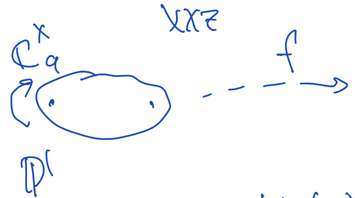
① <u>Motivation</u>

- Enumerative AG [quantum K-theory of Nakajima quiver varieties]
- [Gorsky, K] Quantum/classical duality
  XXZ / trig Ruijsenaars–Schneider (tRS) model
  Bethe
  q-opers
- Geometric Langlands



quantum K-theory

Chern roots S

$1\ 2\ 3$, $z_1\ z_2\ z_2$, $\square\ 4$, $a_1 \ldots a_n$, $\hbar$

$T^*_\hbar Fl_n$

4, 4, $T^*Gr_{k,n}$

$$K(T^*Fl_n) = \frac{\mathbb{C}[\overbrace{S^{\pm1}_{1,a} \ldots S^{\pm1}_{n-1,a}}^{\frac{n(n-1)}{2}},\ a_1^{\pm1} \ldots a_n^{\pm1},\ \hbar^{\pm1},\ z_1^{\pm1} \ldots z_n^{\pm1}]}{(\text{Bethe equs})}$$

XXZ: $e^{\frac{\partial W}{\partial S}} = 1$

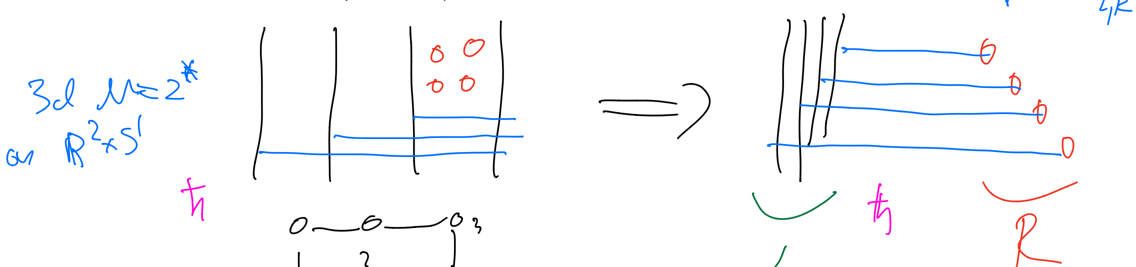
$V \xrightarrow{q\to1} e^{\frac{W}{\log q}}$ — Yang-Yang

[K, Pushkar, Smirnov, Zeitlin]

$$K(T^*Fl) \to \frac{\mathbb{C}[a_1^{\pm1} \ldots a_n^{\pm1},\ \hbar^{\pm1},\ z_1^{\pm1}, \ldots z_n^{\pm1},\ p_1^{\pm1} \ldots p_n^{\pm1}]}{(\text{tRS energy equs})}$$

tRS

$$p_i = \Lambda^i V_i \otimes \Lambda^{i-1} V^*_{i-1} \to \frac{S_{i,1} \cdots S_{i,i}}{S_{i-1,1} \cdots S_{i-1,i-1}}$$



$1\ 2\ 3$, 4

$n=2$

$T^*\mathbb{P}^1$ z, 1, $\square\ 2$, $a_1\ a_2$ $z = \frac{\zeta_1}{\zeta_2}$

$$H_1 = \frac{\zeta_1 - \hbar\zeta_2}{\zeta_1 - \zeta_2} p_1 + \frac{\zeta_2 - \hbar\zeta_1}{\zeta_2 - \zeta_1} p_2$$

$$K(T^*\mathbb{P}^1) = \frac{\mathbb{C}[\zeta_1, \zeta_2, a_1, a_2, \hbar, p_1, p_2]}{(\quad)} \qquad \begin{cases} H_1 = a_1 + a_2 \\ p_1 \cdot p_2 = a_1 \cdot a_2 \end{cases}$$

Let G - simple, simply-connected Lie grp.

q-connection on $\mathbb{P}^1$

$q \in \mathbb{C}^\times$

- principal G-bundle $\mathcal{F}_G$ on $\mathbb{P}^1$
- $M_q : \mathbb{P}^1 \to \mathbb{P}^1$, $\mathcal{F}_G^q$

