Week 7-8 Final: Friday 04/14 (9 hrs) 10:30 am - 1:30 pm 04/14 - 14 min 4/15
Quiz #5 on Thursday 4/6 - next Tuesday
Quiz #6 on Wednesday 4/13
material up to Section 16
Review on Thursday 4/13

Vector Fields (16)

Vector field (field - vector-valued function or mapping)

$\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$

Scalar field

$\varphi: \mathbb{R}^n \to \mathbb{R}$

Ex • Constant field $\vec{v}$: $(x,y) \mapsto \vec{v}$

• Euler vector field $\vec{v}(\vec{x}) = \frac{\vec{x}}{|\vec{x}|}$ (unit vector in direction of $\vec{x}$)

• Magnetic field

• Gradient fields: $\vec{F} = \vec{\nabla}\varphi(x)$, φ - scalar potential

$\vec{F} = \vec{\nabla}\#$ not unique

$\vec{F} = \vec{\nabla} \times \vec{A}$ - vector potential

Integrals over curves

$\int_C f = \lim \sum f(x_i, y_i) \Delta s$

$\int_C f = \int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Ex $\int_C (x+y^2)\, ds$, $C: [0,\pi] \to \mathbb{R}^2$, $t \mapsto (\cos t, \sin t)$

C - smooth (piecewise)

$C = C_1 + C_2$, $\int_C f = \int_{C_1} f + \int_{C_2} f$

Ex $C = C_1 - C_2$

Line integrals over vector fields

$\int_C \vec{F} \overset{def}{=} \int_C \vec{F}(x,y) \cdot \vec{T}(x,y)\, ds$

Ex $C: [0,\pi] \to \mathbb{R}^2$, $t \mapsto (\cos t, \sin t)$

Work: $W = F \cdot L \cos\alpha$

Theorem: $\int_C \vec{F} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|}\, |\vec{r}'(t)|\, dt$

$\vec{F} = \vec{\nabla}\varphi$ - conservative field

Assume C is a closed loop: $\varphi(A) = \varphi(B) \Rightarrow \int_C \vec{F}\, d\vec{r} = 0$

Theorem: If $\vec{F}$ is a vector field, which is continuous on a connected open set D, and if $\int_C \vec{F}\, d\vec{r}$ is independent on $C \subseteq D \Rightarrow \exists\, \varphi(x) \text{ s.t. } \vec{F} = \vec{\nabla}\varphi$

Theorem: Let $C = C(t)$ - smooth curve, $a \le t \le b$, $\vec{F}$ - continuous vector field on C. Then $\int_C \vec{F} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$

Ex $\vec{F} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$, $D = \mathbb{R}^2 \setminus \{0\}$

Theorem: Let $\vec{F} = (P,Q)$ on a simply-connected open set D. Suppose $P_y = Q_x$ for $(x,y) \in D$. Then $\vec{F}$ is conservative.

(16.4) Green's theorem (Stokes's theorem)

$\iint_D (Q_x - P_y)\, dx\, dy = \oint_C P\, dx + Q\, dy$

$C = \partial D$ (boundary of D)

$\oint_{\partial D} \omega = \int_D d\omega$ Stokes' formula

Ex $\int_{\partial D} x^2 dx + xy\, dy$, D: $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$

Ex $P = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) = (P,Q)$

Theorem: $\int_C \vec{F} = \int_{C'} \vec{F}$ if C, C' can be continuously deformed to S^1

Ex $\int_C \vec{F}\, d\vec{r} = \int_0^{2\pi} f(t)\, dt = 2\pi$

Theorem: Let $\vec{F} = (P,Q)$ on a simply-connected open set D. Suppose $P_y = Q_x$ for $(x,y) \in D$. Then $\vec{F}$ is conservative.

Ex $\vec{F} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$, $D = \mathbb{R}^2 \setminus \{0\}$

(16.5) Curl and Divergence

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ - vector field

$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$ (curl)

$\vec{\nabla} = (\partial_x, \partial_y, \partial_z)$

$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F}$

Ex $\vec{F} = (-y, x, 0)$, $\operatorname{curl} \vec{F} = (0,0,2)$

Ex $\vec{E} = \frac{q}{r^3}\vec{r} = \frac{q}{(x^2+y^2+z^2)^{3/2}}(x,y,z)$, $\operatorname{curl} \vec{E} = 0$

Theorem: Let $\vec{F} = \vec{\nabla}\varphi$ for some φ. Then $\operatorname{curl} \vec{F} = 0$

$\vec{\nabla} \times (\vec{\nabla}\varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix} = (\varphi_{yz} - \varphi_{zy}, \varphi_{zx} - \varphi_{xz}, \varphi_{xy} - \varphi_{yx}) = \vec{0}$

$\operatorname{curl}(\vec{\nabla}\varphi) = 0$

Ex $\vec{F} = (xy, yz, -y)$, $\vec{\nabla} \times \vec{F} = (-2y, y, 0 - x) \neq \vec{0}$ - conservative?

Theorem: If $\vec{F}$ has continuous partial derivatives and $\operatorname{curl} \vec{F} = 0$ on $\mathbb{R}^3$, then $\vec{F} = \vec{\nabla}\varphi$ for some φ.

