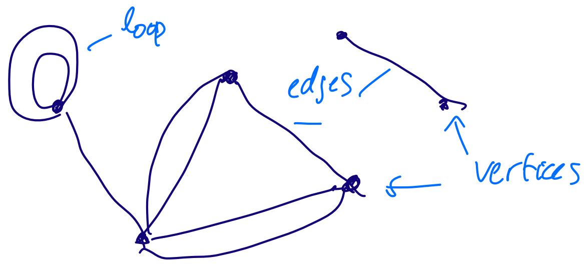
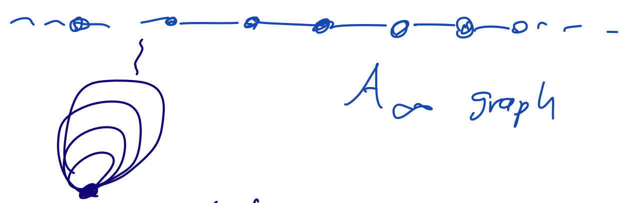
Graphs

Def: A graph $G = (V, E)$, V - set of vertices (nodes), E - set of edges.
Each edge has one or two vertices associated with it, called the endpoints.

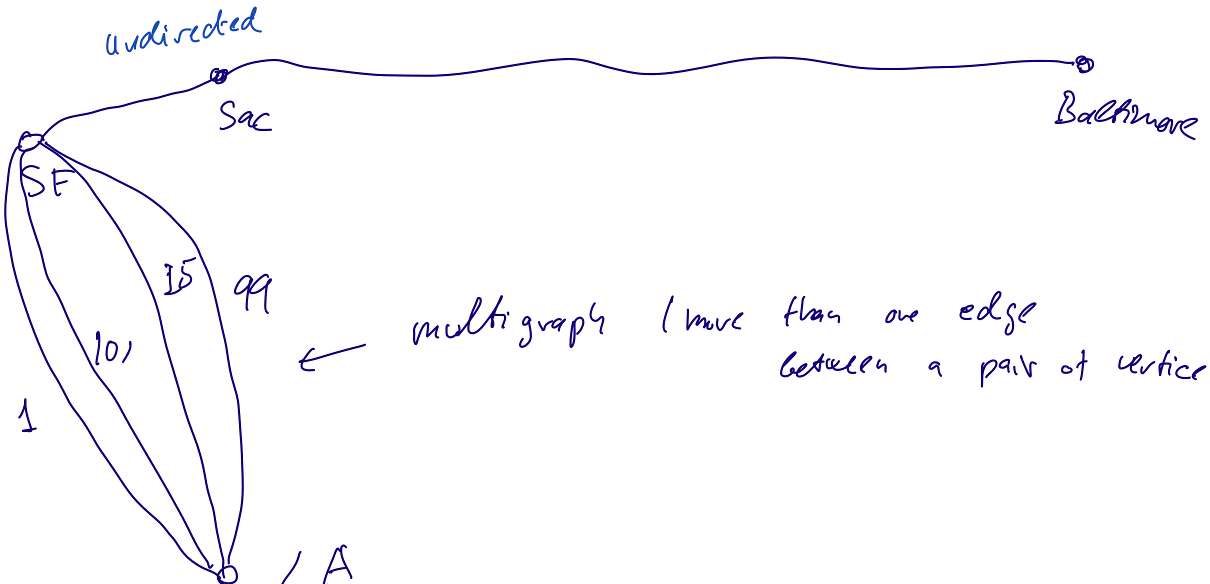


finite

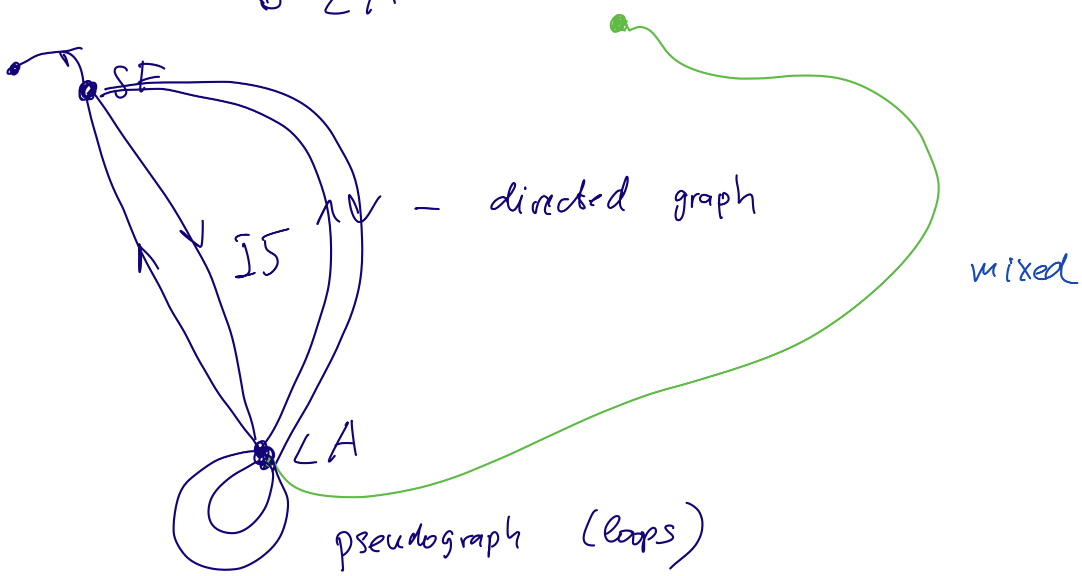


infinite if contains either ∞-many vertices or edges

Examples: Map



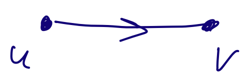
multigraph (more than one edge between a pair of vertices)



– directed graph

pseudograph (loops)

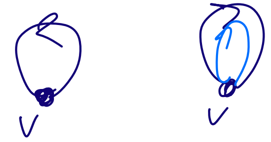
Def: A directed graph (digraph) $G = (V, E)$, V - nonempty set of vertices, E - set of directed edges.



(u, v) - ordered

- starts at u and ends at v

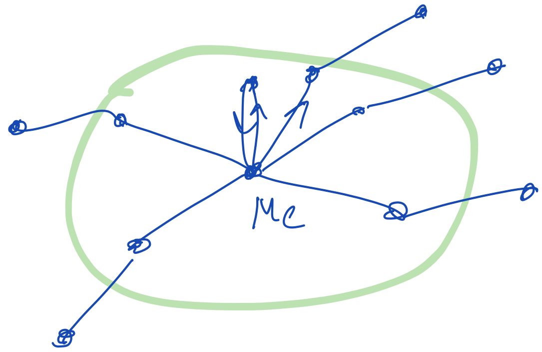
(source, target)



$v \to v$

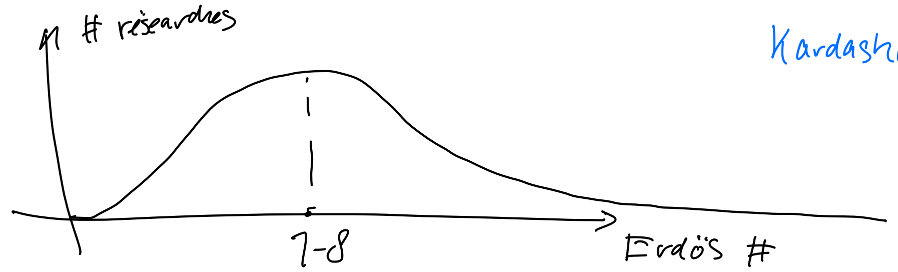
Facebook

Influence graph



Collaboration graph.

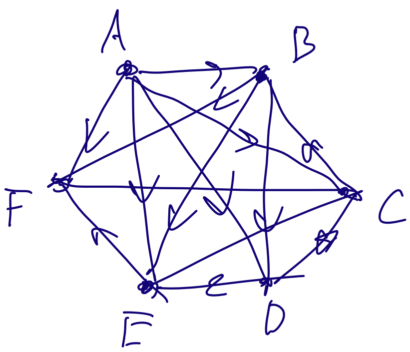
Erdős number, Bacon number



Kardashian's number = $\frac{\text{\# Twitter foll}}{\text{\# citations}}$

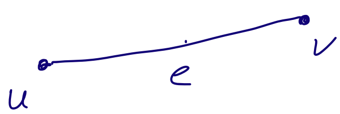
Briane Greene

Elegant Universe



Round-robin turnament

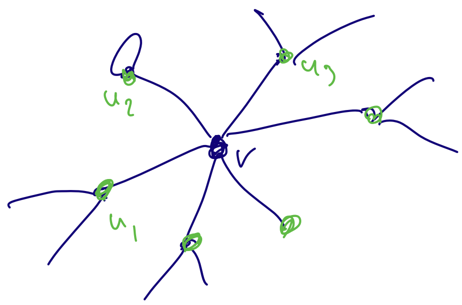
Def: $G=(V,E)$



edge e is incident to vertices u, v.