  $z \mapsto qz$,

Meromorphic (G,q)-connection on $\mathbb{P}^1$ - section A of $Hom_{\mathcal{O}_U}(\mathcal{F}_G, \mathcal{F}_G^q)$

U - open subset of $\mathbb{P}^1$

under change of coordinates $A(z) \longmapsto g(qz) A(z) g(z)^{-1}$

(G,q)-oper on $\mathbb{P}^1$: $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$

oper condition $\quad J_U = U \cap M_q^{-1}(U)$

restriction $A : \mathcal{F}_G \to \mathcal{F}_G^q$ to J_U takes values in

$$B_-(\mathbb{C}[J_U]) \cdot c \cdot B_-(\mathbb{C}[J_U]) \qquad c = \prod_i s_i$$

c — Coxeter element

$\{e_i, f_i, \check{\alpha}_i\}$

$i = 1, \dots, r$

$$A(z) = n'(z) \left(\prod_i \phi_i(z)^{\check{\alpha}_i} s_i \right) \cdot n(z)$$

$$n', \; n(z) \in N_-(z) \qquad N_- = B_-/H$$

Miura (G,q)-opers on $\mathbb{P}^1$: $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$

- $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ - (G,q) oper
- $\mathcal{F}_{B_+}$ preserved by A

Pick x, $\mathcal{F}_{G,x}$ ~ G torsor w/ reductions $\mathcal{F}_{B_-,x}$, $\mathcal{F}_{B_+,x}$

$\mathcal{F}_{G,x} \cong G$, $\quad \mathcal{F}_{B_-,x} \cong a \cdot B_-$, $\quad \mathcal{F}_{B_+,x} \cong b \cdot B_+$

$$a^{-1} \cdot b \in B_- \backslash G / B_+ = W_G$$

generic relative condition at x $\quad$ if $a^{-1} \cdot b \in J \subset \underbrace{B_- \cdot B_+}_{\text{largest Bruhat cell}}$

Structure theorems

Th1: $\forall$ Miura (G,q)-oper s.t. $\mathcal{F}_{B_+}, \mathcal{F}_{B_-}$ are in generic relative position $\forall x \in V \subset \mathbb{P}^1$: if $g(qz)A(z)g(z)^{-1} \in B_+(z)$

then $g(z) \in B_+(z)\cdot N(z)$

Th2:

- $A(z) \in \underbrace{N_-(z)\cdot \prod_i \left(\varphi_i(z)^{\check{\alpha}_i}\cdot s_i\right)\cdot N_-(z) \cap B_+(z)}$
- Any element from $\longrightarrow$ $\prod_i g_i(z)^{\check{\alpha}_i} \exp\left(\frac{\varphi_i(z)}{g_i(z)} t_i \cdot e_i\right)$

  $g_i(z)$ — rational

(G,q)-opers w/ regular singularities

Let $\Lambda_i(z) \in \mathbb{C}[z]$,

roots of Λ - sing. of oper

$$A(z) = n'(z)\cdot \prod_i \left(\Lambda_i^{\check{\alpha}_i}(z)\cdot s_i\right)\cdot n(z)$$

Miura oper: $$A(z) = \prod_i g_i^{\check{\alpha}_i}(z) \exp\left(\frac{\Lambda_i(z)}{g_i(z)} e_i\right)$$

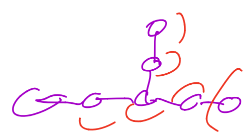
Z-twisted (G,q)-oper:

$$A(z) = g(qz)\, Z\, g^{-1}(z)$$

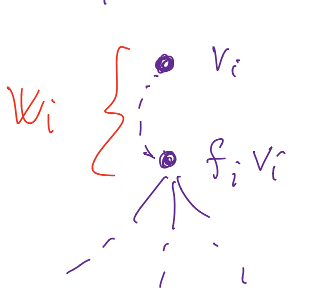
Z - reg. semisimple $\qquad Z = \prod_i z_i^{\check{\alpha}_i}$.

