

$$\Psi_1'' + \left(E - \frac{l(l+1)}{x^2} - x^{2M}\right)\Psi_1 = 0\,.$$

Can be presented in the vector form

$$\Psi_1' + \frac{l}{x}\Psi_1 + \Psi_2 = 0$$

$$\Psi_2' - \frac{l}{x}\Psi_2 + p(x,E)\Psi_1 = 0$$

Symmetries

$$\Lambda : x \mapsto x,\, E \mapsto E,\, l \mapsto -l-1$$

$$\Omega : x \mapsto qx,\, E \mapsto q^{-2}E,\, l \mapsto l$$

$$q = e^{\frac{i\pi}{1+M}}$$

Asymptotics at Infinity

Solution of the Schrödinger equation at $x = \infty$

$$\chi(x, E, l) \sim x^{-\frac{M}{2}} \exp\left(-\frac{x^{1+M}}{1+M} + \dots\right)$$

Let $\chi^+(x, E, l) = \chi(x, E, l)$

Wronskian

Act with Ω to get another solution

$$\chi^-(x, E, l) = iq^{-\frac{1}{2}}\chi(qx, q^{-2}E, l)$$

$$W[\chi^+, \chi^-] = 2$$

$$\Omega : x \mapsto qx,\ E \mapsto q^{-2}E,\ l \mapsto l$$

$$q = e^{\frac{i\pi}{1+M}}$$

Asymptotics cont'd

Solutions around $x = 0$ $\quad$ $\psi(x,E,l) \sim a x^{l+1} + O(x^{l+3})$ $\quad$ $\Lambda : x \mapsto x,\ E \mapsto E,\ l \mapsto -l-1$

Act with Λ to get a different basis

$$\psi^+(x,E) = \psi(x,E,l)$$

$$\psi^-(x,E) = \psi(x,E,-l-1)$$

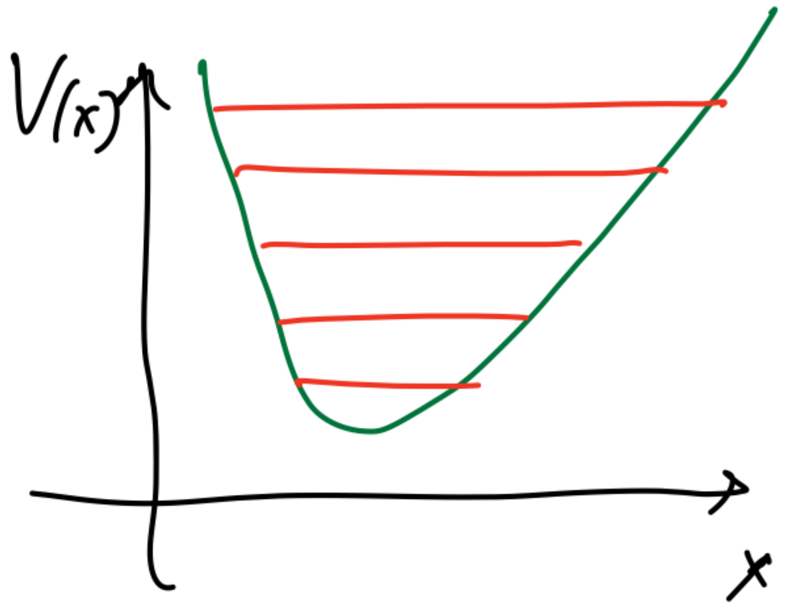
Wronskian

$$W[\psi^+,\psi^-] = 2i(q^{l+\frac{1}{2}} - q^{-l-\frac{1}{2}})$$

Discrete values of energy arise when $\psi^\pm \to 0$ as $x \to \infty$ $\quad$ $\{E_n^\pm\}$

Spectral determinants $\quad$ $D^\pm(E,l) = \prod_{n=1}^{\infty}\left(1 - \frac{E}{E_\pm}\right)$

WKB approximation $\quad$ $E_n^\pm \sim n^{\frac{2M}{1+M}}$



Expansion in χ-basis

Expand ψ in χ basis

$$\psi^+ = C(E,l)\chi^+ + D(E,l)\chi^-$$

Act with Ω, Λ

$$\Lambda\psi^\pm = \psi^\mp \qquad \Lambda\chi^\pm = \chi^\pm$$

$$\Omega\psi^\pm = q^{\frac{1}{2}\pm l \pm \frac{1}{2}}\psi^\pm \qquad \Omega\chi^+ = -iq^{\frac{1}{2}}\chi^- \qquad \Omega\chi^- = -iq^{\frac{1}{2}}\chi^+ + u\chi^-$$

Thus

$$C(E,l) = -iq^{-l-\frac{1}{2}}D(q^{-2}E,l)$$

$$\psi^- = D(E,-l-1)\chi^- - iq^{l+\frac{1}{2}}D(q^{-2}E,-l-1)\chi^+$$

The QQ-System

Wronskian $W[\psi^+,\psi^-] = 2i(q^{l+\frac{1}{2}} - q^{-l-\frac{1}{2}})$

$$q^{l+\frac{1}{2}} D(q^{E}, l) D(E, -l-1) - q^{-l-\frac{1}{2}} D(E, l) D(q^2 E, -l-1) = q^{l+\frac{1}{2}} - q^{-l-\frac{1}{2}}$$

yields the **QQ-system** $\zeta Q^+(q^2E)Q^-(E) - \zeta^{-1}Q^+(E)Q^-(q^2E) = \Delta$

$D(E,l)$ are entire functions — eigenvalues of Baxter Q-operators which appear in the eight-vertex model.