Divergence: $\vec{F} = (F_1, F_2)$ - vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$

$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \sum_{i=1}^{n} \partial_{x_i} F_i$

In $\mathbb{R}^3$: $\vec{F} = (P,Q,R)$, $\operatorname{div} \vec{F} = P_x + Q_y + R_z$

Ex $\vec{E} = \frac{q}{r^3}\vec{r}$, $\operatorname{div} \vec{E} = q\left(\partial_x \frac{x}{(x^2+y^2+z^2)^{3/2}} + \partial_y(\dots) + \partial_z(\dots)\right) = \frac{q}{(x^2+y^2+z^2)^{5/2}}\left(y^2+z^2-2x^2 + x^2-z^2-2y^2 + x^2-y^2-2z^2\right) = 0$

Theorem: $\iiint_B \operatorname{div} \vec{E} = Q$

Theorem: $\vec{F} = (P,Q,R)$, $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = (\vec{\nabla}, \vec{\nabla}, \vec{F}) = (\vec{F}, \vec{\nabla}, \vec{\nabla}) = 0$

$\begin{vmatrix} \partial_x & \partial_y & \partial_z \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = 0$

$\nabla(\operatorname{div}(\vec{\varphi})) = \frac{\partial}{\partial x}\nabla_x + \frac{\partial}{\partial y}\nabla_y + \frac{\partial}{\partial z}\nabla_z = \nabla_{xx} + \nabla_{yy} + \nabla_{zz} = \Delta\nabla\varphi$

Laplace operator $\Delta = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$\operatorname{div} \operatorname{curl} \vec{F} = 0$
$\operatorname{curl} \vec{\nabla}\varphi = 0$
$\operatorname{div} \vec{\nabla}\varphi = \Delta\varphi$

Laplace equation $\Delta\varphi = 0$, $f(x) = \varphi(x,0)$ — Dirac δ-function — $\iint_{\mathbb{R}^2} \delta(x)\delta(y)\, dx\, dy = 1$ — Harmonic function

1d) $\frac{\partial^2}{\partial x^2}\varphi(x) = 0 \Rightarrow \varphi = ax + b$

2d) $\Delta\varphi(x,y) = 0$, $\varphi_{xx} + \varphi_{yy} = 0$, $\varphi = x^3 - 3xy^2$, $\varphi = (x-y)^3$

$\varphi(x,y) = \sin x \cosh y$

Green's theorem in vector form

$\iint_D (Q_x - P_y)\, dx\, dy = \oint_{\partial D} P\, dx + Q\, dy = \oint_{\partial D} \vec{F}\, d\vec{r}$

$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = (0, 0, Q_x - P_y)$

$\iint_D \vec{\nabla} \times \vec{F}\, d\vec{A} = \oint_{\partial D} \vec{F}\, d\vec{r}$, $d\vec{A} = (0,0,dA)$

$\vec{F} = \vec{F}_1 + \vec{F}_2$

$\oint_{\partial D} \vec{F}_1 \cdot d\vec{s}$ - flow of the vector field $\vec{F}$ through the boundary

$\oint_{\partial D} \vec{F}_2 \cdot d\vec{s} = 0$

$\oint_{\partial D} \vec{F} \cdot \vec{n}\, ds = \int_a^b (\vec{F} \cdot \vec{n})(\vec{r}(t))\, |\vec{r}'(t)|\, dt = \int_a^b P(x(t),y(t))\, y'(t)\, dt - Q(x(t),y(t))\, x'(t)\, dt = \oint_{\partial D} P\, dy - Q\, dx = \oint_{\partial D} \vec{H} \cdot d\vec{r} = \iint_D (P_x - (-Q_y))\, dx\, dy = \iint_D \operatorname{div} \vec{F}\, dA$

$\oint_{\partial D} (\vec{F}, \vec{n})\, ds = \iint_D \operatorname{div} \vec{F}\, dx\, dy$ — Maxwell's equations: $\operatorname{div} \vec{E} = \frac{\rho}{\varepsilon_0}$, $\operatorname{div} \vec{B} = 0$

for flow thru ∂D — Integral of the charge density = charge

Grade Exp. — Ex.: 100 %+, A: 90-100, B: 80-90 %, C: 70-80 %, Fund: 70 %

887 % → A-

(16.6) Parametric surfaces

$S = \{\vec{r}(u) = (x(u), y(u), z(u)) \mid (u) = D \subset \mathbb{R}^2\}$

tangent plane at $\vec{r}_0 = (x_0, y_0, z_0)$: $\vec{r}_0 = (x_0, y_0, z_0)$, $\vec{r} = (x, y, z)$, $\vec{n} = \vec{r}_u \times \vec{r}_v$, $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

Ex $\vec{r} = (u^2, u^3, u+v)$, $\vec{r}_u = (1, 1, 1)$, $\vec{r}_v = (2u, 0, 1)$, $\vec{r}_v = (0, 3v^2, 1)$, $\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 0 & 1 \\ 0 & 3v^2 & 1 \end{vmatrix} = (-3v^2, -2u, 6uv)\Big|_{\substack{u=1 \\ v=1}} = (-3, -2, 6)$

