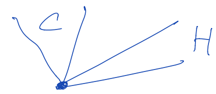
with A. Zeitlin

- **3d mirror symmetry.** Physics origins: 3d $\mathcal{N}=4$ gauge theories. Moduli spaces of vacua:

Coulomb branch	Higgs branch
Vect mult.	hyper mult.
receives quantum corrections	is classical

[BFN]

C_A, H_A $\xleftrightarrow{\text{3d mirror}}$ C_B, H_B



C, H hyper Kähler cones

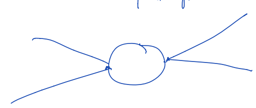
$C = (\mathbb{C}\times\mathbb{C}^\times)^{rk\,G}/W_G$

Ex: 2 hypers $U(1)$ gauge group

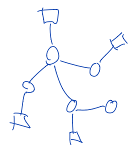
$C \simeq H \simeq \mathbb{C}^2/\mathbb{Z}_2 \qquad uv = w^2$

branches admit deformations and resolutions
mass — FI parameters

$H = \mathrm{Rep}(v,w)/G$



- **Quiver varieties and quantum K-theory**



$Y_{v,w}$, $G = \prod_i GL(v_i) \subset \mathrm{Rep}(v,w)$

$\mu : T^*\mathrm{Rep}(v,w) \to \mathrm{Lie}(G)^*$

$Y_{v,w} = \mu^{-1}(0)/\!/_\theta G = \mu^{-1}(0)_s/G$

$\prod_i GL(w_i)\times\mathbb{C}^\times_\hbar \subset Y$

$K_T(Y)$ – tensorial polynomials of tautological bundles $\mathcal{V}_i, \mathcal{W}_i$ associate G-bundles to reps V_i, W_i

K-theory classes: diagonal operators acting in the basis of fixed pts of $K_T^{loc}(Y)$

$K_T^q(Y)$ – deformation [PSZ] [KPSZ]

Def: 3d Mirror

$Y, Y^!$ are 3d mirror to each other if $\boxed{K_T^q(Y_{v,w}) \simeq K_T^q(Y^!_{v^!,w^!})}$

s.t. $\exists$ 1–1 correspondence between

$A\times T \leftrightarrow T^!\times A^!$

$\{\xi_i\} \leftrightarrow \{a_i^!\}$

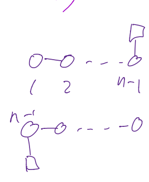
$\{\xi_i^!\} \leftrightarrow \{a\}$ $\quad \hbar \leftrightarrow \hbar^! = \hbar^{-1}$

Instead, today I give an alternative definition using q-opers

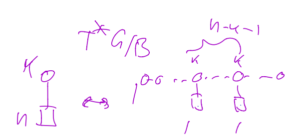
$\mathbb{C}^\times_q$, $\xrightarrow{x} Y$

quantum class: $\hat{\tau}(z) = \sum_d z^d\, ev_{\infty,*}\left(QM^d_{relat}, \hat{\mathcal{O}}_{vir}\,\tau(\mathcal{V}_i|_0)\right)$

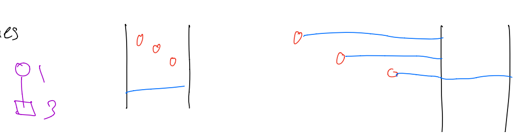
- **Examples of 3d mirrors:** Examples of mirrors: T^*G/B



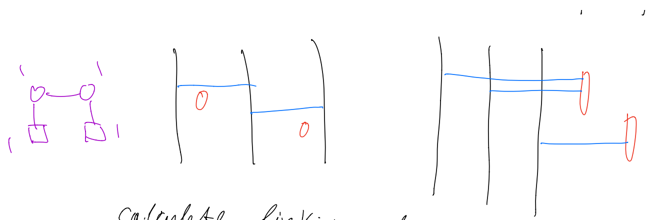
T^*G/B



Branes in Type A:

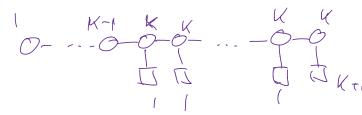


ADHM:

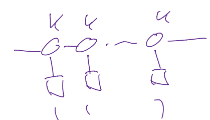


- calculate linking numbers for NS5 and D5 branes
- switch them to calculate mirror