[BLZ] description of $c<1$ CFT as completely integrable theory

QQ-system is ubiquitous to quantum integrable systems (XXX, XXZ, XYZ)

Examples

Consider $M = 1$

$c = -2$ CFT, i.e. the "free fermion" theory

$$E_n^+ = 4n + 2l - 1\,, \qquad n = 1,\, 2\ldots\;,$$

$$D^+(E,l) \qquad = \frac{\Gamma(\frac{3}{4}+\frac{l}{2})\; e^{\mathcal{C}E}}{\Gamma(\frac{3}{4}+\frac{l}{2}-\frac{E}{4})}$$

Consider $M \to \infty$

The Schrödinger potential becomes spherically symmetric rigid well

$$\begin{cases} 0, & \text{if } 0 < x < 1 \\ +\infty, & \text{if } x > 1 \end{cases}$$

$$D^+(E,l) \qquad = \Gamma(l+3/2)\,\left(\sqrt{E}/2\right)^{-l-\frac{1}{2}}\; J_{l+\frac{1}{2}}\left(\sqrt{E}\right)$$

The ODE/IM Correspondence

$$\psi^- = D(E, -l-1)\chi^- - iq^{l+\frac{1}{2}} D(q^{-2}E, -l-1)\chi^+$$

$$\zeta Q^+(q^2 E) Q^-(E) - \zeta^{-1} Q^+(E) Q^-(q^2 E) = \Delta$$

Spectral determinant of the ODE $\longleftrightarrow$ Q-function

Monodromy properties of solutions $\longleftrightarrow$ QQ-system

How can we understand this geometrically?

II. Affine Opers

Study analytic solutions of the following linear problem

$$\Psi_1' + \frac{l}{x}\Psi_1 + \Psi_2 = 0$$
$$\Psi_2' - \frac{l}{x}\Psi_2 + p(x,E)\Psi_1 = 0$$

$$\Psi_1'' + \left(E - \frac{l(l+1)}{x^2} - x^{2M}\right)\Psi_1 = 0\,.$$

Can rewrite it using the following ***affine*** connection

$$\mathcal{L}(x,E) = \partial_x + \frac{\ell}{x} + e + p(x,E)e_0$$

where $$p(x,E) = x^{Mh^\vee} - E, \text{ with } M > 0 \text{ and } E \in \mathbf{C}.$$

Differential Operator

Consider connection $$\mathcal{L} = \partial_x + \frac{\ell}{x} + A(x, E)$$ $A(x, E) \in \mathfrak{g}[x^M, E][[x]]$

Proposition 2.1. *There exist an element $U \in G[[x, x^M, E]]$, such that*

$$U^{-1}\mathcal{L}U = \partial_x + \frac{\ell}{x}$$

and for any finite-dimensional representation V of $\mathfrak{g}$, $U(x, x^M, E)v = v + x\tilde{v}(x)$, where $\tilde{v}(x) \in V[E, x^M][[x]]$.

Allows to find a formal solution

Theorem 2.2. *There following expression $W_g(x, E) = x^{-\ell}U(x)g$, where $g \in G$ and $U \in G[[x, x^M, E]]$ constructed as in Proposition 2.1 gives a formal group-valued solution to the problem $\mathcal{L}W = 0$.*

Canonical Solutions

Corollary 2.3. *In any highest weight representation V, choosing a standard basis $\{v_i\}_{i=1}^{dim(V)}$ accoriding to the weight decomposition, so that $\lambda_i = \mathrm{wt}(v_i)$, there is a family of V-valued solutions of equation $\mathcal{L}\rho = 0$, namely $\{\varphi^{\lambda_i,v_i}(x,E)\}_{i=1}^{dim(V)}$, so that*

$$\varphi^{\lambda_i,v_i}(x,E) = W_e(x,E)v_i = x^{-\langle \ell,\lambda_i\rangle}(v_i + x\tilde{v}_i(x)), \quad i = 1,\ldots,dim(V),$$

where $\tilde{v}_i(x) \in V[E, x^M][[x]]$.

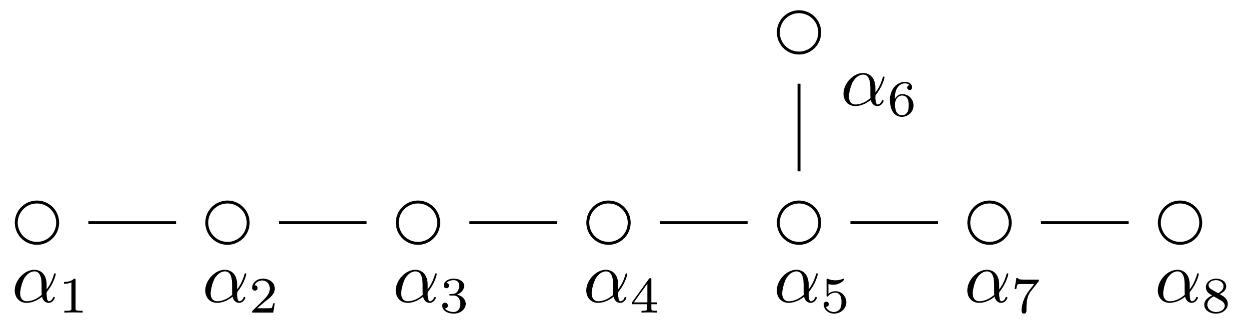
Proposition 2.4. *Any analytic solution $\Psi(x,E)$ in x, E of $\mathcal{L}\Psi(x,E) = 0$ on $x \in D \in \mathbb{C}\backslash\mathbb{R}$ and $E \in \mathbb{C}$ can be decomposed in terms of formal solutions φ^{λ_i,v_i} in the following way:*

$$\Psi(x,E) = \sum_{i=1}^{dim(V)} Q_{v_i}(E)\varphi^{\lambda_i,v_i}(x,E),$$

where $Q_{v_i}(E)$ are analytic functions of E.