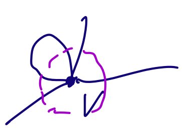
$u, v \in V$

Def:



The neighborhood of $v \in V$ $N(v)$ is the subset of V whose vertices are adjacent to v

Def:

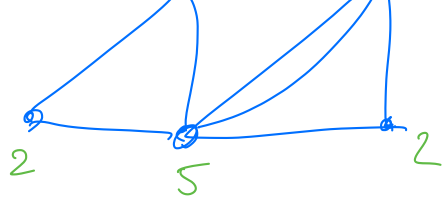


A degree of a vertex in an udirected graph $\deg(v)$ is the number of edges incident with it.

Note that a loop counts as 2

$\deg v = 5$

Add all degrees: 18



The handshake theorem: Let $G = (V, E)$ - undirected graph with m edges ($|E| = m$). Then $2m = \sum_{v \in V} \deg(v)$

Theorem: An undirected graph has an even number of vertices of odd degree

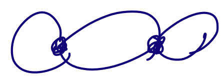
Proof: By the handshake theorem
$$2m = \sum_{v \in V_{even}} \deg(v) + \sum_{v \in V_{odd}} \deg(v)$$

even, even, must be even

Thus $|V_{odd}|$ is even.

Def: A graph is called simple if no pair of vertices is connected with more than one edge and there are no loops.

Simple



not

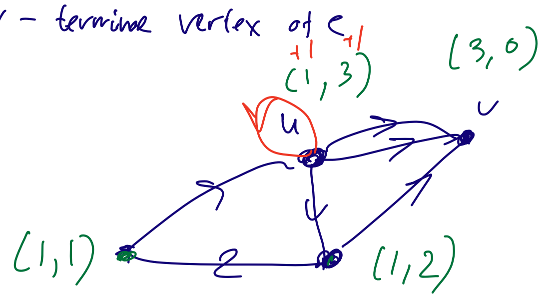
Def: Let $G = (V, E)$ be a directed graph, edge $e = (u, v)$

We say that e is adjacent from u or adjacent in v

u - initial vertex of e

v - terminal vertex of e

Def:



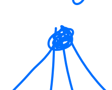
In a graph with directed edges the in-degree of a vertex v $\deg^-(v)$ is the number of edges which have v as the terminal vertex.

The out-degree $\deg^+(v)$ - the number of edges which have v as initial vertex

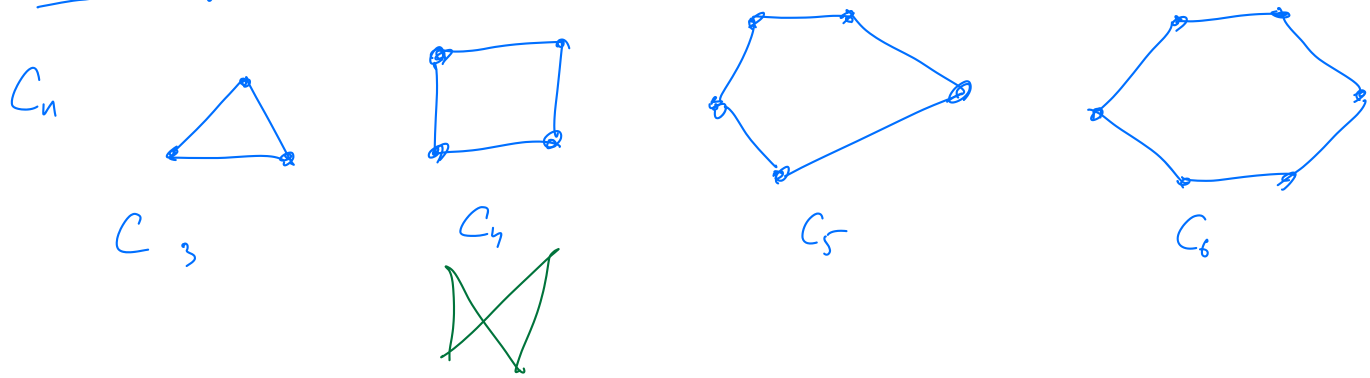
$$\sum_{v \in V} \deg^+ v = 6, \quad \sum_{v \in V} \deg^- v = 6$$

Theorem: Let $G = (V, E)$ - directed graph then
$$\sum_{v \in V} \deg^- v = \sum_{v \in V} \deg^+ v = |E|$$

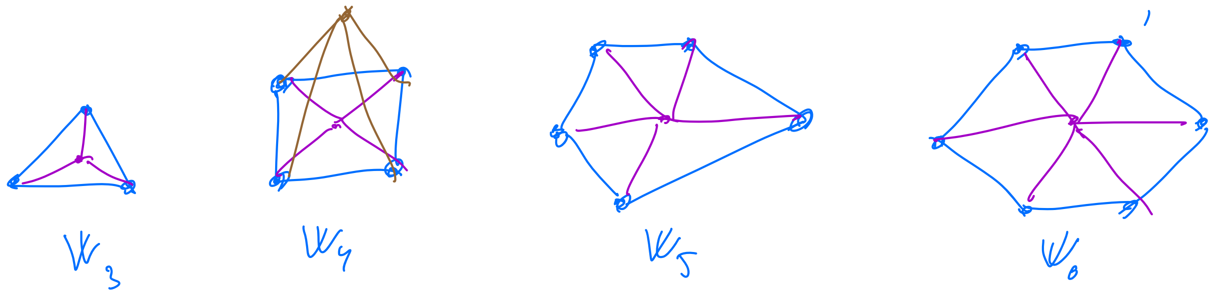
Ex: Complete graph K_n, $n = |V|$. Connect every vertex with every other



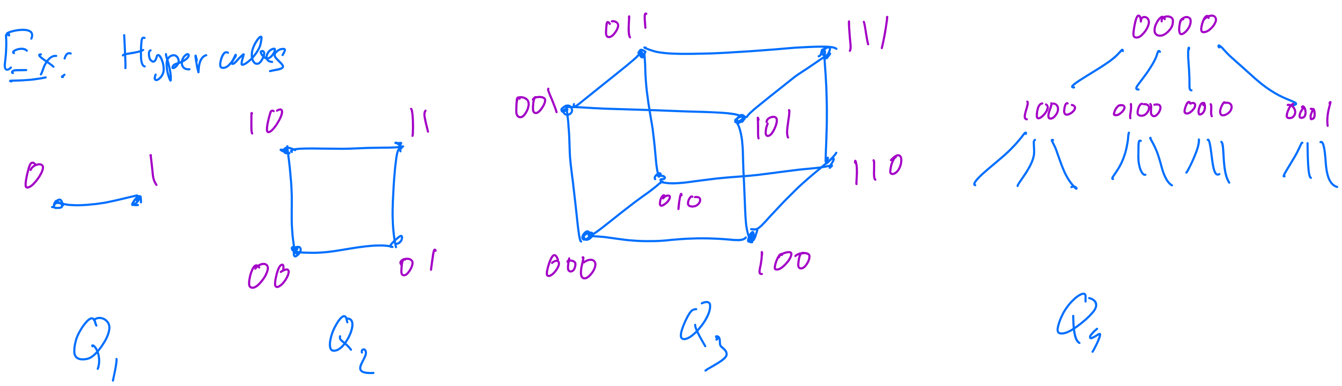
Ex: Cyclic graphs. Order vertices $\{v_1, v_2 \dots v_n\}$, edges $(v_1, v_2), (v_2, v_3), \dots (v_{n-1}, v_n)$



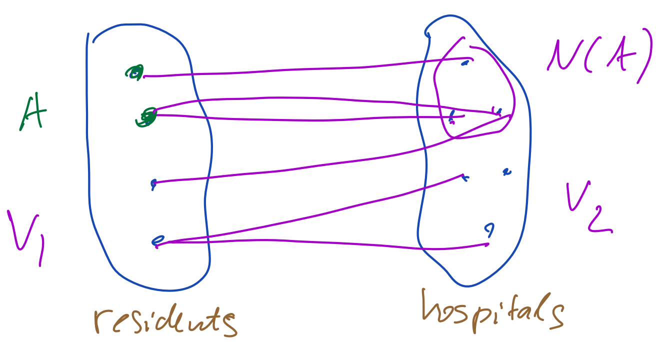
Ex: Wheels W_n. Take a cyclic graph C_n, add one more vertex, connect to every vertex in C_n



Ex: Hypercubes



Bipartite graphs



$|V_1| = m$

$|V_2| = n$

$K_{m,n}$

disjoint union

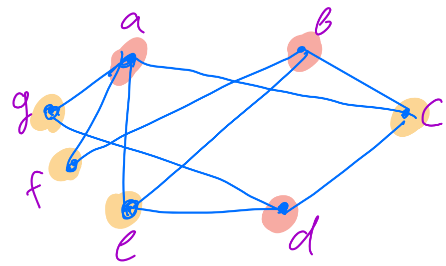
$V_1 \cap V_2 = \phi$

$V = V_1 \sqcup V_2$

bipartition of V

Def: A simple graph G is called bipartite if and every edge connects a vertex in V_1 to a vertex in V_2 (no two elements in V_1 or V_2 are connected by edges)

Ex:



$V = \{a,b,c,d,e,f,g\}$

$V_1 = \{g,f,e,c\}$

$V_2 = \{a,d,b\}$

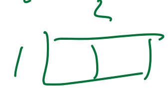
This graph is bipartite.

