Discrete Probability

An experiment is a procedure that yields one of possible outcomes. The sample space (space of states) of the experiment is the set of all possible outcomes. An event is a subset of sample space.

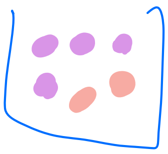
Ex: H , T Sample space $\{H, T\} = S$

$E = \{H\}$ — event

$\{T\}$

Def: If S is a finite nonempty sample space of equally likely outcomes and E is an event which is a subset of S, then the probability of E is

$$p(E) = \frac{|E|}{|S|}$$

Ex: 

$S = \{$ ● ● ● ● ● ● $\}$

$E = \{$ ● ● $\}$

$p($●$) = \frac{2}{2+4} = \frac{1}{3}$

Ex: 2 dice  What's the probability that the sum of numbers is equal to 7 ?

2 \ 1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$|E| = 6$

$|S| = 36$

$p(7) = \frac{1}{6}$

$P(9) = \frac{1}{9}$

Ex: 1 coin

	outcome	(#H, #T)	P
1	H	1	1/2
1	T	1	1/2

2 coins

	outcome	(#H, #T)	P
1	HH	2, 0	1/4
2	HT TH	1, 1	1/2
1	TT	0, 2	1/4

$S = \{HH, HT, TH, TT\}$

$S = \{HH, HT, TH, TT\}$

$|S| = 4$ $\quad E = \{HH\}$

$E' = \{HT\}$ $\quad P(E') = \frac{1}{4}$

$\lvert E \rvert$	3 coins	(#H, #T) $\swarrow E$	P
1	HHH	3, 0	1/8
3	HHT HTH THH	2, 1	3/8
3	TTH THT HTT	1, 2	3/8
1	TTT	0, 3	1/8

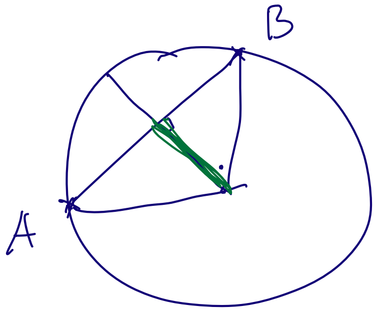
$$P(\underbrace{H \dots H}_{50}) = \frac{1}{2^{50}}$$

Binomial distribution

$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$2^n \leftarrow 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

$$P(kH, (n-k)T) = \frac{\binom{n}{n-k}}{2^n}$$

Probability is hard



continuous probability

Q: $\quad P\left(|AB| < \frac{R}{3}\right) -$

S

not well posed ...

Ex: $\quad$ 52 cards $\quad$ 5 cards. $\quad P$(4 out of 5 of the same kind) - ?

13 kinds

ways $|E|$

$$\underbrace{\binom{13}{1}}_{\text{\# kinds}} \cdot \underbrace{\binom{4}{4}}_{\text{4 of that kind}} \cdot \underbrace{\binom{48}{1}}_{\text{5th card}} = |E|$$

$$|S| = \binom{52}{5}$$

$13 \cdot 1 \cdot 48$

$$p(E) = \frac{\ldots}{\binom{52}{5}} \approx 0.00024\ldots$$

Complements

$E, \quad \bar{E} = S - E$ – complement set to E

Def: $$p(\bar{E}) = \frac{|\bar{E}|}{|S|} = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E)$$

$$p(E) + p(\bar{E}) = 1$$

Ex: A sequence of 10 bits randomly generated.

p (to get at least one 1)

E

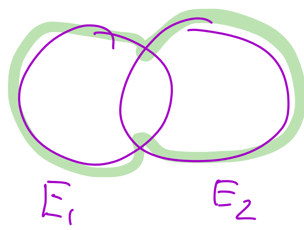
$\bar{E} = \{0 \ldots 0\}$

$$p(\bar{E}) = \frac{1}{2^{10}}$$

$$p(E) = 1 - p(\bar{E}) = 1 - \frac{1}{1024}$$

Theorem: E_1, E_2 – events in sample space S

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$



Proof: $$p(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|}$$

$$= p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Ex: $1 \le n \le 100$. $\quad p(2 | n \vee 5 | n)$

$E_1 \quad 2 | n$ $\qquad E_1 \cap E_2 \quad 2 | n \wedge 5 | n$

$E_2 \quad 5 | n$

$|E_1| = 50, \quad |E_2| = 20, \quad |E_1 \cap E_2| = 10$

$$p(E_1 \cup E_2) = \frac{50 + 20 - 10}{100} = \frac{3}{5}$$

Conditional probability

Ex: Monty Hall 3-door puzzle (car, goat)

Ex: G, B

Two children in the family, one is a boy

p(the other child is a boy?)