Lie-Theoretic Data

ADE-type E_8



Incidence matrix $B = 2\mathbb{1}_n - C$

$$[h_i, h_j] = 0\,, \qquad [h_i, e_j] = C_{ij} e_j\,, \qquad [h_i, f_j] = -C_{ij} f_j\,, \qquad [e_i, f_j] = \delta_{ij} h_i$$

Fundamental weights $\omega_i(h_j) = \delta_{ij}$

Simple roots $\omega_i = \sum_{j\in I} (C^{-1})_{ji} \alpha_j, \qquad i \in I$

Fundamental representations $L(\omega_i),\, i \in I$

Affine Kac-Moody algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$

$$[a \otimes f(t), b \otimes g(t)] = [a, b] \otimes f(t)g(t) + (a|b)\,\mathrm{Res}_{t=0}\,(f'(t)g(t)dt)\,c$$
$$[c, \widehat{\mathfrak{g}}] = 0\,.$$

Evaluation representation: as a vector space we take $V(\zeta) = V$

$$(a \otimes f(t))v = f(\zeta)(av)\,, \qquad c\,v = 0\,, \qquad \text{for } a \in \mathfrak{g}\,, f(t) \in \mathbb{C}[t, t^{-1}]\,, v \in V$$

Twisted Connection

Consider the evaluation representation $V^{(i)}$ corresponding to the highest weight representation V_{ω_i} of $\mathfrak{g}$ with evaluation parameter $t = e^{\pi i p(i)}$

Here $p(i)$ is a homomorphism from the ordered set of vertices of the Dynkin diagram of $\mathfrak{g}$ to $\mathbb{Z}$

Denote $\mathscr{L}_k$ the twisted differential operator by automorphism of $\widehat{\mathfrak{g}}\colon\ t \longrightarrow te^{2\pi i k}$

$$\omega = e^{\frac{2\pi i}{h^\vee (M+1)}}, \quad \Omega = e^{\frac{2\pi i M}{M+1}} = \omega^{h^\vee M}$$

Proposition 3.2. *If $\phi(x, E)$ is the solution of the linear problem $\mathcal{L}(x, E)\phi(x, E) = 0$, then $\phi_k(x, E) = \omega^{-k\rho^\vee}\phi(\omega^k x, \Omega^k E)$ is a solution of linear problem $\mathcal{L}_k(x, E)\phi_k(x, E) = 0$.*

WKB Analysis

Proposition 3.8. *Let $\mathfrak{L}(x,E)$ be the differential operator defined above. Then, we have the following gauge transformation of $\mathfrak{L}$:*

$$q(x,E)^{\mathrm{ad}\rho^\vee}\mathfrak{L}(x,E) = \partial_x + q(x,E)\Lambda + \frac{\ell - M\rho^\vee}{x} + O(x^{-1-\delta}) \tag{3.10}$$

$$p(x,E)^{\frac{1}{h^\vee}} = q(x,E) + O(x^{-1-\delta})$$

$$\Lambda = e_0 + e$$

$$\rho^\vee = \textstyle\sum_{i=1}^r \omega_i^\vee$$

$$[\rho^\vee, e] = e, \quad [\rho^\vee, e_0] = -(h^\vee - 1)e_0$$

Let

$$S(x,E) = \textstyle\int_0^x q(y,E)dy$$

Theorem: There exist a unique solution

$$\Psi(x,E) = e^{-\lambda S(x,E)} q(x,E)^{-h}\big(\psi + o(1)\big), \qquad \textit{in the sector } |\arg x| < \frac{\pi}{2(M+1)} - \delta$$

$$\Lambda\psi = \lambda\psi$$

Family of Solutions

Consider $\Psi_k(x,E) = \omega^{-k\rho^\vee}\Psi(\omega^k x, \Omega^k E)$

Proposition 3.10. *For any* $k \in \mathbb{R}$ *such that* $|k| < \frac{h^\vee(M+1)}{2}$*, on the positive real semi-axis the function* Ψ_k *has the asymptotic behavior*

$$\Psi_k(x,E) = e^{-\lambda\gamma^k S(x,E)} q(x,E)^{-h}\gamma^{-kh}(\psi + o(1))\,, \qquad x \gg 0\,,$$

where $\Lambda\psi = \lambda\psi$ *and* $\gamma = e^{\frac{2\pi i}{h^\vee}}$*. Then, the functions* Ψ_s*,* $s \in \mathbb{Z}, s \in [\frac{1}{2} - \frac{h^\vee}{2}, \frac{h^\vee}{2} + \frac{1}{2}]$*, are solutions in the representation* $V_{\frac{h^\vee}{2}+\frac{1}{2}}$*.*

Ψ-System

Theorem: *Let $\mathfrak{g}$ be a simple Lie algebra of ADE type, and let the solution* $\Psi(x, E)$