Theorem: A simple graph is bipartite iff we can assign two different colors to the vertices such that no two adjacent vertices are assigned with the same color.

Coloring problems in graph theory (Chromatic number)

Hall's Marriage theorem: The bipartite graph $G=(V,E)$ with bipartition (V_1, V_2) has a complete matching (for every $v \in V_1$ there is an edge ending in it) from V_1 to V_2 iff $\forall A \subset V_1$, $|N(A)| \geq |A|$.

quivers



Ex:

$\mathbb{CP}^1 = \square^2 \leftarrow \circ^1$ $\quad \mathbb{C}^2 \leftarrow \mathbb{C}$ $\quad \{z, w\}$, $z \neq 0$, $w \neq 0$ $\quad \begin{pmatrix} z \\ w \end{pmatrix}$ $\quad \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$\{\lambda \neq 0\}$ $\quad \underbrace{(z, w)}_{\text{edge}} \mapsto (z \cdot \lambda, w \cdot \lambda)$

Consider equivalence classes of multiplication of (z,w) by $\lambda \neq 0$.

$z \neq 0$

i.e. $(z, 0)$ is not in this class

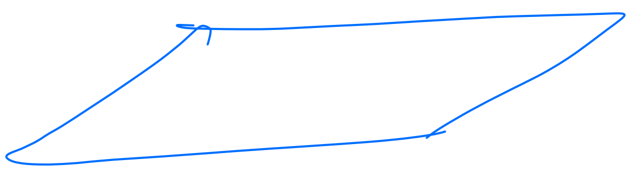
$\mathbb{C}^\times$ - multiplicative group of complex numbers

$(1, 0)$

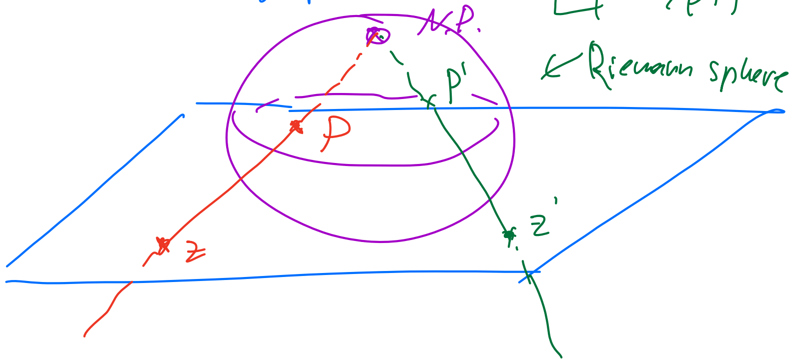
$S^2 \sim \mathbb{CP}^1 = \dfrac{\mathbb{C}^2 - \{0\}}{\mathbb{C}^\times} = \mathbb{C} \sqcup \mathbb{C}^0$ $\quad (z, 1) \in \mathbb{C}$

complex projective space $\quad \cdot\, pt$ $\quad$ pt

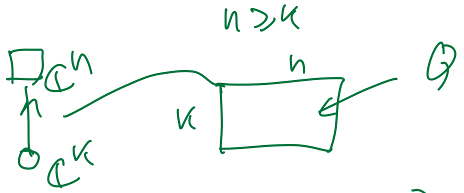
$= \mathbb{C} \sqcup \{pt\}$



Stereographic projection $\sqcup \{pt\}$ $= \infty$



$n \geq k$

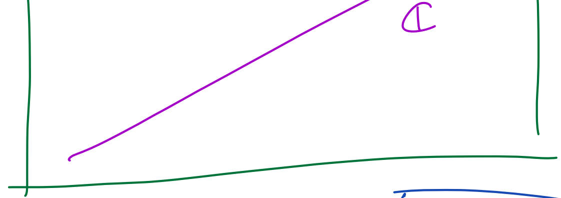


Let $G \in Mat_{k \times k}(\mathbb{C})$

$G \cdot Q$ $\quad$ Grassmannian $\quad$ $Gr_{k,n}$ - space of k-dim planes inside $\mathbb{C}^n$

$\mathbb{C}^n$ $\quad$ $SU(n)$

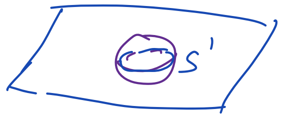
$S(U(k) \times U(n-k))$



$A^\dagger A = 1 \qquad A^\dagger = (A^t)^*$

$\begin{pmatrix} 1 & i \\ 7 & 5 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 7 \\ -i & 5 \end{pmatrix}$

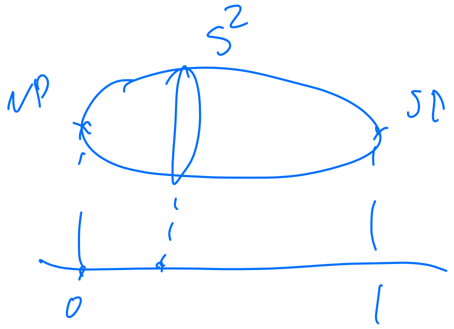
$\underset{\sim}{\mathbb{R}^3 - \{0\}} \sim \underset{\sim}{S^2}$

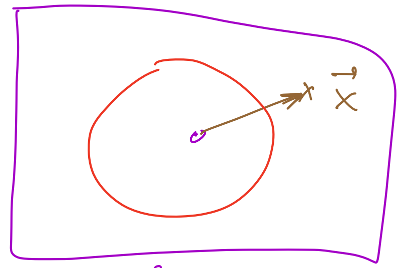


$\mathbb{C}^2 - \{0\} = \mathbb{R}^4 - \{0\} \simeq S^3$

$\mathbb{C}\mathbb{P}^1 = \dfrac{\mathbb{C}^2 - \{0\}}{\mathbb{C}^\times} = S^3/S^1 \cong S^2$

toy example $\quad \mathbb{R}^2 - \{0\} \simeq S^1$



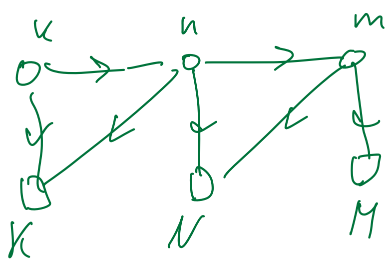


$\vec{x} \in \mathbb{R}^2$ homotopy

Nakajima

$F(\vec{x}, t) = \dfrac{\vec{x}}{|\vec{x}|} \cdot t + (1-t)\,\vec{x}$

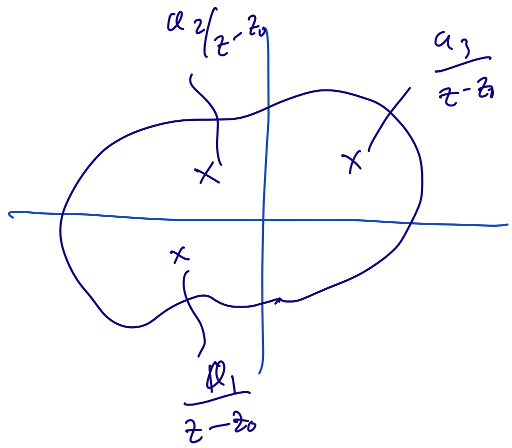
$0 \le t \le 1$

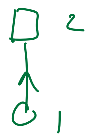


quiver variety

$\int f(z)\,dz$

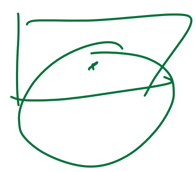
$= (a_1 + a_2 + a_3)\, 2\pi i$

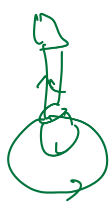




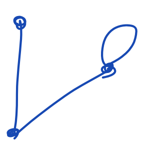
$\sim S^2$

$\sim T^* S^2$

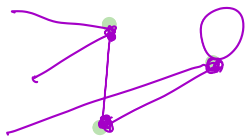




Def: A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subset V$ and $F \subset E$. H is a proper subgraph if $H \neq G$