Th: $\forall\ Y$ of type A 3d mirror $Y^!$ always exists provided that
$$v_{i-1} + v_{i+1} + w_i \geq 2 v_i$$

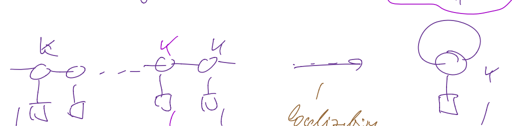
- Plan of the proof for ADHM:
  - Consider family $X_{k,\ell}$ 

    (q-Langlands / q-opers)

    Th: $X^!_{k,\ell} \simeq X_{k,\ell}$
  - Take direct limit $\ell \to \infty$  $X_{k,\infty}$

    Th: $\cdot\ K^q_T(X_{k,\infty}) = \varinjlim K_T^{red,q}(X_{k,\ell})$

    $\cdot\ X_{k,\infty}$ - self-mirror
  - Periodic boundary conditions on data $X_{k,\infty}$?

    $L(X) \to$ 

    localization to constant loops

    Th: ADHM is self-mirror X

- Proof of step 1: Idea: express relations in equivariant K-theory of $X_{k,\ell}$ in two different ways: a) 'magnetic frame' - using Kähler tori
  b) 'electric frame' - using equivariant parameters

  Then prove that $X_{k,\ell}$ is self-mirror by interchanging the two sets.

- Magnetic frame. $(SL(r+1), \hbar)$-opers

  $M_\hbar : \mathbb{P}^1 \to \mathbb{P}^1$, $\hbar \in \mathbb{C}^\times$, $\hbar^N \neq 1$, $\mathbb{C}^\times_\hbar \circlearrowright$
  $z \mapsto \hbar z$,

  Def: A meromorphic $(GL(r+1), \hbar)$-oper on $\mathbb{P}^1$ is $(A, E, \mathcal{L}_\bullet)$
  $\mathcal{L}_{r+1} \subset \mathcal{L}_r \subset \ldots \subset \mathcal{L}_1 \subset E$

'Lie bundle, $A \in \mathrm{Hom}_{\mathcal{O}_U}(E, E^{\hbar})$, i) $A \cdot \mathcal{L}_i \subset \mathcal{L}_{i-1}$

ii) $\exists\, U \subset \mathbb{P}^1$ - Zariski open dense subset s.t. restriction of $A \in \mathrm{Hom}(\mathcal{L}_\bullet, \mathcal{L}_\bullet^{\hbar})$ to $U \cap M_\hbar^{-1}(U)$

$$\bar{A} : \mathcal{L}_i / \mathcal{L}_{i+1} \xrightarrow{\sim} \mathcal{L}^{\hbar}_{i-1} / \mathcal{L}^{i\hbar}_{\hbar} \quad \text{on } U \cap M_\hbar^{-1}(U)$$

$$\mathcal{L}_{r+1} = \mathrm{Span}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \qquad A = \begin{pmatrix} * & \varphi_1 & & & \\ & * & \varphi_2 & & 0 \\ & & * & \ddots & \\ & * & & \ddots & \varphi_r \\ & & & & * \end{pmatrix}$$

no zeros & poles on

if they do have 0 then we have an oper w/ reg. singularities

Def: A Miura $(SL(r+1), \hbar)$-oper on $\mathbb{P}^1$ is $(E, A, \mathcal{L}_\bullet, \hat{\mathcal{L}}_\bullet)$

preserved by A

Theorem: [FKSZ]
$$A = \prod_i g_i^{\check{\lambda}_i} \exp\left(\frac{\varphi_i(z)}{g_i(z)}\, e_i\right), \quad g_i(z) \in \mathbb{C}(z)$$

Def: Z-twisted $(SL(r+1), \hbar)$-oper if $\exists\; g(z) \in SL(r+1)(z)$

s.t. $A(z) = g(\hbar z)\, Z\, g(z)^{-1}$, $Z = \mathrm{diag}(\xi_1, \dots, \xi_{r+1})$ $\quad \zeta_i = \frac{\xi_i}{\xi_{i+1}}$ Kähler par.

