Partial derivatives

Let $f(x,y)$ be a function of $(x,y)\in D$. Then if the limits exist then the are called partial derivatives w.r.t. x and y

$$\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} = \frac{\partial f(x,y)}{\partial x} = \partial_x f(x,y) = f'_x = f_x$$

$$\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h} = \frac{\partial f(x,y)}{\partial y} = \partial_y f(x,y) = f'_y = f_y$$

Ex $f(x,y) = \sin(xy^2)+yx$ $\quad \frac{\partial f}{\partial x} = \cos(xy^2)\cdot y^2 + y$, $\quad \frac{\partial f}{\partial y} = \cos(xy^2)\,2xy + x$

gradient $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

Higher derivatives

$f(x,y)$: $\quad f_x = \frac{\partial f}{\partial x}$, $\quad f_y = \frac{\partial f}{\partial y}$

$(f_x)_x = f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$

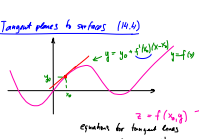
$(f_x)_y = f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

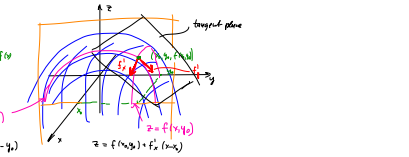
$(f_y)_x = f_{yx} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$(f_y)_y = f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

Th Let $f(x,y)$ is defined on an open disk D and that $f(x,y)$ is continuous on D. If $(a,b)\in D$ then $f_{xy}(a,b) = f_{yx}(a,b)$

Tangent planes to surfaces (H,L)

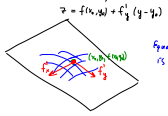




$z = f(x,y)$

Equation for tangent lines

$z = f(x_0,y_0) + f'_y(y-y_0)$



Equation for the tangent plane at (x_0,y_0) is

$$\boxed{z - f(x_0,y_0) = f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)}$$

Ex $z = \sqrt{1-x^2-y^2} = f(x,y)$, tangent plane at $\left(\frac{1}{2},\frac{1}{2}\right)$, $f\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$

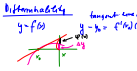
$f'_x = \frac{-2x}{2\sqrt{1-x^2-y^2}} = -\frac{x}{\sqrt{1-x^2-y^2}}$ $\qquad f'_x\left(\frac{1}{2},\frac{1}{2}\right) = f'_y\left(\frac{1}{2},\frac{1}{2}\right) = \frac{-\frac{1}{2}}{\sqrt{1-\frac{1}{2}-\frac{1}{2}}} = -\frac{1}{\sqrt{2}}$

$f'_y = -\frac{y}{\sqrt{1-x^2-y^2}}$

$$z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x-\frac{1}{2}\right) - \frac{1}{\sqrt{2}}\left(y-\frac{1}{2}\right)$$

Differentiability

$y=f(x)$ $\quad$ tangent line: $y-y_0 = f'(x_0)(x-x_0)$



for to function $y-y_0 = f'(x_0)(x-x_0) + \varphi(x)$ $\leftarrow$ $\frac{f''(x_0)}{2!}(x-x_0)^2 + \dots$

$\Rightarrow f \underset{x\to x_0}{\lim}\frac{\varphi(x)}{(x-x_0)} = 0$

Let $z=f(x,y)$. We say that f is differentiable at (x_0,y_0) if for all (x,y) inside a small disc centered at (x_0,y_0)

$$z = f(x_0,y_0) + f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0) + \varphi(x,y)$$

$\Rightarrow$ if $\lim_{(x,y)\to(x_0,y_0)}\frac{\varphi(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$

$\Delta z = f'_x\,\Delta x + f'_y\,\Delta y + \varphi(x,y)$

$\Delta g = g'(x_0)\,\Delta x + \varphi(x)$ $\quad$ holomorphic $\left(\begin{array}{l} z = x^2+y \\ f(x,y) \to g(x) \end{array}\right)$ 145

Theorem If f_x, f_y exist near (x_0,y_0) and continuous at (x_0,y_0) then $f(x,y)$ is differentiable at (x_0,y_0)

$\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$

Ex $f(x,y) = xe^{xy}$ $\quad (x_0,y_0) = (1,0)$ $\quad$ linear approximation at $(1,-0.1)$