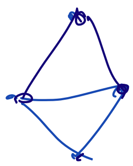


Def: Let $G = (V, E)$ be a simple graph. The subgraph induced by $W \subset V$ is graph (W, F), where $F \subset E$ contains edges iff their endpoints belong to W

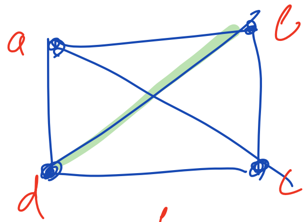


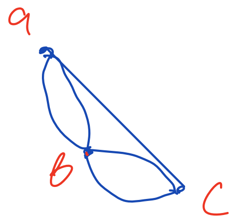
– subgraph induced by W

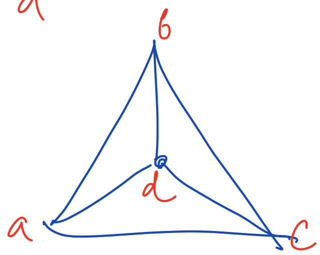
We can remove, add edges and/or vertices to a given graph



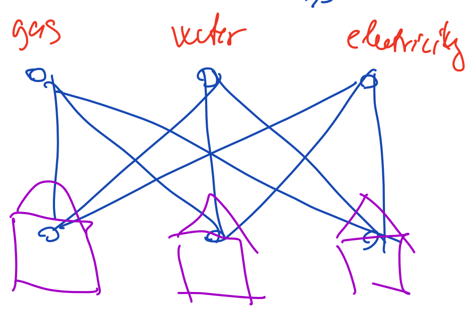
Edge contraction



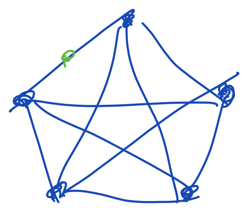




$K_{3,3}$

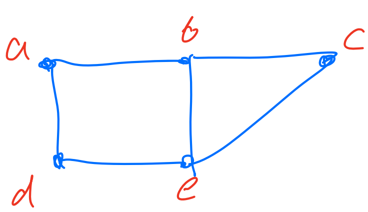


Theorem: A graph is nonplanar iff it contains $K_{3,3}$ or K_5 as a subgraph

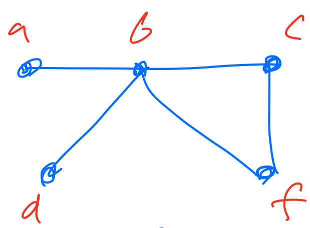


K_5

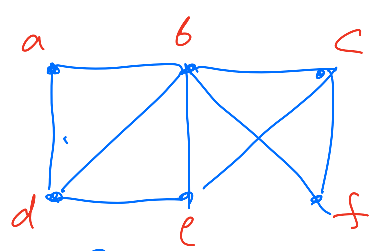
Def: The union of two simple graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ is the simple graph w/ vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, denoted $G_1 \cup G_2$



G_1



G_2



$G_1 \cup G_2$

Def: Tensor product of graphs

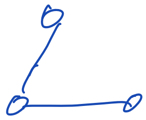
$V_1 \otimes V_2$

$\begin{pmatrix} e_1 \\ \vdots \\ e_m \end{pmatrix} \quad \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$

$$\begin{matrix} e_1 \otimes f_1 & \cdots & e_1 \otimes f_n \\ \vdots & & \vdots \\ e_m \otimes f_1 & \cdots & e_m \otimes f_n \end{matrix}$$

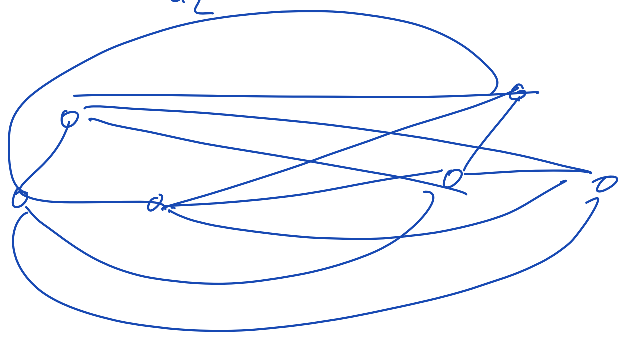
$$G_1 \otimes G_2 = (V_1 \otimes V_2,\ E_1 \otimes E_2)$$

G_1

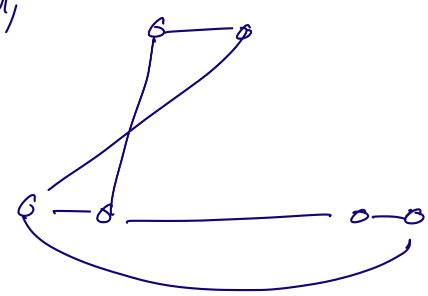


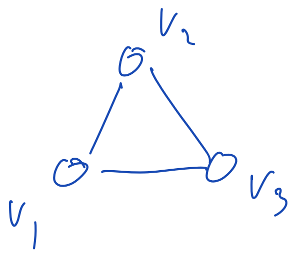
G_2

$G_1 \otimes G_2$



$G_2 \otimes G_1$





	v_1	v_2	v_3
v_1	0	1	1
v_2	1	0	1
v_3	1	1	0

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

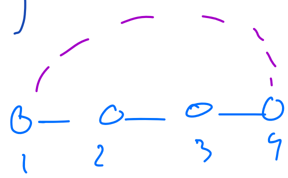
Def: Suppose $G = (V, E)$ is a graph, $|V| = n$, $V = \{v_1, v_2 \dots v_n\}$

The adjacency matrix $A = (a_{ij})$, $a_{ij} = 1$ if there is one edge between i and j (k if there are k edges)

(Note: $A^t = A$ for an undirected graph)

Ex:

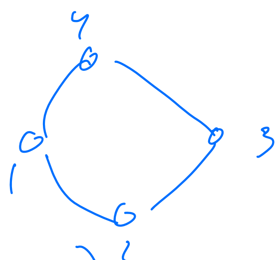
$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$



A_4

Ex:

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 1 & 0 & 1 & 0 \end{array}$$



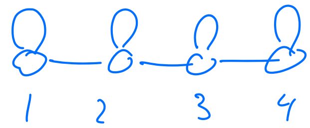
C_4 $(\hat{A}_3)$

$$\begin{pmatrix} 1 & \ddots & & 1 \\ & & 0 & \\ 1 & & & 1 \end{pmatrix}$$

1 2 3 4

Ex:

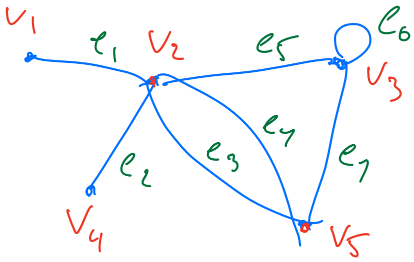
$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$



Def: Let $G = (V, E)$ - undirected graph, $\{v_1, \dots v_n\}$ - vertices, $\{e_1, \dots e_m\}$ - edges

Then the incidence matrix for G $M = (m_{ij})$

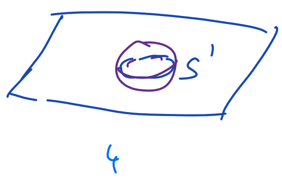
$$m_{ij} = \begin{cases} 1, & e_j \text{ is incident with } v_i \\ 0, & o/w \end{cases}$$



	e_1	e_2	e_3	e_4	e_5	e_6	e_7
v_1	1	0	0	0	0	0	0
v_2	1	1	1	1	1	0	0
v_3	0	0	0	0	1	1	1
v_4	0	1	0	0	0	0	0
v_5	0	0	1	1	0	0	1