$S = \{\underbrace{BB}_{E}, BG, GB, \cancel{GG}\}$

$p(BB) = \frac{1}{4}$

$p = \frac{1}{3}$

Def: Let events E, F, $p(F) > 0$. The conditional probability of E given F is

$$p(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{1/4}{3/4} = \frac{1}{3}$$

$|E| = 1$ $\quad E = BB$

$|F| = 3$ $\quad F = BB, BG, GB$

Def: The events E and F are independent if $P(E \cap F) = P(E) \cdot P(F)$

Ex: E – bit string begins with 1

F – bit string contains even number of 1's

16 bit string

Are E, F independent if the strings of length of 4 are equally likely?

1000	0000
1001	0011
1010	0101
1011	0110
1100	1010
1101	1100
1110	1001
1111	1111
8	8

$p(E) = \frac{8}{16} = \frac{1}{2}$

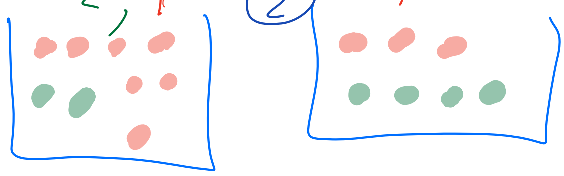
$p(F) = \frac{8}{16} = \frac{1}{2}$

$p(E \cap F) = \frac{4}{16} = \frac{1}{4}$

$p(E) \cdot p(F) = p(E \cap F)$

Bayes' Theorem

Ex: (1) 2, 7 (2) 4, 3



Choice: 1 select a box
2 select a ball

If a red ball is selected what is the probability that it came from the first box?

E red $\qquad F$ first box $\qquad p(F|E)$

$\bar{E}$ green $\qquad \bar{F}$ second box

Theorem:

$$p(F|E) = \frac{p(E|F)\cdot p(F)}{p(E)}$$

$$p(E) = p(E|F)p(F) + p(E|\bar{F})p(\bar{F})$$

$$p(F) = p(\bar{F}) = \frac{1}{2}$$

$$p(E|F) = \frac{7}{9}, \quad p(E|\bar{F}) = \frac{3}{7}$$

$$p(E) = \frac{7}{9}\cdot\frac{1}{2} + \frac{3}{7}\cdot\frac{1}{2} = \frac{7}{18} + \frac{3}{14} = \frac{76}{126} = \frac{38}{63}$$

$$p(F|E) = \frac{\frac{7}{9}\cdot\frac{1}{2}}{\frac{38}{63}} = \frac{7\cdot \cancel{63}^{7}}{38\cdot\cancel{18}_{2}} = \frac{49}{76}.$$

Proof:

$$p(E|F) = \frac{p(E\cap F)}{p(F)} \quad \text{- conditional probability}$$

$$p(F|E) = \frac{p(E\cap F)}{p(E)} \quad \Rightarrow \quad p(F)\,p(E|F) = p(E)\,p(F|E)$$

so

$$p(F|E) = \frac{p(E|F)\,p(F)}{p(E)}$$

Find $p(E)$

Let S - sample set $\qquad E, F \subset S \qquad \bar{F} = S - F$

$$F \cup \bar{F} = S$$

$$E = E\cap S = E\cap(F\cup\bar{F})$$

$$= (E\cap F)\cup(E\cap\bar{F})$$

disjoint since $F\cap\bar{F} = 0$

So $p(E) = p(E \cap F) + p(E \cap \bar{F})$

$$= p(E|F)\,p(F) + p(E|\bar{F})\,p(\bar{F})$$

Generalized Bayes' theorem. Let $E \subset S$ be an event in the sample space, $F_1, F_2, \dots, F_n$ – mutually exclusive events $(F_i \cap F_j = \emptyset)$ and $\bigcup_{i=1}^{n} F_i = S$. Assume $p(E) \neq 0$, $p(F_i) \neq 0$ $\forall i$, then

$$p(F_i|E) = \frac{p(E|F_i)\cdot p(F_i)}{\sum_{i=1}^{n} p(E|F_i)\,p(F_i)}$$

Prove as an exercise!

