Dualities in integrable many-body systems and integrable probabilities

A. Gorsky

IITP, Moscow

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M. Vasyl'ev, A. Zotov , A.G. Arxiv 2109.05562
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Plan of the talk

- Motivation for dualities
- Calogero-Ruijsenaars family
- Spin chain family
- Spectral duality(within the families)
- QC and QQ dualities(between the families)
- Application for the integrable probabilities
- TASEP versus Goldfish in some details
- Toda and qKZ
- General comments

Calogero-Moser systems

$$H = \sum_{k=1}^{N} \frac{p_k^2}{2} - \nu^2 \sum_{i<j}^{N} U(q_i - q_j),$$

$$U(x) = 1/x^2$$

$$U(x) = 1/\sinh^2(x)$$

Calogero-Moser system → Integer QHE

Calogero-Moser system → Particles near black hole horizon

Calogero-Moser system → Dynamics of defects in SU(N) 4d Super YM

Calogero-Moser system → SU(N) 2d YM with heavy fermion on the cylinder

Calogero-Moser systems.Reminder

$$L\Psi = \Psi\Lambda\,, \quad \Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_N)$$

Conserved quantities

$$[Q, L] = \mathcal{O}\,, \quad \mathcal{O}_{ij} = \nu(1 - \delta_{ij})\,.$$

Moment map for coadjoint action of GL_N Lie group on the space $T^*\mathrm{gl}_N$

$$H_k = (1/k)\sum_{i=1}^N \lambda_i^k, \text{ and } \{\lambda_i, \lambda_j\} = 0$$

$$L_{ij} = \delta_{ij}p_i + \nu\frac{1 - \delta_{ij}}{q_i - q_j}\,,$$

Lax operator

Ruijsenaars-Schneider family

$$H = \sum_{i=1}^{N} e^{p_i/c} \prod_{j:j\neq i}^{N} \frac{h(q_i - q_j + \eta)}{h(q_i - q_j)},$$

$$h(x) = \sinh(x)$$

$$h(x) = x$$

$c \to \infty$ $\quad \eta = \nu/c.$ ⟶ Calogero model

$\eta \to \infty$ with $c = 1$ ⟶ Goldfish model(Calogero 80')

$$\dot{L} = \{H, L\} = [L, M], \quad L, M \in \mathrm{Mat}(N, \mathbb{C}).$$

$$L_{ij} = \frac{\eta \dot{q}_j}{q_i - q_j + \eta}, \quad \dot{q}_j = e^{p_j/c} \frac{1}{c} \prod_{k:k\neq j}^{N} \frac{q_j - q_k + \eta}{q_j - q_k}.$$

Ruijsenaars-Schneider'86

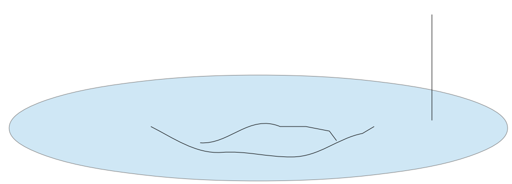
Ruijsenaars-Schneider family

RS family
N particles

N-soliton sector in sin-Gordon theory

Dynamics of defects in SU(N)
5d SuperYM theory

SU(N) 3d Chern-Simons theory on the torus with
marked point



XXZ and Gaudin spin chains

$$L(z) = \Lambda + \sum_{k=1}^{N} \frac{S^k}{z - z_k} \in \mathrm{Mat}_n\,, \quad S^k = \sum_{a,b=1}^{n} E_{ab} S^k_{ab}$$

$$\{S^i_{ab}, S^j_{cd}\} = \delta^{ij}(S^i_{ad}\delta_{cb} - S^i_{cb}\delta_{ad})\,, \quad i,j = 1,\ldots,N; \quad a,b,c,d = 1,\ldots,n\,.$$

$$\{L_1(z), L_2(w)\} = [r_{12}(z-w), L_1(z) + L_2(w)]\,,$$

$$\frac{1}{2}\mathrm{tr}(L^2(z)) = \frac{1}{2}\sum_{k=1}^{N} \frac{\mathrm{tr}\Big((S^i)^2\Big)}{(z-z_i)^2} + \sum_{k=1}^{N} \frac{H_i}{z-z_i}\,,$$