Miura-Plücker (G,q)-oper

$SL(2) \subset G$



V_i - irrep of G, ω_i - weight, highest weight vector v_i



$$A_i(z) = v(qz)\, Z\, v(z)^{-1}\Big|_{W_i} \qquad i = 1, \dots, r$$

$$= \begin{pmatrix} g_i(z) & \Lambda_i(z) \prod_{j>i} g_j(z)^{-a_{ji}} \\ 0 & g_i^{-1}(z)\cdot \prod_{j\neq i} g_j(z)^{-a_{ji}} \end{pmatrix}$$

Th: $MP_qOp \longleftrightarrow QQ$-System

Th: $MP_q Op \subset M_q Op$

Cartan connection $\quad A^H(z) = \prod_i g_i(z)^{\check{\alpha}_i} \qquad A^H \sim Z$

$$A^H(z) = \prod_i y_i(qz)^{\check{\alpha}_i} \cdot Z \cdot y_i(z)^{-\check{\alpha}_i}$$

$$g_i(z) = z_i \frac{y_i(qz)}{y_i(z)}$$

Theorem:

$\left\{ \begin{array}{c} Z\text{-twisted Miura-Plücker} \\ (G,q)\text{-opers} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{set of nondegenerate} \\ \text{polynomial solutions} \\ \text{of } QQ_G\text{-system} \end{array} \right\}$

$$\tilde{\xi}_i Q_i^-(z) Q_i^+(qz) - \xi_i Q_i^-(qz) Q_i^+(z) = \Lambda_i(z) \cdot \prod_{j>i} (Q_j^+(qz))^{-a_{ji}} \prod_{j<i} (Q_j^+(z))^{-a_{ji}}$$

$$\tilde{\xi}_i = z_i \prod_{j>i} z_j^{a_{ji}}$$

$$\xi_i = z_i^{-1} \prod_{j<i} z_j^{-a_{ji}}$$

$$\left[\begin{array}{l} Q_i^+(z) = y_i(z) \\ v(z) = \prod_{i=1}^{r} y_i(z)^{\check{\alpha}_i} \prod_{i=1}^{r} e^{-\frac{Q_i^-(z)}{Q_i^+(z)} e_i} \cdots \end{array} \right]$$

Theorem: $MP(G,q)Op_p \subset M(G,q)Op_p$

Bäcklund transformations:

$$A(z) \longmapsto A^{(i)}(z) = e^{\mu_i(qz) f_i} A(z) e^{-\mu_i(z) f_i}$$

$$\mu_i(z) = \frac{\prod_{j \neq i} (Q_j^+)^{-a_{ji}}}{Q_i^+(z) Q_i^-(z)}$$

results in

$$Q_j^+ \longmapsto Q_j^+, \quad j \neq i$$

$$Q_i^+ \longmapsto Q_i^-, \quad Z \longmapsto s_i(Z)$$

$\left\{ \begin{array}{c} \text{nondegenerate} \\ QQ \text{ system} \end{array} \right\} \longleftrightarrow \{\text{Bethe eqns}\}^{*} \qquad {}^L\hat{\mathfrak{g}}$

Theorem: $w_0 = s_{i_1} \cdots s_{i_\ell}$ – maximal element in W_G

{ nondegenerate Z-twisted }

$\{ W_0\text{-generic } QQ \text{ system} \} \longleftrightarrow \{ \text{Miura } (G,q) \text{ op} \}$

$(SL(2), q)$ $\qquad \mathcal{L} \subset W \qquad \mathcal{L} \to W \to W/\mathcal{L}$

$\bar{A}: \quad \mathcal{L} \xrightarrow{\sim} (W/\mathcal{L})^q$

$S = \begin{pmatrix} Q^-(z) \\ Q^+(z) \end{pmatrix}$

Q^+

$S(qz) \wedge Z\, S(z) = \Lambda(z)$

$\Downarrow$

$\zeta\, Q^+(qz)\, Q^-(z) - \zeta^{-1} Q^+(z)\, Q^-(qz) = \Lambda(z)$

$Q^+ = z - p_1$

$Q^- = \alpha(z - p_2)$

$\Downarrow$

tRS