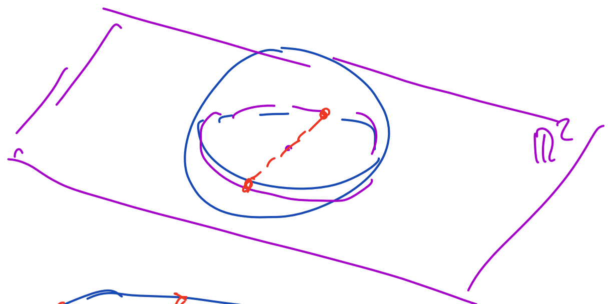
$$\mathbb{R}P^n = \frac{\mathbb{R}^{n+1} - \{0\}}{\{x \sim \lambda x\}} \qquad \lambda \neq 0, \ \lambda \in \mathbb{R}$$

$$\frac{\mathbb{R}^3 - \{0\}}{\sim} \sim \frac{S^2}{\sim}$$



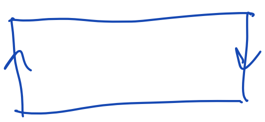
$n = 2$

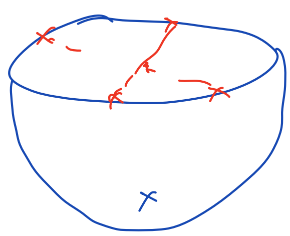
$$\mathbb{R}P^2 = \frac{\mathbb{R}^3 - \{0\}}{\{\vec{x} \sim \lambda \vec{x}\}} \qquad |\vec{x}| = 1$$



Identify the opposite points of S^2

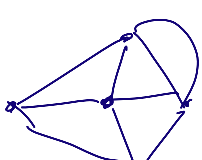
Möbius strip

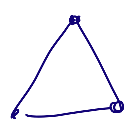


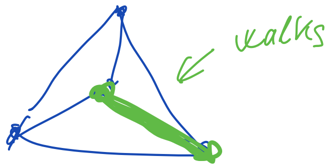


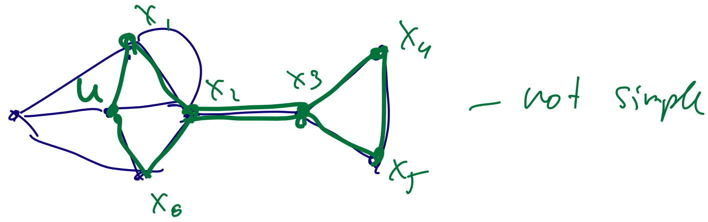
$\mathbb{R}P^2$ - hemisphere glued by a Möbius strip

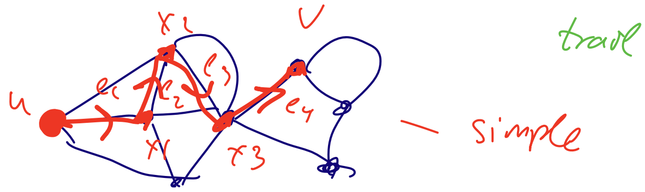
Connectivity in graphs











(open)

Def: A path in an undirected graph $G=(V,E)$ of length n from u to v, $u,v \in V$, is a sequence $e_1, e_2 \dots e_n$ for which there exists a sequence of vertices $u, x_1 \dots x_{n-1}, v$ such that edge e_i connects x_{i-1} with x_i ($u=x_0$, $v=x_n$). If $u=v$ then we have a circuit from u to itself. A path or a circuit is called simple if it does not contain the same edge more than once.

Examples: Erdös # — length of a collaboration path
Bacon #

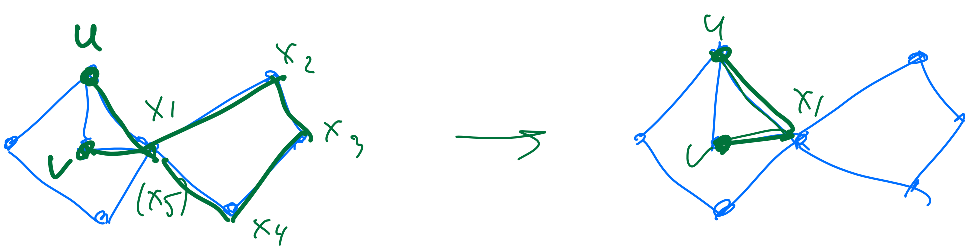
Def: An undirected graph is called connected if there is a path between any pair of distinct vertices on the graph. (Otherwise it is disconnected)

Theorem: There is a simple path between every pair of distinct vertices of a connected undirected graph.

Proof: Let $G=(V,E)$ be a connected graph. There is at least one path between any $u,v \in V$. $x_0, x_1 \dots x_n$, $x_0=u$, $x_n=v$ — path

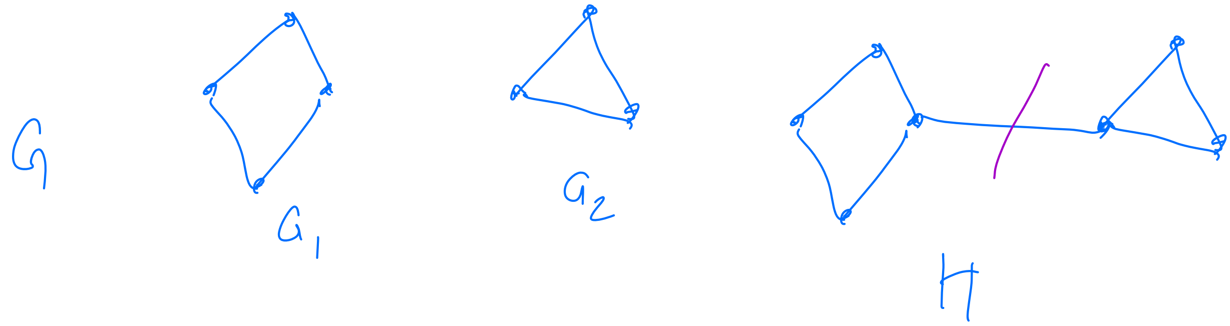
The path of least length is simple. Assume false. Then $x_i = x_j$ for some $0 \le i < j \le n$. This means that path

$x_0, x_1 \dots x_{i-1}, x_j \dots x_n$ - path with shorter length. Contradiction.



Def: A connected component of a graph G is a connected subgraph of G which is not a proper subgraph of another connected subgraph of G.

(i.e. it's a maximal connected component)

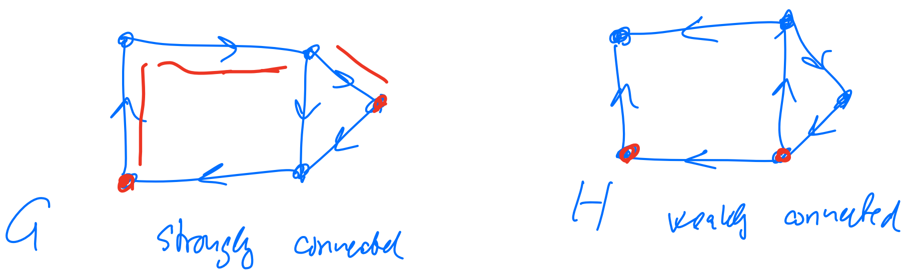


<u>Cutting graphs:</u> <u>Cut edge.</u> Let $e \in E$ of $G = (V, E)$, let G connected. e is called cut edge if $G' = (V, E - \{e\})$ is disconnected.

<u>Cut vertex.</u> Let $v \in V$ of $G\ (V, E)$, G - connected.
v is called cut vertex if $G' = (V - \{v\}, E - f)$
f - set of edges adjacent to v

<u>Def</u>: A directed graph is <u>strongly connected</u> if there is a directed path from u to v $\forall\, u, v \in V$

<u>Def</u>: A directed graph is <u>weakly connected</u> if the corresponding undirected graph ("forget the arrows") is connected.