Nonlocal commuting Hamiltonians →

$$H_i = \mathrm{tr}(S^i\Lambda) + \sum_{k:k\neq i}^{N} \frac{\mathrm{tr}(S^i S^k)}{z_i - z_k}\,, \quad i = 1,\ldots,L\,.$$

Inhomogeneous twisted spin chain

$$T(z) = V L^N(z-z_N)\ldots L^1(z-z_1) \in \mathrm{Mat}_n\,,$$

Monodromy matrix of inhomogeneous Twisted GL(N) spin chain

$V = \mathrm{diag}(V_1, \ldots, V_n)$ is the twist matrix, and $z_1, \ldots, z_N$ are the inhomogeneities.

$$\{T_1(z), T_2(w)\} = [r_{12}(z-w), T_1(z)T_2(w)]\,.$$

$$t(z) = \mathrm{tr}T(z)$$

$$H_i = \operatorname*{Res}_{z=z_i} t(z)\,,$$

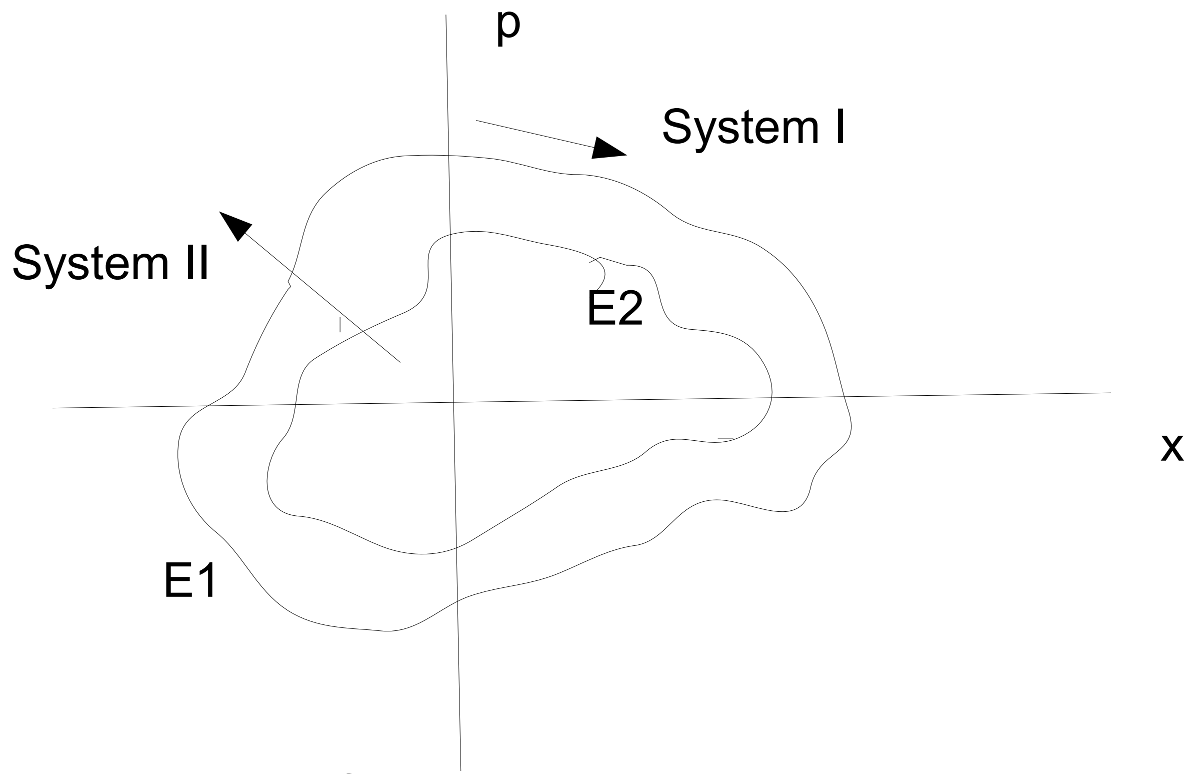
Nonlocal Hamiltonians

$$L^k(z-z_k) = 1_n \frac{\mathrm{tr}(S^k)}{n} + \eta\,\frac{S^k}{z-z_k}\,.$$

Local Lax operator for XXX chain

$\eta \to 0$ together with substitution $V = 1 + \eta\Lambda$ → Gaudin model

Spectral duality. Toy example



H(p,q)=E fixed energy level of some one-particle Hamiltonian.