Let us denote the solution Ψ corresponding to representation $V = V^{(i)}$ as $\Psi^{(i)}$

$$m_i\left(\Psi^{(i)}_{-1/2}(x,E) \wedge \Psi^{(i)}_{1/2}(x,E)\right) = \bigotimes_{j>i\in I} \Psi^{(j)}(x,E)^{\otimes(-a_{ij})} \bigotimes_{j<i\in I} \Psi^{(j)}(x,E)^{\otimes(-a_{ij})}, \qquad \forall i \in I.$$

Here unique morphism of representations

$$m_i : \bigwedge^2 L(\omega_i) \to \bigotimes_{j\in I} L(\omega_j)^{\otimes B_{ij}}$$

Such that

$$m_i(f_i v_i \wedge v_i) = w_i$$

$$w_i = \otimes_{j\in I} v_j^{\otimes B_{ij}}$$

III. q-Opers

G-simple simply-connected complex Lie group

Principal G-bundle $\mathcal{F}_G$ over D

$$M_q : D \to D$$
$$z \mapsto qz$$

A meromorphic **(G,q)-connection** on $\mathcal{F}_G$ is a section A of $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_G, \mathcal{F}_G^q)$

U-open dense set

Choose U so that the restriction $\mathcal{F}_G|_U$ of $\mathcal{F}_G$ to U is isomorphic to a trivial G-bundle

$A(z) \in G(\mathbb{C}(z))$ on $U \cap M_q^{-1}(U)$

Change of trivialization $A(z) \mapsto g(qz)A(z)g(z)^{-1}$

(G,q)-Opers

A meromorphic (G,q)-oper on disc D is a triple $(\mathscr{F}_G, A, \mathscr{F}_{B_-})$

A is a meromorphic (G,q)-connection

$\mathfrak{F}_{B_-}$ is a reduction of $\mathfrak{F}_G$ to B_-

Oper condition: Restriction of the connection on some open dense set U

$$A : \mathfrak{F}_G \longrightarrow \mathfrak{F}_G^q \text{ to } U \cap M_q^{-1}(U)$$

takes values in the double Bruhat cell

$$B_-(\mathbb{C}[U \cap M_q^{-1}(U)])cB_-(\mathbb{C}[U \cap M_q^{-1}(U)])$$

Coxeter element: $c = \prod_i s_i$

Locally

$$A(z) = n'(z) \prod_i (\phi_i(z)^{\check{\alpha}_i} s_i) n(z)$$

$\phi_i(z) \in \mathbb{C}(z)$ and $n(z), n'(z) \in N_-(z)$

Miura (G,q)-Opers

Definition: A *Miura* (G,q)-*oper* on D is a quadruple $(\mathcal{F}_G, A, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$, where $(\mathcal{F}_G, A, \mathcal{F}_{B_+})$ is a meromorphic (G,q)-oper on D and $\mathcal{F}_{B_-}$ is a reduction of the G-bundle $\mathcal{F}_G$ to B_- that is preserved by the q-connection A.

Choose a trivialization $\mathcal{F}_{G,x} \simeq G$ under this isomorphism
$$\mathcal{F}_{B_-,x} \simeq aB_- \subset G$$
$$\mathcal{F}_{B_+,x} \simeq bB_+$$

Then $a^{-1}b$ is a well defined element of the double quotient of $B_-\backslash B/B_+ \simeq W_G$

Flags $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ are in *generic relative position* at $x \in X$ if the corresponding element of the Weyl group assigned to them at x is equal to 1 or $a^{-1}b \in B_- \cdot B_+$

Structure Theorem

Theorem: *i) Each regular Miura (G,q)-oper on a disk can be represented as an element of* $N_+((z))\prod_i s_i N_+((z)) \,\cap\, B_-((z))$

ii) Elements from the above intersection may be written in the form

$$A(z)=\prod_i g_i(z)^{-\check{\alpha}_i}\, e^{\frac{t_i f_i}{g_i(z)}}, \qquad g_i(z)\in\mathbb{C}((z))^\times, \tag{4.3}$$

where $t_i\in\mathbb{C}^\times$ *are complex parameters corresponding to the lift of* c_i *to* s_i.

Adding Singularities and Twists

Consider family of polynomials $\{\Lambda_i(z)\}_{i=1,\dots,r}$

(G,q)-oper with regular singularities can be written as

$$A(z) = n'(z) \prod_i (\Lambda_i(z)^{\check{\alpha}_i}\, s_i) n(z), \qquad n(z), n'(z) \in N_-(z)$$

Using structure theorem every Miura (G,q)-oper with singularities reads

$$A(z) = \prod_i g_i(z)^{\check{\alpha}_i}\, e^{\frac{\Lambda_i(z)}{g_i(z)} e_i}, \qquad g_i(z) \in \mathbb{C}(z)^\times$$

(G,q)-oper is **Z-twisted** if it is equivalent to a constant element of G $Z \in H \subset H(z)$ Z is regular semisimple. There are W_G Miura (G,q)-opers for each (G,q)-opers

$$A(z) = g(qz) Z g(z)^{-1}$$

Z-twisted Miura (G,q)-oper if gauge transform is from Borel

$$A(z) = v(qz) Z v(z)^{-1}, \qquad v(z) \in B_+(z)$$

Plücker Relations

V_i^+ irrep of G with highest weight ω_i Line $L_i \subset V_i$ stable under B_+

Plucker relations: for two integral dominant weights $L_{\lambda+\mu} \subset V_{\lambda+\mu}$ is the image of $L_\lambda \otimes L_\mu \subset V_\lambda \otimes V_\mu$

under canonical projection $V_\lambda \otimes V_\mu \longrightarrow V_{\lambda+\mu}$