$f_x = e^{xy} + xy\,e^{xy}$ $\qquad f_x(1,0) = 1$ $\qquad f(1,0) = 1$

$f_y = x^2 e^{xy}$ $\qquad f_y(1,0) = 1$

tangent plane $\quad z = f_x(x-x_0) + f_y(y-y_0) = x+y$

linear approximation $\quad L(x,y) = x+y = 1 - 0.1 = 1$

true value of the function $\quad f(1,-0.1) = 1.1\cdot e^{-0.11} = 1.1\left(1-0.11+\frac{(0.11)^2}{2}\right)$

$= 0.9812 = 1.1 - 0.121 = 0.98$

Differentials

Notation $\quad dy = f'_x\,dx$

Def $dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

total differential

Ex $z = f(x,y) = x^4 + xy + y^6$ $\qquad dz = (4x^3+y)\,dx + (x+6y^5)\,dy$

Let $z = f(x_1, \dots, x_n)$

$dz = f_{x_1}dx_1 + \dots + f_{x_n}dx_n = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}dx_i$

Chain rule

1-variable $\quad y(x)$, $x(t)$ $\qquad \frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$

Multivariable case: $\quad z = f(x,y)$ – differentiable function, $x = g(t)$, $y = h(t)$ – differentiable function of t

then $\boxed{dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}$

Proof Since $f(x,y)$ is differentiable function at (x_0,y_0) $\quad t=t_0$ $\quad (\Delta x, \Delta y)\to 0$ as $\Delta t\to 0$

$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \underbrace{\varphi_1(x,y)\,\Delta x + \varphi_2(x,y)\,\Delta y}_{\Phi(x,y)}$

$\Rightarrow \lim_{(\Delta x,\Delta y)\to 0}\varphi_1(x,y) = \lim_{(\Delta x,\Delta y)\to 0}\varphi_2(x,y) = 0$

$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varphi_1\frac{\Delta x}{\Delta t} + \varphi_2\frac{\Delta y}{\Delta t}$

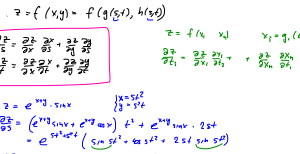
$\Delta t \longrightarrow 0$

$\frac{dz}{dt} = \lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \left(\lim_{\Delta t\to 0}\varphi_1\right)\frac{dx}{dt} + \left(\lim_{\Delta t\to 0}\varphi_2\right)\frac{dy}{dt}$

Ex $z = x^4y + xy^2$, $\quad \begin{cases} x = \sin 2t \\ y = \cos t \end{cases}$ $\qquad \frac{dx}{dt} = 2\cos 2t$, $\quad \frac{dy}{dt} = -\sin t$

$\frac{dz}{dt} = (4x^3y + y^2)\,2\cos 2t + (x^4 + 2xy)(-\sin t)$

$= (4\sin^3 2t\cos t)\,2\cos 2t + (\sin^4 2t + 2\sin t\cos t)(-\sin t)$



Let $z = f(x,y)$ be a differentiable function, $x = g(r,s)$, $y = h(r,s)$ – differentiable functions of r and s

$z = f(g(r,s), h(r,s))$

$$\boxed{\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \end{aligned}}$$

$z = f(x_1, \dots, x_n)$ $\qquad x_i = g_i(t_1, \dots, t_m)$

$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \dots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i} = \sum_{j=1}^{n}\frac{\partial z}{\partial x_j}\frac{\partial x_j}{\partial t_i}$, $\quad i = 1, \dots, m$

Ex $z = e^{x+y}\cdot\sin x$ $\qquad \begin{cases} x = 5t^2 \\ y = 5t^2 \end{cases}$

$\frac{\partial z}{\partial s} = \left(e^{x+y}\sin x + e^{x+y}\cos x\right)\cdot t^2 + e^{x+y}\sin x\cdot 2st$

$= e^{5t^2+5t^2}\left(\sin 5t^2 + \cos 5t^2 + 2s\sin 5t^2\right)$

$\frac{\partial z}{\partial t} = e^{x+y}(\sin x + \cos x)\,2st + e^{x+y}\sin x\cdot s^2$

$= e^{5t^2+5t^2}\left(\sin 5t^2 + \cos 5t^2 + \sin 5t^2\right)s^2$

Ex Let $g(r,s) = f(\overbrace{s^2-r^2}^{x}, \overbrace{2rs}^{y})$, f – differentiable

Show that g satisfies $s\frac{\partial g}{\partial r} + r\frac{\partial g}{\partial s} = 0$