Bayes spam filters: Flagged word "Apple Watch" in 250 out of 2000 messages known to be spam and 5 in 1000 are known not to be spam. Filter is set at 90%. Will a message clear the filter?

$$p(\text{Apple Watch}) = \frac{250}{2000} = 0.125 \quad \text{– spam probability}$$

E

$$q(\text{Apple Watch}) = \frac{5}{1000} = 0.005 \quad \text{– not spam probability}$$

$\bar{E}$

Assume F – event that an incoming message is spam

$\bar{F}$ – not spam $\qquad p(F) = p(\bar{F}) = \frac{1}{2}$

$$r(\text{Apple Watch}) = \frac{p(E|F)\cdot p(F)}{p(E|F)\,p(F) + p(E|\bar{F})\,p(\bar{F})}$$

$$= \frac{p(\text{Apple Watch})\cdot\frac{1}{2}}{p(\text{Apple Watch})\,\frac{1}{2} + q(\text{Apple Watch})\,\frac{1}{2}} = \frac{0.125}{0.125+0.005}$$

$$= \frac{125}{130} > 0.9$$

Def: A random variable is a function from sample space S to reals

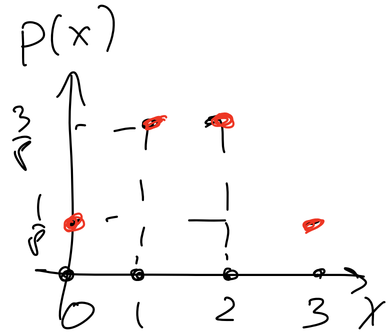
$$S \to \mathbb{R}$$

$$S = \{H, T\} \to \{0, 1\}$$

E

3 coins

$\lvert E \rvert$		$(\#H, \#T)$	P	X – random variable
1	HHH	3, 0	1/8	3
3	HHT HTH THH	2, 1	3/8	2
3	TTH THT HTT	1, 2	3/8	1
1	TTT	0, 3	1/8	0



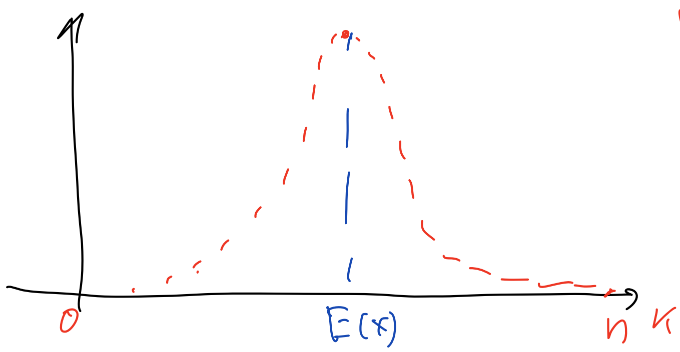
Def: A distribution of random variable X on sample space S is $(r, p(X=r))$, $r \in X(S)$, $p(X=r)$ – probability that X takes value r.

The Binomial distribution

$$B(p, n, k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$n \to \infty$?

$$p_1 + p_2 + \dots + p_n = 1$$

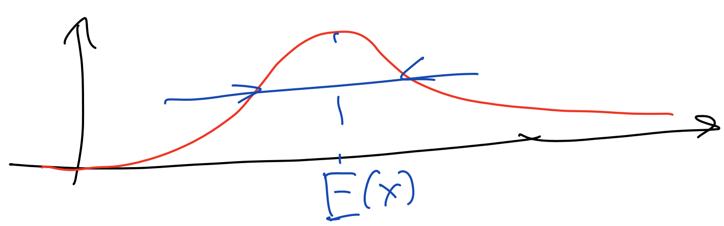


Normal distribution

$$p(x, x_0, \sigma) \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-x_0)^2}{2\sigma}}$$

$$\int_{-\infty}^{+\infty} p(x, x_0, \sigma)\, dx = 1$$

σ – standard deviation



The central limit theorem

Def: The <u>expectation value</u> (expected value) of the random variable X on the sample space S is

$$\left(\bar{X} = \langle X \rangle = \right) E(X) = \sum_{s \in S} p(s) \cdot X(s)$$