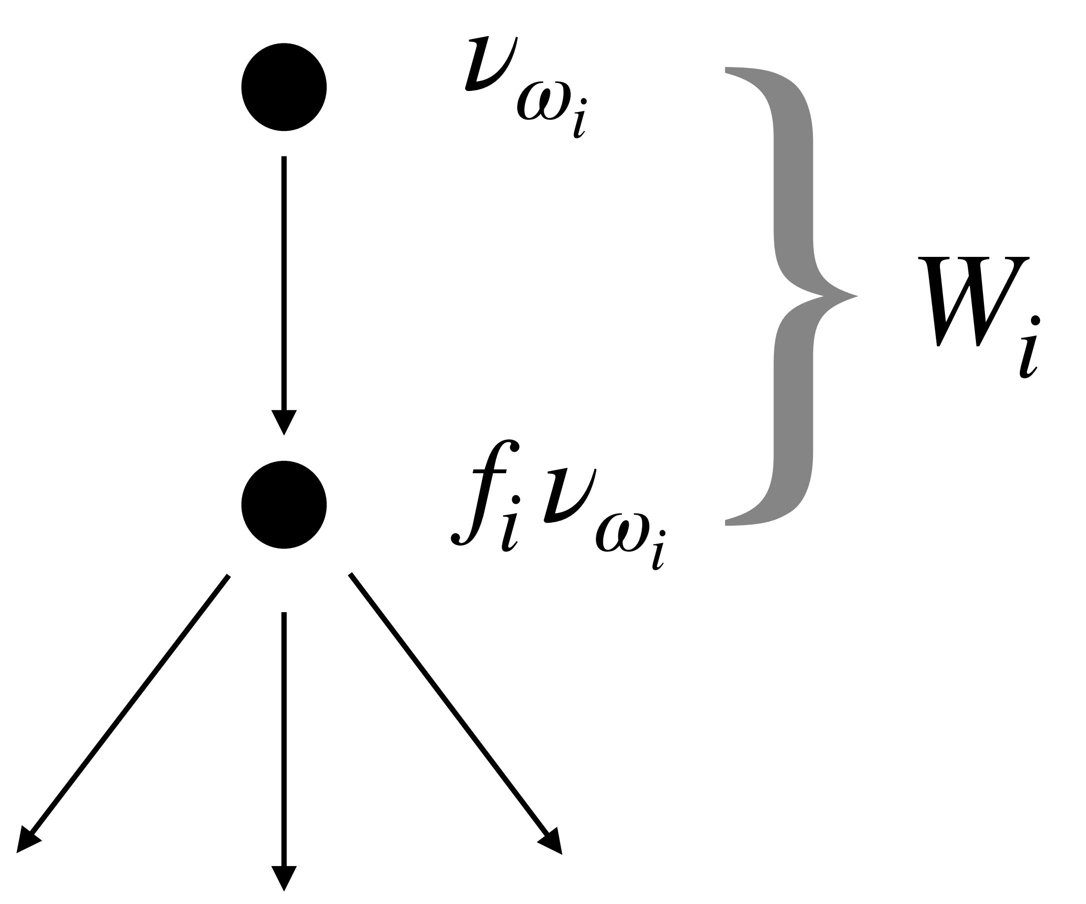
Conversely, for a collection of lines $L_\lambda \subset V_\lambda$ satisfying Plucker relations $\exists B \subset G$ such that L_λ is stabilized by B for all λ

A choice of B is equivalent to a choice of B_+-torsor in G

Let ν_{ω_i} be a generator of the line $L_i \subset V_i$. This is a vector of weight ω_i wrt $H \subset B_+$

The subspace of V_i of weight $\omega_i - \alpha_i$ is one-dimensional and spanned $f_i \cdot \nu_{\omega_i}$

Thus the 2d subspace spanned by $\{\nu_{\omega_i}, f_i \cdot \nu_{\omega_i}\}$ is a B_+-invariant subspace of V_i



Miura-Plücker (G,q)-Opers

let $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$ be a Miura (G,q)-oper with regular singularities $\{\Lambda_i(z)\}_{i=1,\dots,r}$

Associated vector bundle $\mathcal{V}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} V_i = \mathcal{F}_G \underset{G}{\times} V_i$ contains rank-two subbundle $\mathcal{W}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} W_i$

associated to $W_i \subset V_i$, and $\mathcal{W}_i$ in turn contains a line subbundle $\mathcal{L}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} L_i$

Using structure theorems we obtain **r** Miura (GL(2),q)-opers

$$A_i(z) = \begin{pmatrix} g_i(z) & \Lambda_i(z) \prod_{j>i} g_j(z)^{-a_{ji}} \\ 0 & g_i^{-1}(z) \prod_{j\neq i} g_j(z)^{-a_{ji}} \end{pmatrix}$$