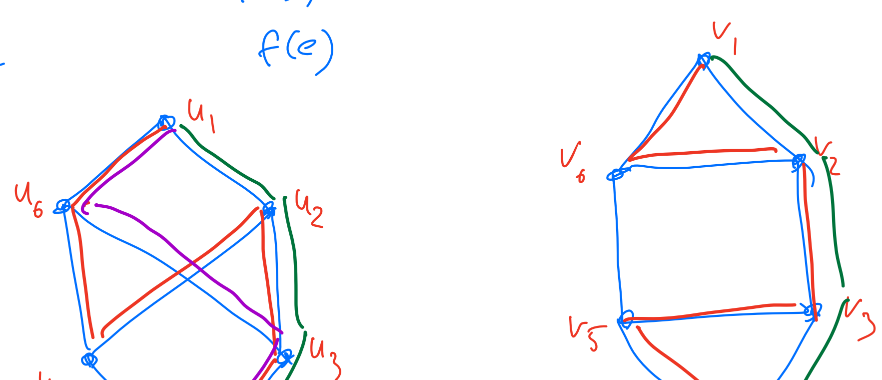


$G \xrightarrow{f} H$ f - graph isomorphism

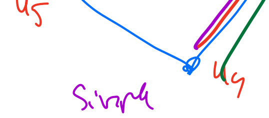
u, v $f(u), f(v)$

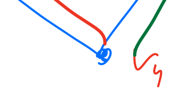
e $f(e)$

Ex:



$f: u_i \mapsto v_i \ \forall i$





Simple
4 paths of length 3

2 simple paths of length 3

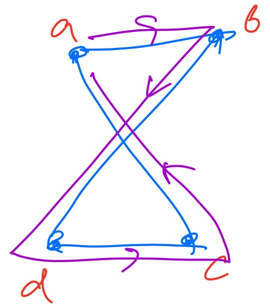
not isomorphic

Theorem: Let A be the adjacency matrix of G, $V = \{v_1, \dots, v_n\}$ - vertices

$$\left(\#\text{ paths from } v_i \text{ to } v_j \text{ of length } r\right) = (a^r)_{ij}$$

$$A^r - r\text{th power} \quad (\underbrace{A \cdot A \dots A}_{r \text{ times}})$$

Ex:



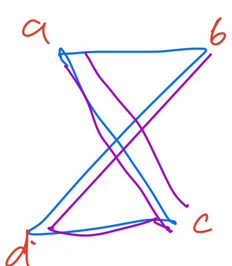
$$A = \begin{array}{c|c} & \begin{matrix} a & b & c & d \end{matrix} \\ \hline \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$r = 4$

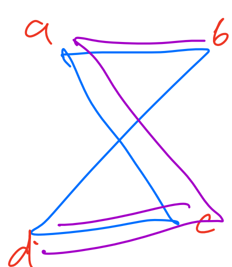
$$A^2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

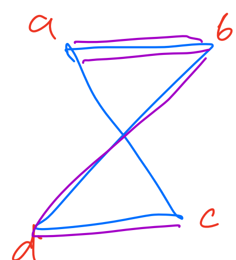
$$A^4 = A^2 \cdot A^2 = \begin{array}{cc} & \begin{matrix} & & c & \end{matrix} \\ b & \begin{pmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{pmatrix} \end{array}$$



bdcac



bacdc



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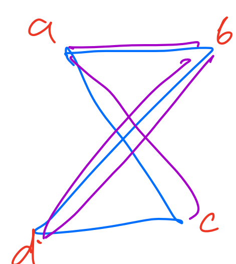

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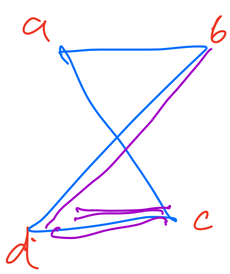
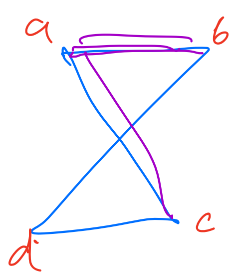
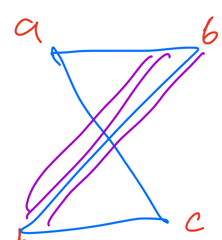
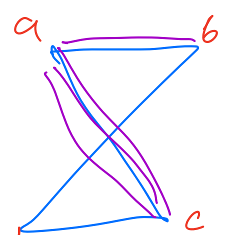








bdcdc   babac   d bdbdc   d bacac

Proof: • Base $r=1$ A - adjacency matrix $|V|=n$

• Assumption. Assume for some r $(A^r)_{ij}$ - # of paths from v_i to v_j of length r

b_{ij} - matrix elements of A^r

• Step $A^{r+1} = A^r \cdot A$

(i, j) element $\Rightarrow$ $b_{i1}a_{1j} + b_{i2}a_{2j} + \dots + b_{in}a_{nj}$

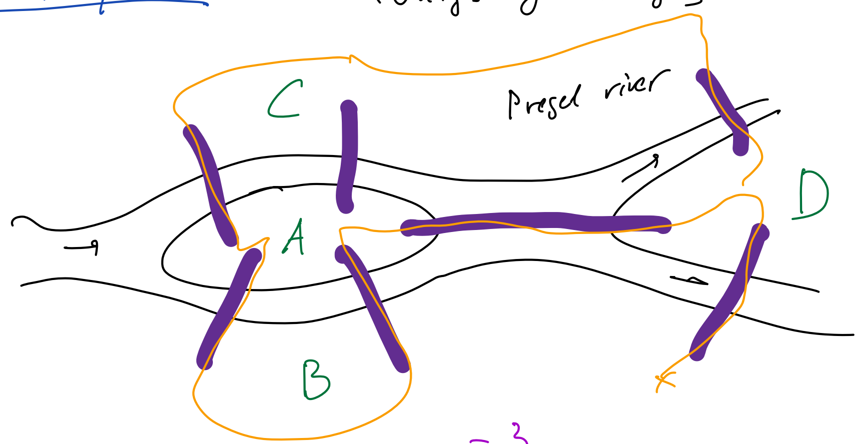
paths of length r from i → l

edges (paths of length 1) from l j

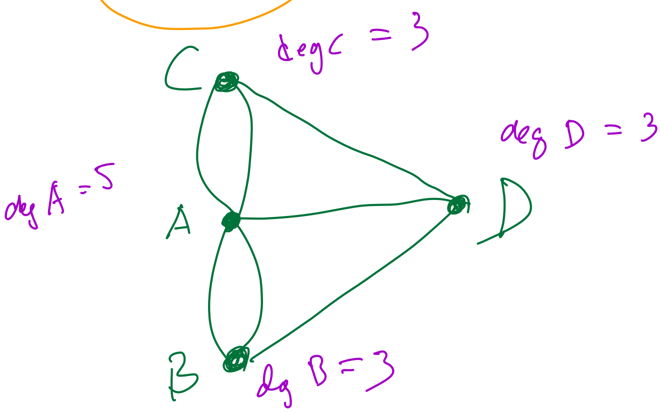
Counts paths of length r+1 from v_i to v_j via all possible vertices, which are all possible paths of length r+1 from v_i to v_j.

Euler paths

Königsberg bridges

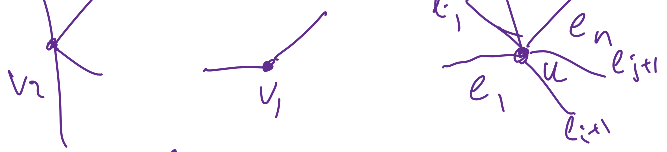


Q: Can you make a circuit which passes each bridge only once? (Euler circuits)



Theorem: A connected graph with at least two vertices has an Euler circuit iff each of its vertices has an even degree.

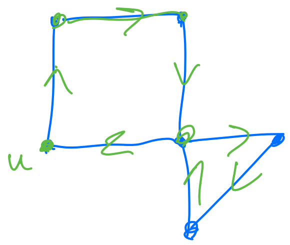
(⇒)

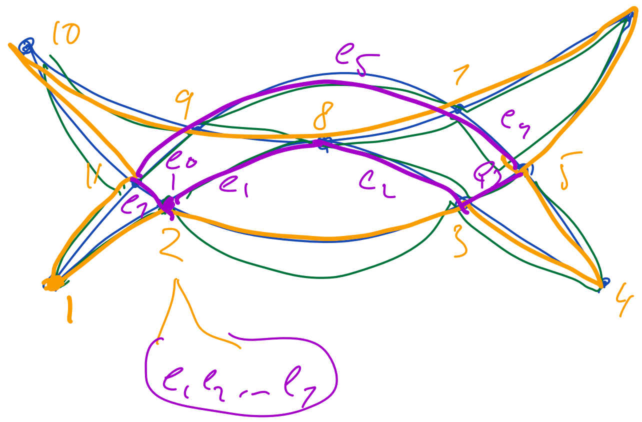


e_1, e_n contribute 2 to deg u

If an Euler circuit exists then all degrees are even:
for u we have e_1 and e_n (first and last edges)
for all other vertices in the path every passage through the vertex contributes degree 2.

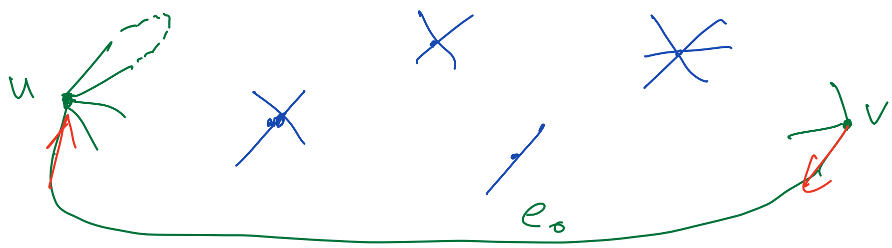
($\Leftarrow$)





Def: An Euler path in G is a path from u to v, $u, v \in V$ s.t. it traverses all edges only once.