$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}\cdot 2s + \frac{\partial f}{\partial y}(-r)$

$\frac{\partial g}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}(-2r) + \frac{\partial f}{\partial y}(2s)$

0

Implicit differentiation

$F(x,y) = 0$ $\qquad$ Calculate $\frac{dy}{dx} = 0$

$\frac{d}{dx}F(x,y) = 0 \Rightarrow \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$

$F(x,y) = y^5 + x^4 + 7xy$

$y' = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{5x^4+7y}{5y^4+7x}$

$z = f(x,y)$ given implicitly by $F(x,y,f(x,y)) = 0$, $(x,y)\in D$

Calculate partial derivatives f'_x, f'_y

$\frac{\partial}{\partial x}F(x,y,f(x,y)) = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$

$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial f}{\partial x} = 0 \Rightarrow \boxed{f'_x = -\frac{\partial F/\partial x}{\partial F/\partial z}}$

$\frac{\partial}{\partial y}F(x,y,f(x,y)) = \frac{\partial F}{\partial x}\overbrace{\frac{\partial x}{\partial y}}^{0} + \frac{\partial F}{\partial y}\overbrace{\frac{\partial y}{\partial y}}^{1} + \frac{\partial F}{\partial z}f'_y \Rightarrow \boxed{f'_y = -\frac{\partial F/\partial y}{\partial F/\partial z}}$

Ex $F(x,y,z) = \underbrace{xy^2}_{0} + \underbrace{x^3y}_{0} + z^2 - 4 = 0$ $\qquad$ (Marbor cubic)

$z = f(x,y)$ $\qquad 4+2+2-4=0 \Rightarrow z=0$

f'_x, f'_y

$f'_x = -\frac{3x^2+2y^2}{xy+2z}$ $\qquad \theta = \frac{\pi}{2}\Big|_{y=0=0}$ $\qquad u = (\theta, \varphi)$ – (a,θ)

$f'_y = -\frac{3x^2+2xy}{xy+2z}$ $\qquad \boxed{\partial_\theta f = f'_x\,a + f'_y\,b} = 0$

1) $(x,y) = (0,0)$

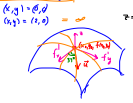
2) $(x,y) = (1,0)$ $\qquad = \infty$

(H,G) Directional derivatives and gradients

$z = f(x,y)$

$f'_x(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$

$f'_y(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0,y_0+h) - f(x_0,y_0)}{h}$



Def The directional derivative of f at (x_0,y_0) in the direction $\vec{u} = (a,b)$

$$\left(D_{\vec{u}}f(x_0,y_0)\right) \to \partial_{\vec{u}}f(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

Theorem If f is differentiable at (x_0,y_0) then the directional derivative exists for any unit vector $\vec{u} = (a,b)$ and

$\partial_{\vec{u}}f(x_0,y_0) = f_x(x_0,y_0)\,a + f_y(x_0,y_0)\,b$ $\qquad |\vec{u}| = 1$

Proof $g(h) = f(\underbrace{x_0+ha}_{x}, \underbrace{y_0+hb}_{y})$

$\partial_{\vec{u}}f(x_0,y_0) = \lim_{h\to 0}\frac{g(h)-g(0)}{h} = g'(0)$

$g' = \frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} = f'_x\,a + f'_y\,b$

$g'(0) = f'_x(x_0,y_0)\underbrace{a}_{\cos\theta} + f'_y(x_0,y_0)\underbrace{b}_{\sin\theta}$

Gradient vector

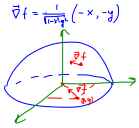
$\partial_{\vec{u}}f = (f'_x, f'_y)\cdot(a,b) = f'_x\,a + f'_y\,b$

$\nabla f(x,y)$ – gradient

Def Let $f(x,y)$ have f'_x, f'_y, then the gradient of f is $\vec{\nabla f} = (f_x, f_y) = f_x\vec{i} + f_y\vec{j}$

$$\boxed{\partial_{\vec{u}}f = (\vec{\nabla} f\cdot\vec{u})}$$

$f(x,y) = \sqrt{1-x^2-y^2}$ $\qquad \begin{cases} f'_x = -\frac{x}{\sqrt{1-x^2-y^2}} \\ f'_y = -\frac{y}{\sqrt{1-x^2-y^2}} \end{cases}$ $\qquad \vec{\nabla} f = \frac{1}{\sqrt{1-x^2-y^2}}(-x, -y)$