Z-twisted Miura-Plücker (G,q)-oper is meromorphic Miura (G,q)-oper on P1 such that for each Miura (GL(2),q)-oper

$$A_i(z) = v(zq)Zv(z)^{-1}|_{W_i} = v_i(zq)Z_i v_i(z)^{-1}$$

$$\text{where } v_i(z) = v(z)|_{W_i} \text{ and } Z_i = Z|_{W_i}$$

QQ-System

Theorem: *There is a one-to-one correspondence between the set of nondegenerate Z-twisted Miura-Plücker (G,q)-opers and the set of nondegenerate polynomial solutions of the QQ-system*

$$\widetilde{\xi}_i Q^i_-(z)Q^i_+(qz) - \xi_i Q^i_-(qz)Q^i_+(z) = \Lambda_i(z)\prod_{j>i}\left[Q^j_+(qz)\right]^{-a_{ji}}\prod_{j<i}\left[Q^j_+(z)\right]^{-a_{ji}}, \qquad i=1,\dots,r,$$

$$\widetilde{\xi}_i = \zeta_i\prod_{j>i}\zeta_j^{a_{ji}}, \qquad \xi_i = \zeta_i^{-1}\prod_{j<i}\zeta_j^{-a_{ji}}$$

Proof uses $$v(z) = \prod_{i=1}^r y_i(z)^{\check{\alpha}_i}\prod_{i=1}^r e^{-\frac{Q^i_-(z)}{Q^i_+(z)}e_i}\dots, \qquad g_i(z) = \zeta_i\frac{Q^i_+(qz)}{Q^i_+(z)}$$

SL(2) Example

$$A(z) = \begin{pmatrix} g(z) & \Lambda(z) \\ 0 & g(z)^{-1} \end{pmatrix}$$

Z-twisted q-oper condition

$$A(z) = v(zq) Z v(z)^{-1}, \qquad Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

Gauge transformation reads

$$v(z) = \begin{pmatrix} y(z) & 0 \\ 0 & y(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\frac{Q_-(z)}{Q_+(z)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y(z) & -y(z)\frac{Q_-(z)}{Q_+(z)} \\ 0 & y(z)^{-1} \end{pmatrix}$$

We find

$$g(z) = \zeta_i y(zq) y(z)^{-1} \qquad \Lambda(z) = y(z)y(zq)\left(\zeta\frac{Q_-(z)}{Q_+(z)} - \zeta^{-1}\frac{Q_-(zq)}{Q_+(zq)}\right)$$

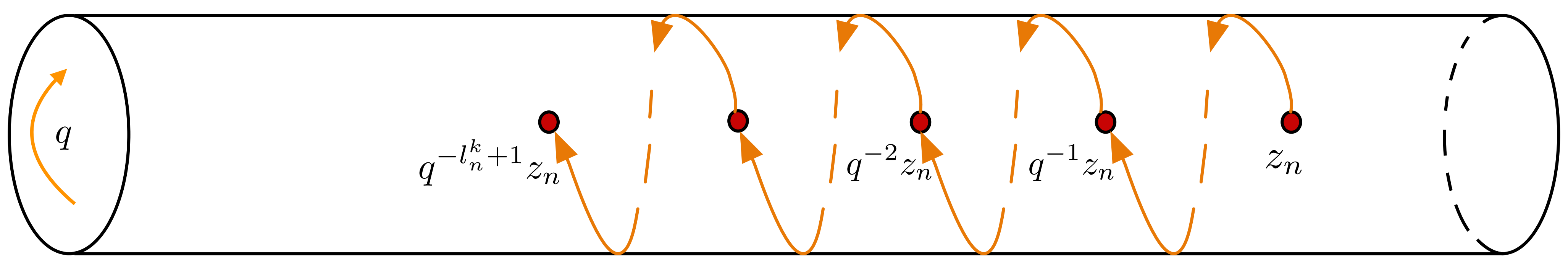
The q-oper condition becomes the **SL(2) QQ-system**

$$\zeta Q_-(z)Q_+(zq) - \zeta^{-1}Q_-(zq)Q_+(z) = \Lambda(z)$$

To get Bethe equations $Q_+(z) = \prod_{k=1}^{m}(z - w_k)$ evaluate at roots of Q

$$\frac{\Lambda(w_k)}{\Lambda(q^{-1}w_k)} = -\zeta^2\frac{Q_+(qw_k)}{Q_+(q^{-1}w_k)}, \qquad k = 1, \dots, m.$$

Singularities $\Lambda(z) = \prod_{p=1}^{L}\prod_{j_p=0}^{r_p-1}(z - q^{-j_p}z_p)$



XXZ Bethe equations

$$q^r\prod_{p=1}^{L}\frac{w_k - q^{1-r_p}z_p}{w_k - qz_p} = -\zeta^2 q^m\prod_{j=1}^{m}\frac{qw_k - w_j}{w_k - qw_j}, \qquad k = 1, \dots, m.$$

XXZ Bethe Ansatz Equations for G

roots of Q+

$$\frac{Q^i_+(q w^k_i)}{Q^i_+(q^{-1} w^k_i)} \prod_j \zeta_j^{a_{ji}} = - \frac{\Lambda_i(w^i_k) \prod_{j>i} \left[Q^j_+(q w^i_k)\right]^{-a_{ji}} \prod_{j<i} \left[Q^j_+(w^i_k)\right]^{-a_{ji}}}{\Lambda_i(q^{-1} w^i_k) \prod_{j>i} \left[Q^j_+(w^i_k)\right]^{-a_{ji}} \prod_{j<i} \left[Q^j_+(q^{-1} w^i_k)\right]^{-a_{ji}}}$$

Space of nondegenerate solutions of QQ-system for G

Nondegenerate ***Z-twisted Miura-Plucker*** (G,q)-opers with regular singularities

?

Space of nondegenerate solutions of XXZ for G

?