System I — motion along the fixed energy level

System II- motion across the fixed energy levels

Spectral duality in CM-RS family

$p \backslash q$	*rational*	*trigonometric*
rat.	rational CM (self-dual) ↻	trig. CM ↗
trig.	↙ rational RS	trig. RS (self-dual) ↻

Action and coordinates get interchanged in dual systems

Ruijsenaars 88-90

In gauge theories Fock, Nekrasov Roubtsov, A.G. 1999,

Differential operator In q

$$\hat{H}^{CM}\Psi_{\vec{k}}(\vec{q}) = E\Psi_{\vec{k}}(\vec{q})\,, \quad E = \sum_{i=1}^{N}\frac{k_i^2}{2}\,,$$

$$\hat{H}^{RS}\Psi_{\vec{k}}(\vec{q}) = \tilde{E}\Psi_{\vec{k}}(\vec{q})\,, \quad \tilde{E} = \sum_{i=1}^{N} e^{2\omega q_i}\,,$$

Difference operator In k

Spectral duality for the inhomogeneous spin chains

$$\frac{\det\limits_{N\times N}(\lambda - L(z))}{\det\limits_{N\times N}(\lambda - \Lambda)} = \frac{\det\limits_{n\times n}(z - \tilde{L}(\lambda))}{\det\limits_{n\times n}(z - Z)}\,.$$

Classical spectral curves for the dual systems are identical

$\{L \overset{\otimes}{,} L\}_r \diagdown r_{12}$	XXX	XXZ
linear	rational Gaudin (self-dual) ↻	trig. Gaudin ↗
quadratic	↙ XXX spin chain	XXZ spin chain (self-dual) ↻

Twists and inhomogenities in the dual systems get interchanged

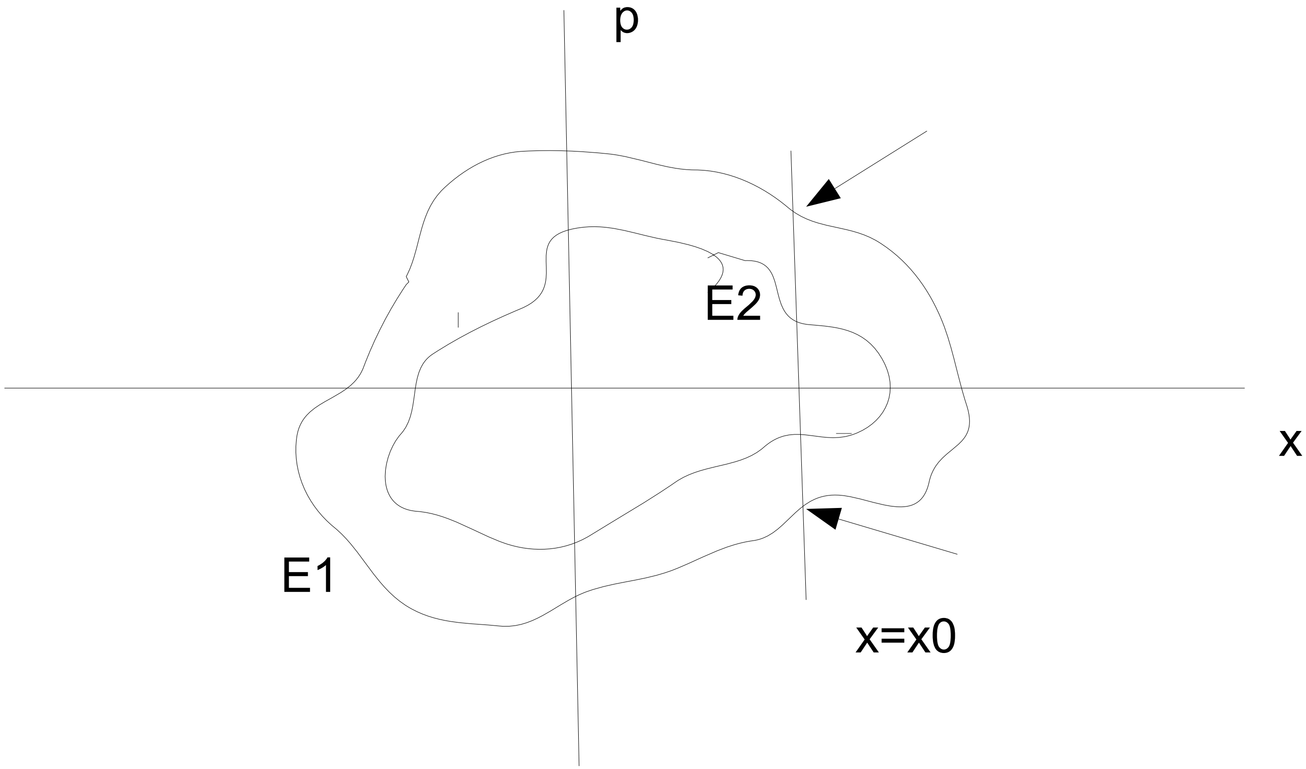
Harnad,Adams. Hurtubise 90'