Theorem Let f is a differentiable function of two variables. Then the maximum value of $\partial_{\vec{u}}f$ is equal to $|\vec{\nabla} f|$ and it occurs when $\vec{u} \parallel \vec{\nabla} f$

Proof $\partial_{\vec{u}}f = \vec{\nabla} f\cdot\vec{u} = |\vec{\nabla} f|\cdot|\vec{u}|\cos\theta = |\vec{\nabla} f|\cos\theta$

$\partial_{\vec{u}}f$ is maximal if $\theta = 0$

Ex $f(x,y) = xe^{yz}$ $\qquad P(2,0)$

$\vec{\nabla} f = (f_x, f_y) = (e^{yz}, xe^{yz})$, $\quad \vec{\nabla} f(2,0) = (1,2)$

$F(x,y,z) = \frac{\theta D}{1+x^2+2y^2+3z^2}$ $\qquad P(1,1,-2)$ $\quad$ In which direction does the temperature change the most?

$\vec{\nabla} f = \left(F_x, F_y, F_z\right) = \left(\frac{2x}{(1+x^2+2y^2+3z^2)^2}, -\frac{4y}{(\ )^2}, -\frac{6z}{(\ )^2}\right)$

$= -\frac{160}{(1+x^2+2y^2+3z^2)^2}(x, 2y, 3z)\Big|_{\substack{x=1\\y=1\\z=-2}} = -\frac{160}{(1+1+2+12)^2}(1, 2, -6)$

$= -\frac{160}{256}(1,2,-6) = -\frac{5}{8}(1,2,-6) = \frac{5}{8}(-1,-2,6)$

$|\vec{\nabla}| = \frac{5}{8}\sqrt{1+4+36} = \frac{5}{8}\sqrt{41}$

$V(R)$ – scalar potential

$\boxed{\vec{F} = -\vec{\nabla} V(R)}$

Gravity force $\quad M \xrightarrow{\vec{r}} \xleftarrow{\vec{F}} m$ $\qquad$ Potential energy of m: $E_{pot} = V(r)\,m$

$\boxed{\vec{F} = -\frac{G_N M m}{r^2}\cdot\vec{u}}$

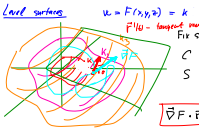
$V(r) = -\frac{G_N M}{r}$

$E_{pot} = -\frac{G_N M m}{r}$

$\frac{\partial V}{\partial r} = \frac{G_N M m}{r^2}$, $\quad \frac{\partial V}{\partial \theta} = 0$

Level surfaces

$u = F(x,y,z) = k$



$\vec{F}'(t)$ – tangent vector to C

For surface S, curve on S, call it C

$C \quad \vec{r}(t) = (x(t), y(t), z(t))$

$S \quad F(x,y,z) = k$

$F(x(t), y(t), z(t)) = k$

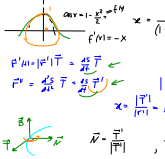
$\boxed{\vec{\nabla} F\cdot\vec{r}'(t) = 0}$ $\qquad \vec{r}_0 = (x_0, y_0, z_0)$, $t=t_0$

$(x-x_0, y-y_0, z-z_0)$

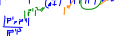
$F_x(x-x_0) + F_y(y-y_0) + F_z(z-z_0) = 0$

Vector functions Review

$\vec{r} = (x(t), y(t), z(t))$ $\qquad a \le t \le b$

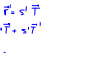


$|\vec{r}'(t)| = v(t)$ $\qquad L = \int_a^b v(t)\,dt$



$\frac{ds}{dt} = v(t) \iff s = \int_a^t v(t)\,dt$

$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ $\qquad \boxed{\varkappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{\vec{T}'(t)}{v(t)}\right| = \frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}}$



$\vec{r}(t) = (x, f(x))$ $\qquad \varkappa = \frac{f''(x)}{(1+(f')^2)^{3/2}}$

$\cos\alpha = \frac{1}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+(f')^2}}$

$f'(x) = -x$ $\qquad \varkappa = \frac{1}{(1+x^2)^{3/2}}\Big|_{x=0} = 1$

$\vec{r}'(t) = |\vec{r}'|\vec{T} = \frac{ds}{dt}\vec{T}$ $\qquad \vec{r}'\times\vec{r}'' = \left(\frac{ds}{dt}\right)^2\vec{T}\times\vec{T}'$