· The <u>deviation</u> of X at $s \in S$ is $X(s) - E(X)$

Ex: roll one dice

X - number on the dice

$$E(X) = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6 = \frac{21}{6}$$

$$= \frac{7}{2} = 3.5$$

Ex: 3 coin tosses. X - total number of heads

$$E(X) = \frac{1}{8}\Big(\overset{3}{X(HHH)} + \overset{2}{X(HHT)} + \overset{2}{X(HTH)} + \overset{2}{X(THT)} + \overset{1}{X(TTH)} + \overset{1}{X(THT)} + \overset{1}{X(HTT)} + \overset{0}{X(TTT)} \Big)$$

$$= \frac{12}{8} = \frac{3}{2}$$

$$= \frac{1}{8} \underbrace{X(3H)}_{=3} + \frac{3}{8} \underbrace{X(2H, 1T)}_{=2} + \frac{3}{8} \underbrace{X(1H, 2T)}_{=1} + \frac{1}{8} \underbrace{X(3T)}_{=0}$$

Theorem: If X is a random variable, $p(X=r) = \sum_{\substack{s \in S \\ X(s = r)}} p(s)$

Then

$$E(X) = \sum_{r \in X(S)} p(X=r) \cdot r$$

Follows from definition

Ex: 2 dice problem

1/36 1/18

2 \ 1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8

$$E(X) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{18} + 4 \cdot \frac{1}{12}$$

$$+ 5 \cdot \frac{1}{9} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{1}{6}$$

3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$$+ 8\cdot\frac{5}{36} + 9\cdot\frac{1}{9} + 10\cdot\frac{1}{12}$$

$$+ 11\cdot\frac{1}{18} + 12\cdot\frac{1}{36} = \ldots 7$$

Bernoulli trial Successive independent experiments with favorable probability p

Theorem: The expectation value of successes in n Bernoulli trials w/ probability p is equal to $n\cdot p$

Proof: $$E(X) = \sum_{k=1}^{n} k\cdot p(X=k) = \sum_{k=1}^{n} k\binom{n}{k} p^k(1-p)^{n-k}$$

$$= \sum_{k=1}^{n} n\binom{n-1}{k-1} p^k(1-p)^{n-k}$$

$$k\binom{n}{k} = \cancel{k}\,\frac{\overbrace{(n-1)!\cdot n}^{n!}}{\underbrace{(k-1)!\cdot\cancel{k}}_{k!}\cdot(n-k)!}$$

$$= n\,\frac{(n-1)!}{(k-1)!\,(n-k)!} = n\binom{n-1}{k-1}$$

$$= np\sum_{k=1}^{n}\binom{n-1}{\underbrace{k-1}_{\ell}} p^{\overbrace{k-1}^{\ell}}(1-p)^{n-k}$$

$k-1=\ell \quad k=\ell+1$

$k=1 \quad \ell=0$

$$= n.p\,\underbrace{\sum_{\ell=0}^{n-1}\binom{n-1}{\ell} p^{\ell}(1-p)^{n-1-\ell}}_{(p+1-p)^{n-1}=1}$$

$$\sum_{m=0}^{N}\binom{N}{m} a^m b^{N-m} = (a+b)^N$$

$= n.p$

Theorem: Let X_i, $i=1,\ldots,n$ - random variables for sample set S, $a, b \in \mathbb{R}$

then

- $E(X_1+\ldots+X_n) = E(X_1)+\ldots+E(X_n)$
- $E(aX_i+b) = a\,E(X_i)+b$

Proof: $$E(X_1+X_2) = \sum_{s\in S} p(s)\big(X_1(s)+X_2(s)\big) = \sum_{s\in S} p(s)X_1(s)$$