$$\hat{T}_0(z) = V_0 \hat{L}_0^1(z - z_N)...\hat{L}_0^N(z - z_1) = V_0 R_{01}(z - z_N)...R_{0N}(z - z_1)\,,$$

Quantum monodromy matrix. generating function for the Quantum Hamiltonians

The dual quantum systems have identical up to multiplicities systems of nested Bethe anzatz equations. The off-shell Bethe vectors in the dual systems obey the dual pair of Knizhnik-Zamolodchikov equations
Mukhin,Tarasov,Varchenko 07'-22'

QC duality. Toy example



Bethe Anzatz equations at the quantum spin chain side are equivalent to the intersection of two Lagrangian submanifolds H=E and x=x0 at the classical Calogero-Ruijsenaars side

QC duality between families

<u>Quantum-classical dualities:</u>

class. rat. Calogero-Moser	$\longleftrightarrow$	quant. rat. Gaudin
class. trig. Calogero-Moser	$\longleftrightarrow$	quant. trig. Gaudin
class. rat. Ruijsenaars-Schneider	$\longleftrightarrow$	quant. XXX chain
class. trig. Ruijsenaars-Schneider	$\longleftrightarrow$	quant. XXZ chain

$$\eta = \hbar\,,$$
$$q_i = z_i\,,$$
$$\dot{q}_i = H_i/\hbar\,,$$

Identification of parameters under QC duality

Zabrodin,Zotov, A.G. 13'
Gaiotto,Koroteev 13'
Beketov,Lyashuk, Zabrodin,Zotov 14'

QC duality

Bethe anzatz equations at the quantum spin chain side =

Intersection of two Lagrangian submanifolds at the

Calogero-Ruijsenaars side

Two Lagrangian submanifolds are as follows

1. Fixed coordinates(=fixed inhomogeneities)

2. Fixed Hamiltonians (=fixed twists)

BA equations= equations for classical momenta at CM-RS side)

QQ duality between families

Quantum-quantum dualities:

quant. rat. Calogero-Moser	$\longleftrightarrow$	rational KZ
quant. trig. Calogero-Moser	$\longleftrightarrow$	trigonometric KZ
quant. rat. Ruijsenaars-Schneider	$\longleftrightarrow$	rational qKZ
quant. trig. Ruijsenaars-Schneider	$\longleftrightarrow$	trigonometric qKZ

Full picture for QQ -- Zabrodin,Zotov 17
Role of Matsuo-Cherednik in QQ – Bulycheva, A.G. 14

$$\kappa\partial_{z_i}\Big|\Psi\Big\rangle = \left(V^{(i)} + \hbar\sum_{j\neq i}^{n}\frac{P_{ij}}{z_i - z_j}\right)\Big|\Psi\Big\rangle$$