Theorem: A connected graph has an Euler path but not an Euler circuit iff the graph has exactly two vertices of odd degree.



Proof: by adding edge e_0 connecting u and V we come back to the proof of theorem for Euler circuit.

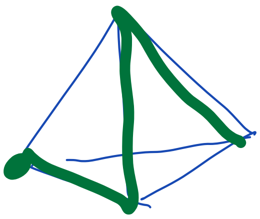
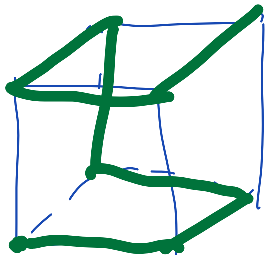
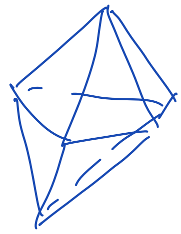
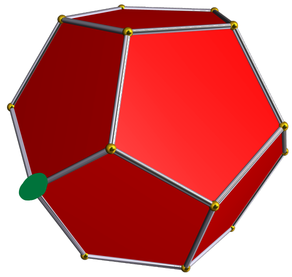
Hamilton paths

$q = a + ib + jc + kd$ $\qquad$ i, j, k $\qquad$ $i^2 = j^2 = k^2 = -1$

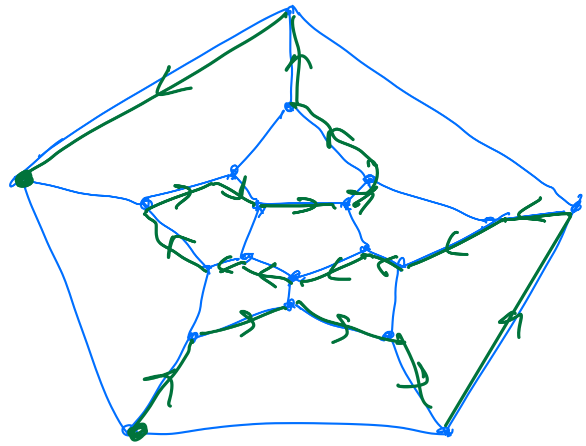
$ij = k$

$ij = -ji$

Def: A simple path (circuit) in graph G that passes through every vertex exactly once is called a Hamilton path (circuit). That is if $V = \{v_1, \dots, v_n\}$ then the path goes through $v_{i_1}, \dots, v_{i_n}$, where $i_1 \dots i_n$ is a permutation of $1 \dots n$.

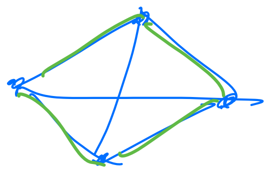
icosahedron



Conditions for existence of Hamilton paths are not known in general

Ex: K_n

$n \geq 3$



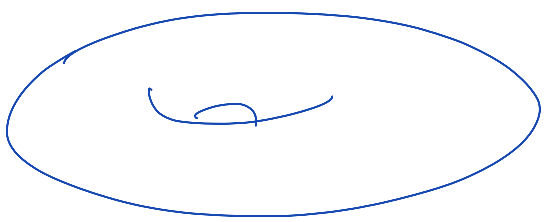
K_4

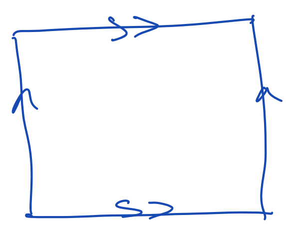
K_2

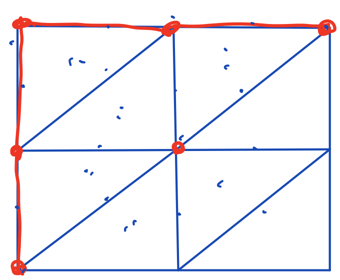
<u>Theorem:</u> (Dirac) Let $G = (V, E)$ - simple graph, $|V| = n \geq 3$. Then G admits a Hamilton circuit if $\deg(v_i) \geq \frac{n}{2}$ $\forall i = 1 \dots n$

<u>Theorem (Ore):</u> If $G = (V, E)$ - simple graph, $n \geq 3$ such that $\deg u + \deg v \geq n$ for every pair of non adjacent vertices u and $v \in V$, then G has a Hamiltonian circuit.

<u>Torus:</u>





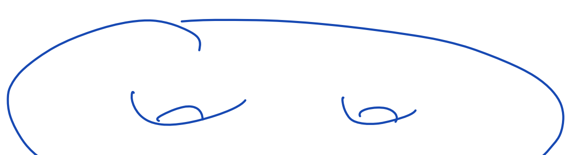


$V = 6$

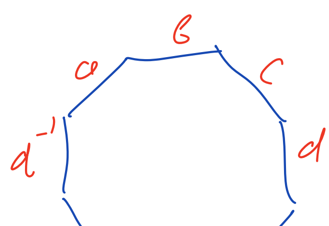
$e = 12$

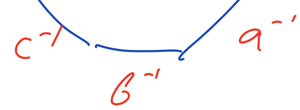
$f = 6$

$\chi = V - e + f = 0$



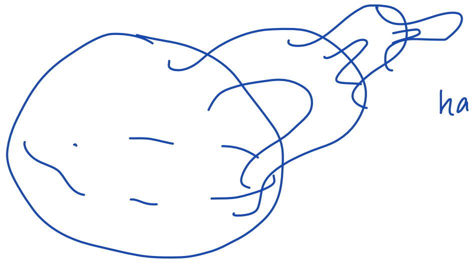
$g = 2$





$\chi = -2$

$\chi_g = 2 - 2g$

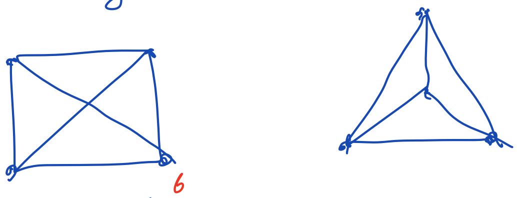


handle body decomposition

Planar graphs:

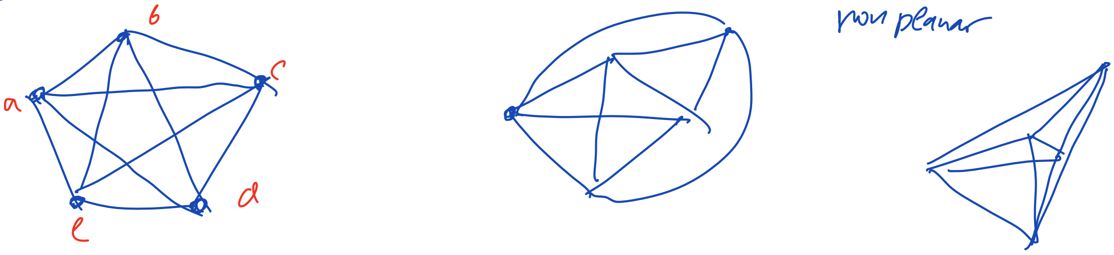
Def: A graph is called planar if it can be drawn on a plane without any edges crossing.

K_4



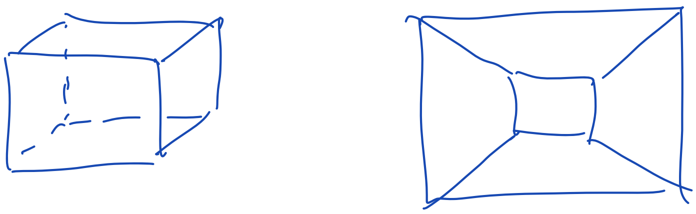
planar

K_5

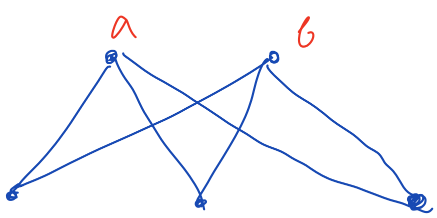


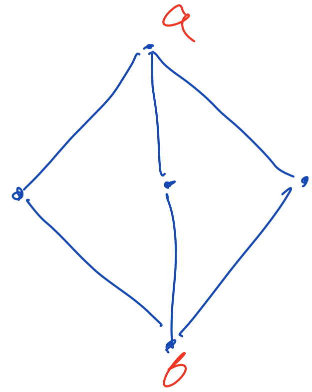
non planar

Q_3



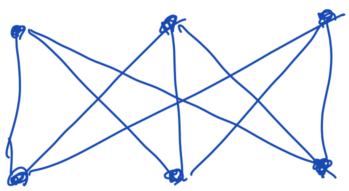
$K_{2,3}$





planar

$K_{3,3}$

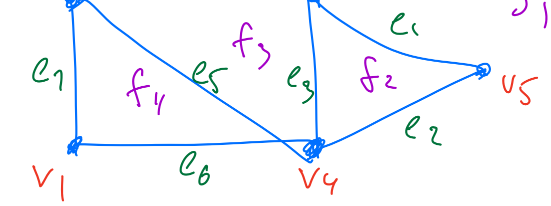


nonplanar

Theorem (Euler) Let G be a connected planar simple with e edges, v vertices and f faces (connected regions on the plane) then