$$+\sum_{s\in S} p(s)X_2(s)$$

$= E(X_1) + E(X_2)$

Ex: 2 dice: $\qquad$ 1 dice $\qquad$ 1 dice

$$E(X_1+X_2) = 7 \qquad E(X_1) = \frac{7}{2} \qquad E(X_2) = \frac{7}{2}$$

$$E(X_1+X_2) = E(X_1) + E(X_2)$$

<u>Def:</u> <u>Variance:</u> Let X be a random variable on S

variance $$V(X) = \sum_{s\in S} (X(s) - E(X))^2 p(s)$$

The standard deviation $\sigma(X) = \sqrt{V(X)}$

<u>Theorem:</u> $V(X) = E(X^2) - E(X)^2$

<u>Proof:</u> $$V(X) = \sum_{s\in S} (X(s) - E(X))^2 p(s)$$

$$= \underbrace{\sum_{s\in S} X(s)^2 p(s)}_{= E(X^2)} - 2E(X) \overbrace{\sum_{s\in S} X(s)\, p(s)}^{E(X)}$$

$$+ E(X)^2 \underbrace{\sum_{s\in S} p(s)}_{= 1}$$

$$= E(X^2) - 2E(X)^2 + E(X)^2$$

$$= E(X^2) - E(X)^2$$

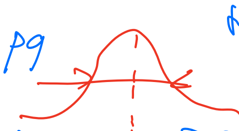
<u>Ex:</u> Bernoulli trial w/ probability p

X: success $\mapsto 1$ $\qquad$ $t \in S = \{\text{success, failure}\}$

failure $\mapsto 0$ $\qquad$ $t^2 = t$

$$E(X^2) = p, \quad E(X) = p \qquad q = 1-p$$

$$V(X) = p - p^2 = p(1-p) = pq$$



for n tosses: $V(X) = npq$

<u>Ex:</u> dice: $X: n \mapsto n$ $\qquad S = \{1, 2, 3, 4, 5, 6\}$

$$E(X) = \frac{7}{2}, \quad (E(X))^2 = \frac{49}{4}$$

$$E(X^2) = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{6}$$

$$V(X) = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{35}{12} \sim 2.9$$

$$\sigma = \sqrt{V(X)} \sim 1.7$$

Ex: 2 dice, $X: (i,j) \mapsto 2i$ $\qquad$ $X: (i,j) \mapsto i+j$ $\qquad$ $V(X) = \frac{35}{6}$, $\sigma = \sqrt{\frac{35}{6}} \sim 2.2$

$2, 4, 6, 8, 10, 12$

$$E(X) = \frac{1}{6}(2+4+6+8+10+12) = 7$$

$$E(X^2) = \frac{1}{6}(2^2+4^2+6^2+8^2+10^2+12^2) = \dots = \frac{182}{3}$$

$$V(X) = E(X^2) - E(X) = \frac{182}{3} - 49 = \frac{35}{3} \sim 12$$

$$\sigma \sim 3.5$$

Theorem: If X, Y are random variables for two independent events then

$$V(X+Y) = V(X) + V(Y)$$

also, if $X_1, X_2, \dots X_n$ are pairwise independent random variables then

$$V\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n V(X_i)$$

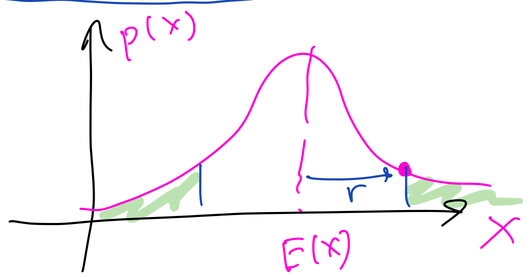
Proof:
$$V(X+Y) = E((X+Y)^2) - (E(X+Y))^2 = E(X^2+2XY+Y^2) - (E(X)+E(Y))^2$$

$$= E(X^2) + \cancel{2E(XY)} + E(Y^2) - E(X)^2 - \cancel{2E(X)E(Y)} - E(Y)^2$$

$V(X)$ $\qquad$ $V(Y)$

$$= V(X) + V(Y)$$

Chebyshev's Inequality



Q: $p(|X(s) - E(X)| \geq r)$?