Nondegenerate ***Z-twisted Miura*** (G,q)-opers with regular singularities

Quantum Bäcklund Transformation

Theorem: Consider the following q-gauge transformation

$$A \mapsto A^{(i)} = e^{\mu_i(qz) f_i} A(z) e^{-\mu_i(z) f_i}, \quad \text{where} \quad \mu_i(z) = \frac{\prod_{j \neq i} \left[Q_+^j(z) \right]^{-a_{ji}}}{Q_+^i(z) Q_-^i(z)}$$

changes the set of Q-functions

$$Q_+^j(z) \mapsto Q_+^j(z), \qquad j \neq i,$$
$$Q_+^i(z) \mapsto Q_-^i(z), \qquad Z \mapsto s_i(Z)$$

$$\{\widetilde{Q}_+^j\}_{j=1,\dots,r} = \{Q_+^1, \dots, Q_+^{i-1}, Q_-^i, Q_+^{i+1} \dots, Q_+^r\}$$

$$\{\widetilde{z}_j\}_{j=1,\dots,r} = \{z_1, \dots, z_{i-1}, z_i^{-1} \prod_{j \neq i} z_j^{-a_{ji}}, \dots, z_r\}$$

Now the strategy is to successively apply Bäcklund transformations according to the reduced decomposition of the element of the Weyl group

Consider longest element $\quad w_0 = s_{i_1} \dots s_{i_\ell}$

Theorem: Every Z-twisted Miura-Plucker (G,q)-oper is Z-twisted Miura (G,q)-oper

The proof based on properties of double Bruhat cells addresses the existence of the diagonalizing element $v(z)$ (to be constructed later)

Generalized Wronskians

Consider big cell in Bruhat decomposition

$$G_0 = N_- H N_+$$

$$g = n_- \; h \; n_+$$

V_i^+ irrep of G with highest weight ω_i

$$h\nu^+_{\omega_i} = [h]^{\omega_i} \nu^+_{\omega_i}$$

Define **principal minors** for group element g

For SL(N) they are standard minors of matrices

$$\Delta^{\omega_i}(g) = [h]^{\omega_i}, \quad i = 1, \ldots, r$$

Then **generalized minors** are regular functions on G

$$\Delta_{u\omega_i, v\omega_i}(g) = \Delta^{\omega_i}(\tilde{u}^{-1} g \tilde{v}) \qquad u, v \in W_G$$

Proposition 4.5. *For a W-generic Z-twisted Miura-Plücker (G, q)-oper with q-connection $A(z) = v(qz)Zv(z)^{-1}$, where $v(z) \in B_-(z)$ we have the following relation:*

$$\Delta_{w\cdot\omega_i,\omega_i}(v^{-1}(z)) = Q_+^{w,i}(z) \tag{4.5}$$

for any $w \in W$.

Generalized Minors and QQ-system

The set of generalized minors $\{\Delta_{w\cdot\omega_i,\omega_i}\}_{w\in W;i=1,\ldots,r}$ creates a set of coordinates on G/B^+, known as *generalized Plücker coordinates.* In particular, the set of zeroes of each of $\Delta_{w\cdot\omega_i,\omega_i}$ is a uniquely and unambiguously defined hypersurface in G/B.

Proposition *For a W-generic Z-twisted Miura-Plücker (G,q)-oper with q-connection $A(z)=v(qz)Zv(z)^{-1}$, where $v(z)\in B_-(z)$ we have the following relation:*

$$\Delta_{w\cdot\omega_i,\omega_i}(v^{-1}(z)) = Q_+^{w,i}(z)$$

for any $w\in W$.

Proof: Since $\Delta^{\omega_i}(v^{-1}(z)) = Q^i_+(z)$ Diagonalizing gauge transformation $v^{-1}(z) = \prod_{i=1}^r e^{\frac{Q^i_-(z)}{Q^i_+(z)}f_i} \prod_{i=1}^r \left[Q^i_+(z)\right]^{\check{\alpha}_i} \ldots$

$$v^{-1}(z)\nu^+_{\omega_i} = Q^i_+(z)\nu^+_{\omega_i} + Q^i_-(z)f_i\nu^+_{\omega_i} + \ldots$$

Generalized Wronskians

The approach is similar to Miura-Plucker q-Opers

Let $\nu^+_{\omega_i}$ be a generator of the line $L^+_i \subset V^+_i$ $\qquad$ V^+_i irrep of G with highest weight ω_i

The subspace $L^+_{c,i}$ of V_i of weight $c^{-1}\cdot\omega_i$ is one-dimensional and is spanned by $s^{-1}\nu^+_{\omega_i}$

Associated vector bundle $\qquad \mathcal{V}^+_i = \mathcal{F}_{B_+} \underset{B_+}{\times} V^+_i = \mathcal{F}_G \underset{G}{\times} V^+_i$

Contains line subbundles $\qquad \mathcal{L}^+_i = \mathcal{F}_H \underset{H}{\times} L^+_i, \quad \mathcal{L}^+_{c,i} = \mathcal{F}_H \underset{H}{\times} L^+_{c,i}$

Define **generalized Wronskian** as quadruple $(\mathcal{F}_G, \mathcal{F}_{B_+}, \mathscr{G}, Z)$