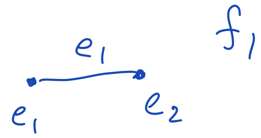
$$\chi := v - e + f = 2.$$

$$\chi = 5 - 7 + 4 = 2$$



Proof: • Bases

G_2:

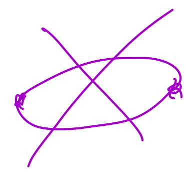


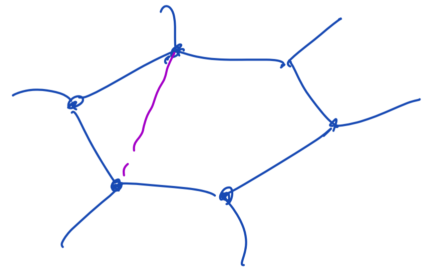
$\chi = 2 - 1 + 1 = 2$

• Assumption: for graph G_n $\chi = v - e + f = 2$

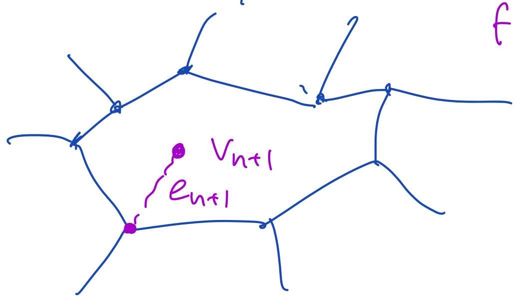
• Inductive step:

G_n





$e \mapsto e+1$
$v \mapsto v$ $\Rightarrow$ $\chi \mapsto \chi$
$f \mapsto f+1$

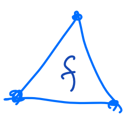


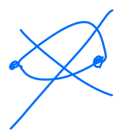
$e \mapsto e+1$
$v \mapsto v+1$ $\Rightarrow$ $\chi \mapsto \chi$
$f \mapsto f$

Corollary 1: If $v \geq 3$ then $e \leq 3v - 6$ for a connected planar simple graph

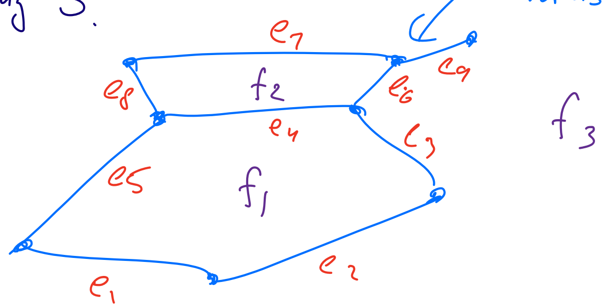
Corollary 2: If G is a connected planar simple graph then it has a vertex of degree not exceeding 5.

Proof of 1:





Contains 2



Def: Degree of a face - # edges on the boundary of the face.

$\deg f_1 = 5$

$\deg f_2 = 4$

$\deg f_3 = 9$

$\sum_{i\text{-all faces}} \deg(f_i) = 2e$, but $\deg f_i \geq 3$

$$\sum_{i \in \text{faces}} \deg(f_i) \geq 3f$$

$$\underline{3f \geq 2e}$$

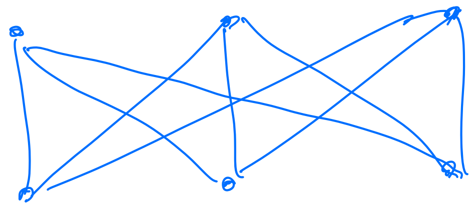
$$f = 2 + e - v$$

$$6 + 3e - 3v \geq 2e$$

$$\underline{e \leq 3v - 6}$$

Theorem: If G is a simple connected planar graph with no circuits of length 3 then $\underline{e \leq 2v - 4}$

Ex: $K_{3,3}$



no triangles

$e = 9$

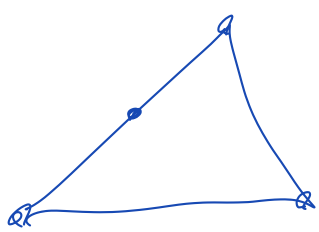
$v = 6$

9 vs $12 - 4 = 8$

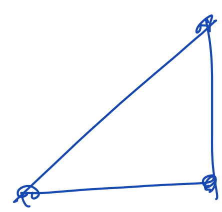
$\not\leq$

Theorem (Kuratowski).

We say that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are homeomorphic if one can be obtained from the other by a sequence of elementary subdivisions:



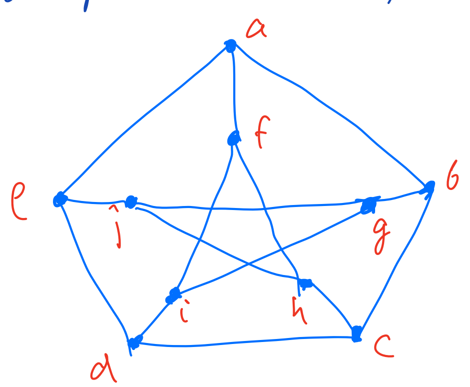
G_1



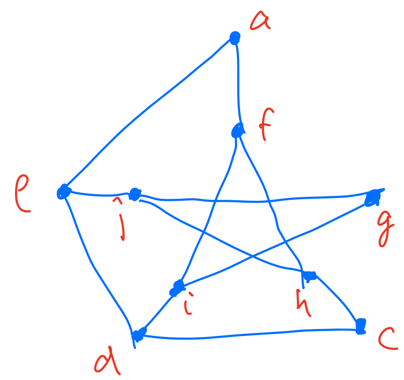
G_2

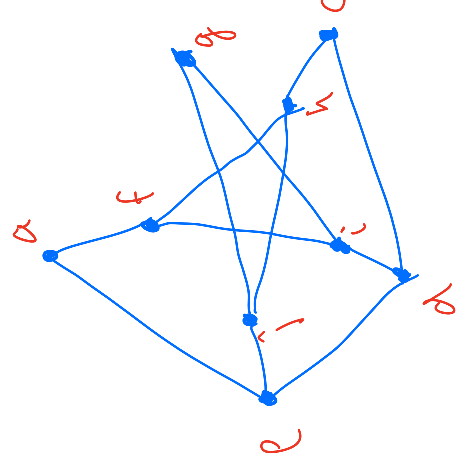
Theorem: A graph is nonplanar iff it contains a subgraph which is homeomorphic to $K_{3,3}$ or K_5.

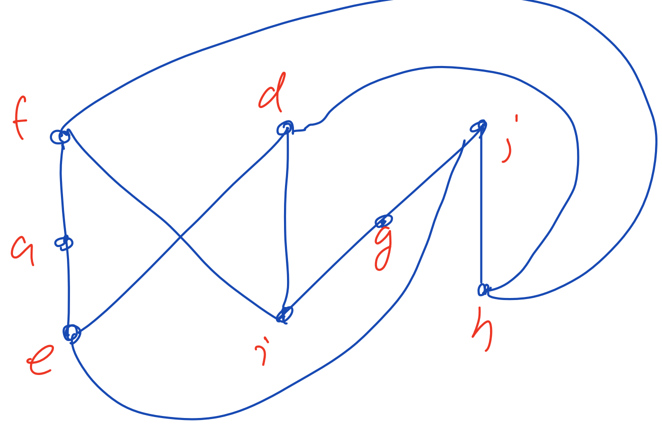
Ex:

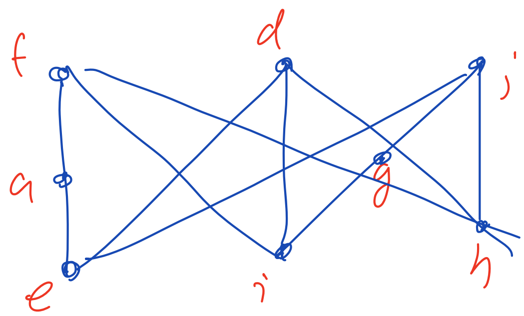


to









$K_{3,3}$