$-2(x-1) - 1(y-1) + 6(z-3) = 0$
$-2x - y + 6z = 15$
$x + y - 3z = -5$

Surface area: $S = \{\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v)) \mid (u,v) = D \subset \mathbb{R}^2\}$

$\sum_{ij} A(S_{ij}) \xrightarrow{\Delta u_i, \Delta v_j \to 0} \operatorname{Area}(S)$

$A(S_{ij}) \approx |\vec{r}_u \times \vec{r}_v|(u_i, v_j)\, \Delta u_i \Delta v_j$

$\operatorname{Area}(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\, du\, dv$

Ex Area of S^2 of radius R

$\vec{r}(\varphi,\theta) = (R\sin\varphi\cos\theta, R\sin\varphi\sin\theta, R\cos\varphi)$, $D = [0,\pi] \times [0, 2\pi]$

$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ R\cos\varphi\cos\theta & R\cos\varphi\sin\theta & -R\sin\varphi \\ -R\sin\varphi\sin\theta & R\sin\varphi\cos\theta & 0 \end{vmatrix} = R^2(\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi\cos^2\theta + \sin\varphi\cos\varphi\sin^2\theta)$

$|\vec{r}_\varphi \times \vec{r}_\theta| = R^2\sqrt{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta + \sin^2\varphi\cos^2\varphi} = R^2\sqrt{\sin^2\varphi} = R^2\sin\varphi$ (since $\sin\varphi \ge 0$ on $[0,\pi]$)

$A = \int_0^{2\pi} d\theta \int_0^{\pi} R^2 \sin\varphi\, d\varphi = 2\pi R^2 \int_0^{\pi} \sin\varphi\, d\varphi = 4\pi R^2$

Explicit formula $z = f(x,y)$

$\vec{r}(x,y) = (x, y, f(x,y))$, $\vec{r}_x = (1, 0, f_x)$, $\vec{r}_y = (0, 1, f_y)$, $\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1)$

$A = \iint_S \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$

Ex $z = x^2 + y^2$, $z \le 9$, $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 9\}$

$A = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy = \int_0^{2\pi} d\theta \int_0^3 dr\, r\sqrt{1+4r^2}$

$= \frac{2\pi}{8} \int_1^{37} u^{1/2}\, du = \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2}\Big|_1^{37} = \frac{\pi}{6}(37\sqrt{37} - 1)$ ($u = 1 + 4r^2$, $du = 8r\, dr$; $r=0 \Rightarrow u=1$, $r=3 \Rightarrow u=37$)

(16.7) Surface integrals

$S \subset \mathbb{R}^3$, $f(x,y,z): \mathbb{R}^3 \to \mathbb{R}$

$\int_S f\, dA = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$

After taking the limit: $\sum_{ij} A(S_{ij}) \cdot f(P_{ij}) \xrightarrow{\Delta u_i, \Delta v_j \to 0} \int_S f(x,y,z)\, dA$

Integral of f over the surface S

$\overline{f} = \frac{1}{A(S)} \int_S f\, dA$ (mean)

Ex $\iint_S x^2\, dA$, $S = \{x^2 + y^2 + z^2 = 1\}$, $x = \sin\varphi\cos\theta$, $y = \sin\varphi\sin\theta$, $z = \cos\varphi$, $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$

$|\vec{r}| = 1$

$\iint_S x^2\, dA = \int_0^{2\pi} \sin^2\varphi\cos^2\theta\, \sin\varphi\, d\varphi\, d\theta = \int_0^{2\pi} \cos^2\theta\, d\theta \int_0^{\pi} \sin^3\varphi\, d\varphi = \pi\left(-\cos\varphi + \frac{1}{3}\cos^3\varphi\right)\Big|_0^{\pi} = \pi\left(1 - \frac{1}{3} - (-1 + \frac{1}{3})\right) = \frac{4\pi}{3}$

$\|f\|_S = \int_S g\, dA$, then $\int_S f(x,y,z)\, dA = \int_D f(x,y,g(x,y))\, \sqrt{1 + g_x^2 + g_y^2}\, dx\, dy$

Surface integrals of vector fields

$\vec{F}$ - vector field $\mathbb{R}^3 \to \mathbb{R}^3$

$\int_S \vec{F} \cdot \vec{n}\, dA = \int_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\, |\vec{r}_u \times \vec{r}_v|\, du\, dv = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv = \iint_D (\vec{F}, \vec{r}_u, \vec{r}_v)\, du\, dv$

Ex S: $x^2 + y^2 + z^2 = 1$, $\vec{F} = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$, $\vec{r}_\varphi \times \vec{r}_\theta = (\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi)$

$I = \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \left(\sin^3\varphi\cos^2\theta + \sin^3\varphi\sin^2\theta + \sin\varphi\cos^2\varphi\right)$ ($\sin^3\varphi$ and $\sin\varphi$)

$= 2\pi \int_0^{\pi} \sin^3\varphi\, d\varphi = 2\pi\left(\frac{4}{3} - \frac{2}{3}\right)$