Theorem: Let X be a random variable on a sample space S, p-probability function, $r > 0$. Then

$$p(|X(s) - E(X)| \geq r) \leq \frac{V(X)}{r^2} = \left(\frac{\sigma}{r}\right)^2$$

Proof: Event $A = \{s \in S \mid |X(s) - E(X)| \geq r\}$

$\geq r^2$

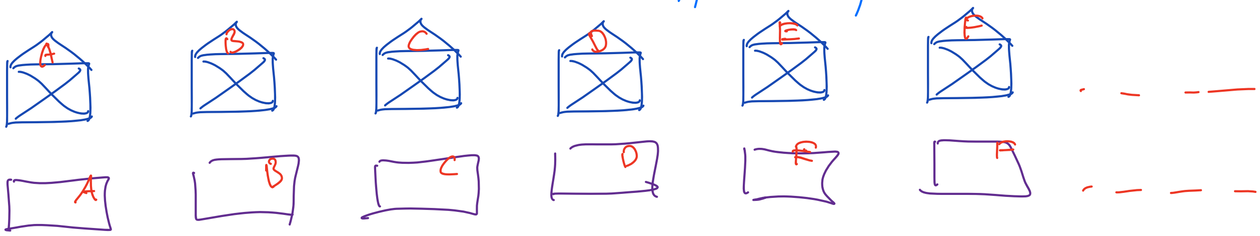
$$V(X) = \sum_{s\in S}\left(X(s)-E(x)\right)p(s) = \sum_{s\in A}\left(X(s)-E(x)\right)p(s) + \overbrace{\sum_{s\in S-A}\left(X(s)-E(x)\right)^2 p(s)}^{\geq 0}$$

$$V(x) \geq r^2\cdot \overbrace{\sum_{s\in A} p(s)}^{P(A)} \Rightarrow P(A) \leq \frac{V(x)}{r^2}$$

Ex: Counting trials. Toss coin n times

$$E(X) = \frac{n}{2} \qquad r = \sqrt{n} \qquad V(X) = \frac{n}{4}$$

$$P\left(\left|X(s) - \frac{n}{2}\right| \geq \sqrt{n}\right) \leq \frac{n/4}{n} = \frac{1}{4}$$



$P(A, B, C, \ldots)$

Ex:

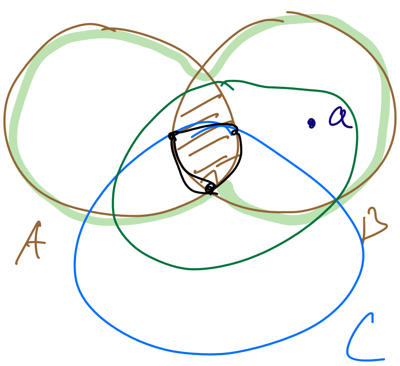
1	2	3	
3	1	2	not ← derangements
2	3	1	
2	1	3	OK
3	2	1	
1	3	2	

$D_3 = 2$

Def: Let $S = \{a_1, \ldots a_n\}$. A permutation of S $\Pi(S) = \{b_1, \ldots b_n\}$ is called derangement of S if $a_i \neq b_i$ $\forall i = 1, \ldots, n$.

D_n – number of derangements.

Small detour to Venn diagrams



$$|A \cup B| = |A| + |B| - |A\cap B|$$

$$|A \cup \underbrace{B \cup C}_{D}| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

Theorem (inclusion - exclusion) Let $A_1 \dots A_n$ - finite sets

Then $$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j|$$

$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \dots$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n|$$

Proof: Suppose a is an element of exactly r sets out of $A_1 \dots A_n$, $1 \le r \le n$. In the first term $\sum_{i=1}^{n} |A_i|$ a is counted $\binom{r}{1} = r$ times

in the second term $-\sum_{i<j} |A_i \cap A_j|$ a is counted $\binom{r}{2} = \frac{r(r-1)}{2}$ times

$\vdots$

in the m'th term a is counted $\binom{r}{m}$ times

So, according to the formula, a is counted

$$B = \binom{r}{1} - \binom{r}{2} + \binom{r}{3} - \dots + (-1)^{r+1} \binom{r}{r} \quad \text{times}$$

Recall:
$$0 = (1-1)^r = \sum_{k=0}^{r} \binom{r}{k} 1^{n-r} \cdot (-1)^r = \binom{r}{0} \underbrace{- \binom{r}{1} + \binom{r}{2} \dots (-1)^r \binom{r}{r}}_{-B}$$

So $B = \binom{r}{0} = 1$

Back to letters. Denote $N(P_1 \dots P_k)$ - # elements w/ properties $P_1 \dots P_k$

Theorem: $$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!}\right) \to \frac{n!}{e}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \dots$$