$\mathscr{G}$ is a meromorphic section of a principle bundle $\mathcal{F}_G$

s.t. for sections $\{v^+_i, v^+_{c,i}\}_{i=1,\dots,r}$ of line bundles $\{\mathcal{L}^+_i, \mathcal{L}^+_{c,i}\}_{i=1,\dots,r}$ on $U \cap M_q^{-1}(U)$

$$\mathscr{G}^q \cdot v^+_i = Z \cdot \mathscr{G} \cdot v^+_{c,i}$$

Adding Singularities

Effectively the above definition means that the Wronskian, written as an element of G(z), satisfies

$$Z^{-1}\mathscr{G}(qz)\ \nu^{+}_{\omega_i} = \mathscr{G}(z)\cdot s_\phi(z)^{-1}\cdot \nu^{+}_{\omega_i}$$

$$s_\phi(z) = \prod_i \phi_i^{-\check{\alpha}_i} s_i$$

Define **generalized Wronskian with regular singularities** if

$$s_\Lambda(z)^{-1} = \prod_i^{\mathrm{inv}} s_i \Lambda_i^{\check{\alpha}_i}$$

q-Opers and q-Wronskians

Theorem 1:

Nondegenerate generalized q-Wronskians with regular singularities $\{\Lambda_i\}_{i=1,\dots,r}$	$\longleftrightarrow$	Nondegenerate Z-twisted Miura (G,q)-opers with regular singularities $\{\Lambda_i\}_{i=1,\dots,r}$

Theorem 2: *For a given Z-twisted (G,q)-Miura oper, there exists a unique generalized q-Wronskian*

$$\mathscr{W}(z) \in B_-(z)w_0B_-(z) \cap B_+(z)w_0B_+(z) \subset G(z),$$

satisfying the system of equations

$$\mathscr{W}(q^{k+1}z)\nu^+_{\omega_i} = Z^k\mathscr{W}(z)s^{-1}(z)s^{-1}(qz)\dots s^{-1}(q^kz)\nu^+_{\omega_i}\,,$$
$$i = 1,\dots,r, \qquad k = 0,1,\dots,h-1,$$

where h is the Coxeter number of G.

Examples: SL(2)

$$\mathscr{W}(qz)\nu_\omega^+ = Z\mathscr{W}(z)s^{-1}(z)\nu_\omega^+$$

$$s^{-1}(z) = \tilde{s}^{-1}\Lambda(z)^{\check{\alpha}} = \begin{pmatrix} 0 & \Lambda(z)^{-1} \\ \Lambda(z) & 0 \end{pmatrix}, \qquad \nu_\omega^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

In terms of Q-polynomials

$$\mathscr{W}(z) = \begin{pmatrix} Q^+(z) & \zeta^{-1}\Lambda(z)^{-1}Q_+(qz) \\ Q^-(z) & \zeta\Lambda(z)^{-1}Q^-(qz) \end{pmatrix}$$

$$\zeta Q^+(z)Q^-(qz) - \zeta^{-1}Q^+(qz)Q^-(z) = \Lambda(z)$$

is equivalent to $\det\mathscr{W}(z) = 1$.

The qDE/IM Correspondence

[PK, Frenkel, Zeitlin]

Proposition 5.1. *i)In any evaluation representation $\bar{V}^i_k$ there is a system of formal solutions of the linear problem*

$$\varphi_k^{i,v_s}(x) = x^{-<\ell,\lambda_s>}(v_s + x\tilde{v}_s(x)),$$

where $\tilde{v}_s(x) \in V[E, x^M][[x]]$. Here $\lambda_s = \mathrm{wt}(v_s)$.
ii) The following transformation properties hold:

$$\omega^{-k\rho^\vee}\varphi_{k'}^{i,v_s}(\omega^k x, \Omega^k E) = \omega^{-k\langle \ell+\rho^\vee,\lambda_s\rangle}\varphi_{k+k'}^{i,v_s}(x)$$

Proposition 5.3. *The solutions $\Psi^i_{-k+1/2}$ have the following form:*

$$\Psi^{(i)}_{-k+1/2}(x,E) = \sum_{s=1}^{dimV} Q^i_{v_s}(q^{-k+1/2}E)\varphi_{1/2}^{i,Z^k v_s}(x,E)$$

Main Theorem

Affine $\mathfrak{g}$ oper on formal disk $\longleftrightarrow$ (G,q)-oper on a disc

Theorem: *1)In case if $M \in \mathbb{Z}_+$ $(q^{M+1}=1)$, the monodromy matrix is represented by regular semisimple element $Z^{(M+1)h^\vee}$ in the basis of φ_0^{i,v_s}:*

$$\varphi_k^{i,v_s}(e^{2\pi i}x,E) = (-1)^{2\langle\rho^\vee,\omega_i\rangle}\varphi_k^{i,Z^{(M+1)h^\vee}v_s}(x,E),$$

2)For $\Psi^{(i)}_{-k+1/2}(x,E)$ solutions, the monodromy operator can be expressed as follows:

$$\Psi^{(i)}_{-k+1/2}(e^{2\pi i}x,E) = (-1)^{2\langle\rho^\vee,\omega_i\rangle}W_e(x,E)Z^{(M+1)h^\vee}W_e^{-1}(x,E)\Psi^{(i)}_{-k+1/2}(x,E)$$

3) The Monodromy matrix is conjugated to the following operator

$$\Big[A(q^ME)A(q^{M-1}E)\ldots A(E)\Big]^{h^\vee} = v(E)Z^{(M+1)h^\vee}v(E)^{-1},$$

where $A(E) = v(qE)Zv(E)^{-1}$ is the Miura (G,q)-oper connection, defined by the QQ-system.