Miura: $A(z) = v(\hbar z)\, Z\, v(z)^{-1}$, $v(z) \in B_+(z)$

Def: Miura, Z-twisted $(SL(r+1), \hbar)$-oper w/ regular singularities $\{\Lambda_i(z)\}$ $i = 1 \dots r$

$$A = \begin{pmatrix} * & \Lambda_1 & & & \\ & \ddots & \ddots & & 0 \\ & * & & \ddots & \\ & & & * & \Lambda_r \\ & & & & * \end{pmatrix}$$

Theorem: $\exists$ 1-1 correspondence

$$\left\{ \begin{array}{c} \text{Nondegenerate } Z\text{-twisted} \\ \text{Miura } (SL(r+1), \hbar)\text{-opers} \\ \text{on } \mathbb{P}^1 \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Nondegenerate polynomial} \\ \text{solutions of } QQ\text{-system} \end{array} \right\}$$

$$A(z) = \prod_i g_i(z)^{\check{\lambda}_i} \exp \frac{\Lambda_i(z)}{g_i(z)}\, e_i$$

$$g_i(z) = \zeta_i \frac{Q_i^+(\hbar z)}{Q_i^+(z)}$$

$$\xi_i\, Q_i^+(\hbar z)\, Q_i^-(z) - \xi_{i+1}\, Q_i^+(z)\, Q_i^-(\hbar z) = \Lambda_i(z)\, Q_{i-1}^+(z\hbar)\, Q_{i+1}^+(z)$$

$i = 1 \dots r$

$$\Updownarrow$$

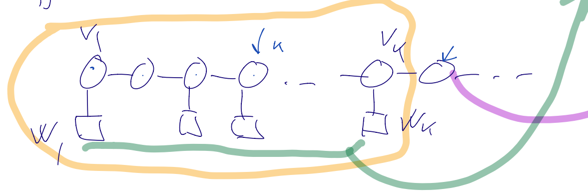
$\left\{ \text{Solutions of XXZ Bethe equations} \right\}$

L relations in $K_T^q(X)$ ✓

Let s - local section of $\mathcal{L}_{r+1}$

$\hbar$-Oper condition:

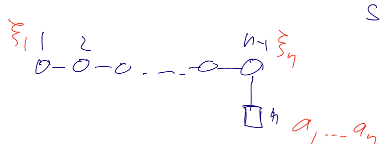
$$\det_{i,j}\left[\xi^{k-j}_{r+1-k+i}\, s_{r+1-k+i}(\hbar^{j-1} z)\right] = W_k(z)\, V_k$$



$S_{r+1}(z) = Q_r^+(z)$, $\quad S_r(z) = Q_r^-(z)$, $\quad S_{r-1}(z) = Q^-_{r-1,r}(z)$ ...

$S_1(z) = Q^-_{1,2,\dots r}(z)$

- Let $\boxed{Y = T^* G/B}$



Lemma: $s_i(z) = z - p_i$, $\quad i = 1, \dots, r+1$ $\qquad p_i\, f(\xi_i) = f(q\,\xi_{\xi_i})$

Th: Let

$$L_{ij} = \frac{\prod_{k\neq j}(\xi_i - \xi_k \hbar)}{\prod_{k\neq i}(\xi_i - \xi_k)}\, \hat{p}_i$$

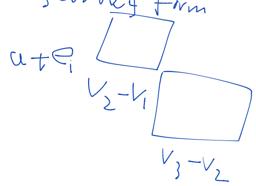
Lax matrix for tRS model. tRS momenta

Then $\hbar$-oper condition is equivalent to

$$\boxed{\det(u - L) = \prod_{i=1}^{n}(u - a_i)}$$

$(\xi_i, p_i^{\xi_i})$ Kähler $\qquad a_i$ equiv

Seesaw form



- Calogero–Moser space

Let $M' \subset GL(V) \times GL(V) \times V \times V^*$

$\dim V_{\mathbb{C}} = N$ $\quad$ s.t. $\quad \boxed{\hbar\, MT - TM = u \otimes v^T}$

$GL(N;\mathbb{C}) \circlearrowright M'$:

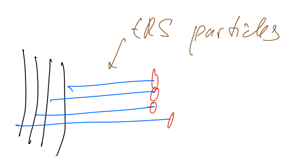
$$(M, T, u, v) \mapsto (gMg^{-1}, gTg^{-1}, gu, vg^{-1})$$

$$\mathcal{M} = \mathcal{M}'/GL(V) \text{ — CM space}$$

Lemma: If M is diagonal $\Rightarrow$ $T_{ij} = \dfrac{u_i v_j}{\hbar\,\xi_i - \xi_j}$, parametrize u, v using freedom.