$\vec{r}'' = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}'$ $\qquad |\vec{r}'\times\vec{r}''| = \left(\frac{ds}{dt}\right)^2|\vec{T}\times\vec{T}'| = v^2\varkappa$

$\varkappa = \frac{|\vec{T}'|}{|\vec{r}'|} = \frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3}$

$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}$, $\quad \vec{B} = \vec{T}\times\vec{N}$

Frenet triple

$\vec{r}' = v\vec{T}$

$\vec{r}'' = v'\vec{T} + v\vec{T}'$

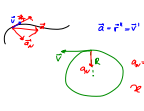
$\vec{r}''' = \ldots$

$\frac{d\vec{T}}{ds} = \varkappa\vec{N}$

$\frac{d\vec{N}}{ds} = -\varkappa\vec{T} + \tau\vec{B}$

$\frac{d\vec{B}}{ds} = -\tau\vec{N}$

$\tau = \frac{(\vec{r}'\times\vec{r}'')\cdot\vec{r}'''}{|\vec{r}'\times\vec{r}''|^2} \sim$ Volume $= 0$



$\vec{a} = \vec{r}'' = \vec{v}'$ $\qquad a_n = \frac{v^2}{R} = \varkappa v^2$

$\vec{v} = v\vec{T}$

$\vec{a} = \vec{v}' = v'\vec{T} + v\vec{T}' = \underbrace{v'\vec{T}}_{a_T} + \underbrace{v\varkappa\vec{N}}_{a_N}$

$\varkappa = \frac{|\vec{T}'|}{v}$ $\qquad \varkappa = \frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3}$

Divergence 7/17

1) Do the limits exist (calculate if they do exist)

a) $\lim_{(x,y)\to(1,1)}\frac{xy-y}{(x-1)^2+y^2}$ $\qquad \frac{y(x-1)}{(x-1)^2+y^2} \sim \begin{cases} 0, & y=0 \\ \frac{y}{x^2+y^2}, & x=y=1 \end{cases}$

b) $\lim_{(x,y)\to(0,0)}\frac{x^3-y^3}{x^2+xy+y^2} = \lim_{(x,y)\to(0,0)}(x-y) = 0$

c) $\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2-y^2+x^2} = 0$

d) $\lim_{(x,y)\to(0,0)}(x^2y)\log(xy) = 0$

2) $\partial_x\partial_y f = x\frac{\partial^2 f}{(\ )}$

- Heat equation

$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$ $\qquad u(x,t)$

For which k $u(x,t) = e^{-a^2k^2t}\sin ax$ is a solution of the heat equation? $\qquad -a^2k^2u = -a^2a^2u$, $\quad a^2 = a^2$, $\quad a = \pm k$

- Laplace equation $\quad u_{xx} + u_{yy} = 0$ $\qquad \Delta u = 0$ $\qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

For which values of a the following functions solve Laplace equation

a) $u = x^2 + axy^2$

d) $u = e^{-x}\cos y + a\,e^{-2}\sin x$

$u = x^3 - 3xy^2$ $\qquad (z^3)' = 3z^2$

$+\begin{cases} u_{xx} = 6x \\ u_{yy} = 6x \end{cases}$ $\Rightarrow a = -3$ $\qquad z = x+iy$

$0 = (6+2a)x$

$z^3 = (x+iy)^3 = x^3 + 3ix^2y + 3i^2xy^2 + i^3y^3$

$= \underbrace{x^3 - 3xy^2}_{u} + i\,(\underbrace{3x^2y - y^3}_{v})$

$z^3 = u + iv$ $\qquad \Delta u = 0$, $\quad \Delta v = 0$ $\qquad$ Harmonic

- Wave equation

$z(x,t)$ $\qquad \frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}$

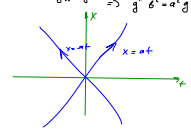
$a \le x \le b$

$c \le t \le d$

Let f', g be twice-differentiable functions

For which values of b $\quad z = f(x+at) + g(x+bt)$ – solution of wave equation

$b = -a$

$z = g(x+bt)$ $\qquad z = f_R(x-at) + f_L(x+at)$

$z'_t = g'\cdot b$, $\quad z''_{tt} = g''\cdot b^2$ $\qquad g_+ = g'$, $\quad g_- = g''$ $\Rightarrow g''\,b^2 = a^2 g''$ $\Rightarrow b = \pm a$ $\quad$ left moving, right moving