Let a permutation have property P_i meaning that i'th element is fixed

property $P_i P_j$ — i and j fixed

$P_{i_1} P_{i_2} \dots P_{i_k}$ — $i_1 \dots i_k$ are fixed

$D_n = N(\overline{P_1}\,\overline{P_2} \dots \overline{P_n})$ - number of permutations where no elements are fixed

According to the inclusion - exclusion principle

$$D_n = \underbrace{N}_{\text{all permutation}} - \sum_i N(P_i) + \sum_{i<j} N(P_i P_j) + \cdots + (-1)^{n+1} N(P_1 \ldots P_n)$$

$N = n!$

$N(P_i) = (n-1)! \leftarrow \binom{n}{1}$ ways

$N(P_i P_j) = (n-2)! \leftarrow \binom{n}{2}$ ways

$\vdots$

$N(P_{i_1} \ldots P_{i_m}) = (n-m)! \leftarrow \binom{n}{m}$ ways

So $D_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \cdots + (-1)^{n-1}\binom{n}{n} \cdot 1!$

$$= n! - \frac{n!}{1! \cdot \cancel{(n-1)!}} \cdot \cancel{(n-1)!} + \frac{n!}{2!\ \cancel{(n-2)!}} \cancel{(n-2)!} - \cdots + \frac{n!}{m!\ \cancel{(n-m)!}} \cancel{(n-m)!} + \cdots (-1)^{n+1}$$

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \cdots (-1)^{n+1} \frac{1}{n!}\right)$$

$n = 7$ $n = \infty$

$$P\left(\begin{array}{c}\text{nobody will get} \\ \text{the letter}\end{array}\right) = \frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k!} = 0.367$$

$$= \frac{1}{e} \approx .368$$

$$P\left(\begin{array}{c}\text{someone will} \\ \text{get a letter}\end{array}\right) = 1 - \frac{D_n}{n!} = 1 - \frac{1}{e}$$

Actually, within $\frac{1}{(n+1)!}$ $\frac{D_n}{n!} = \frac{1}{e}$

Consider binomial distribution

$$f(x) = \binom{n}{x} p^x q^{n-x}$$

$q = 1-p$

X - random variable

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \quad \Big| \ p \cdot \frac{d}{dp}$$

$$p \cdot \frac{d}{dp}(p+q)^n = \sum_{k=0}^{\infty} k \binom{n}{k} p^k q^{n-k}$$

$$p \cdot n(p+q)^{n-1} \quad //$$

Plug in $q = 1-p$

$$n\cdot p = \sum_{k=0}^{n} k\cdot f(k) = E(X)$$

$$p^2 \frac{d^2}{dp^2}(p+q)^n = \sum_{k=0}^{n} k(k-1)\binom{n}{k} p^k q^{n-k}$$

$$\parallel$$

$$p^2\cdot n(n-1)(p+q)^{n-2}$$

Plug in $q = 1-p$

$$p^2 n(n-1) = \sum_{k=0}^{n} (k^2-k)\binom{n}{k} p^k q^{n-k} = E(X^2) - \overset{n\cdot p}{\overset{\parallel}{E(X)}}$$

$$p^2n^2 - p^2n = E(x^2) - np$$

$$E(x^2) = p^2n^2 - p^2n + pn$$

$$V(x) = E(x^2) - (E(x))^2 = \cancel{p^2n^2} - p^2n + pn - \cancel{n^2p^2}$$

$$= np\overbrace{(1-p)}^{=q}$$

$$\sigma = \sqrt{np(1-p)}$$

How does the distribution function behaves as $n \to \infty$?

$$f(x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

Stirling approximation $\quad n! = n^n e^{-n}\sqrt{2\pi n}\left(1 + O\left(\frac{1}{n}\right)\right)$

$$f(x) = e^{\log f(x)}$$

$$f(x) = \left(\frac{p}{x}\right)^x \left(\frac{q}{n-x}\right)^{n-x} \cdot n^n \sqrt{\frac{n}{2\pi x(n-x)}}\left(1 + O\left(\frac{1}{n}\right)\right)$$

$$f(x) \to \frac{1}{\sqrt{2\pi npq}}\, e^{-\frac{(x-\overset{E(x)}{np})^2}{2\underset{\sigma^2}{npq}}}$$

Gaussian normal distribution.