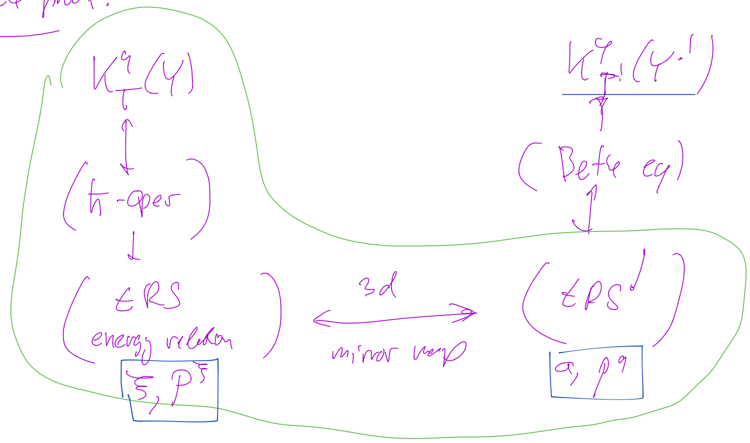
- 3d Mirror symmetry of CM_N space:

$\hbar \to \hbar^{-1}$ $\qquad M \longleftrightarrow T$



$a \leftrightarrow \xi$

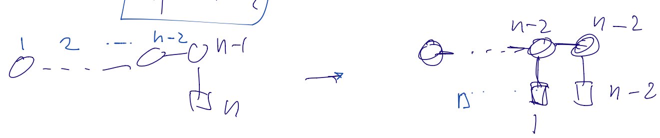
Strategy of the proof:



- **Resonance conditions**

Consider $T^*\mathbb{F}l_n$. All parameters non-degenerate.

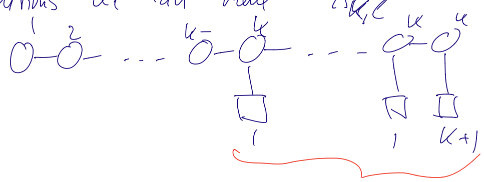
$\boxed{a_1 = \hbar a_2}$



$L_{ij} \to$ Slodowy form

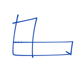
and so on: $a_1 = \hbar a_2 = \hbar^2 a_3 = \dots = \hbar^{\ell-1} a_\ell$.

Using these conditions we can make $X_{k,\ell}$



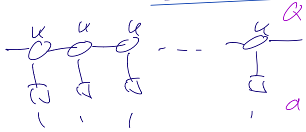
$+$ $k+\ell$ Kähler parameters

$Y_{\lambda, \lambda^!}$



- **Periodic conditions for ADHM quiver**

Take direct $\ell \to \infty$ limit



$Q_i(z)$, a_i

$Q_i^+(z) = \prod_{\ell=1}^{k} (z - s_{i,\ell})$ ← Bethe roots

$\Lambda_i(z) = (z - a_i)$

Prop: Algebra $K^q_T(X_{k,\infty})$ is invariant under translations

$s_{i,k} \mapsto s_{i+m,k}$, $a_i \mapsto a_{i+m}$, $\xi_i \mapsto \xi_{i+m}$,

$m \in \mathbb{Z}$

Impose $\xi_i = \xi_0 \xi^i$, $Q_i^\pm(z) = Q^\pm(t^i z)$, $\Lambda_i(t^i z) = \xi_i \Lambda(z)$

$\xi, t \in \mathbb{C}^\times$

Then the QQ-system for $X_{n,\infty}$ becomes

$$\boxed{\xi\, Q^{+}(\hbar z)\, Q^{-}(z) - Q^{+}(z)\, Q^{-}(\hbar z) = \Lambda(z)\, Q^{+}(t z)\, Q^{-}(t^{-1}\hbar z)}$$

which describes $K^{q}_{T}(\mathrm{ADHM})$