KZ equation for the rational spin model

Parameter k in front of derivative in KZ equation gets identified with Planck constant in the quantum Calogero-Ruijsenaars models

QQ duality between families

$$\left(\kappa^2 \sum_{i=1}^{n} \partial_{z_i}^2 - \sum_{i \neq j}^{n} \frac{\hbar(\hbar - \kappa)}{(z_i - z_j)^2}\right) \Psi = E\Psi \, ,$$

$$\langle \Omega | P_{ij} = \langle \Omega |$$

Solution to the KZ equation symmetric under permutations of inhomogeneities
Matsuo-Cherednik projection 91-92

Solution to the Schrodinger equation at Calogero side coincides with the symmetrized solution to KZ at the spin side.
the spectrum of Calogero is expressed in terms of twists as at a classical level

Consistency of spectral and QC-QQ dualities

$$
\begin{array}{ccc}
\text{spin chain/Gaudin} & \xrightarrow{\text{spectral}} & \text{dual spin chain/Gaudin} \\
\big\downarrow \text{QC} & & \big\downarrow \text{QC} \\
\text{class. many-body} & \xleftarrow{\text{spectral}} & \text{dual class. many-body}
\end{array}
$$

Classical consistency

$$
\begin{array}{ccc}
\text{(q)KZ equations} & \xrightarrow{\text{duality in KZ}} & \text{dual (q)KZ equations} \\
\big\downarrow \text{QQ} & & \big\downarrow \text{QQ} \\
\text{quant. many-body} & \xleftarrow{\text{spectral}} & \text{dual quant. many-body}
\end{array}
$$

Quantum consistency

QC and QC duality via geometry

Bethe anzatz equation at the spin chain side yields the cohomologies
or K-theory or the particular complex manifolds
Nekrasov-Shatashvili 2009

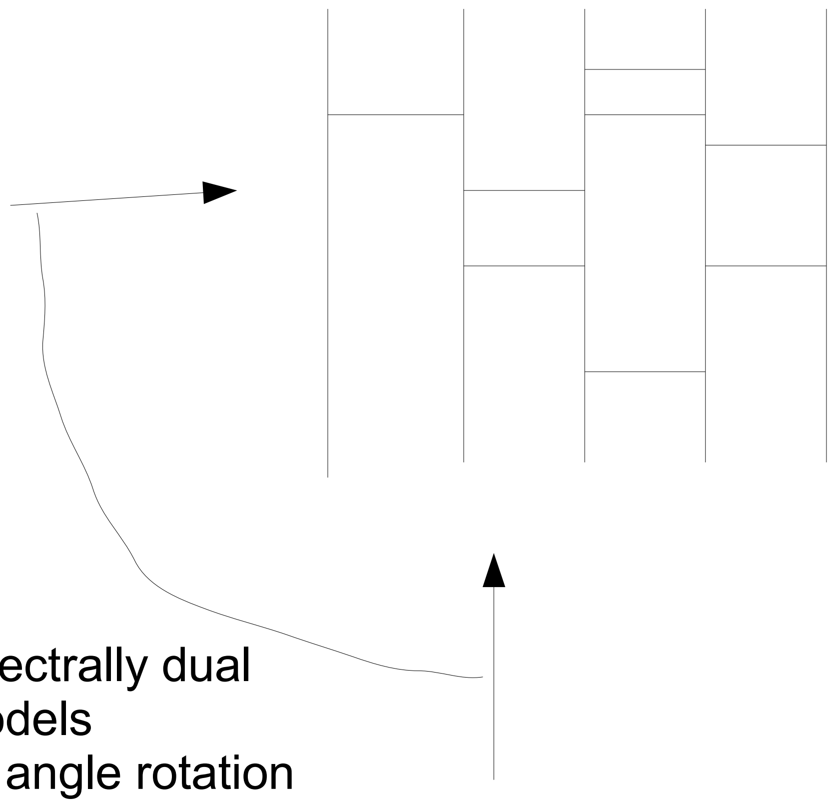
Hence according to QC and QQ duality we could expect that the classical
and quantum CM-RS systems describe the cohomologies and K-theory
as well

It is true indeed

Open Toda - flags Givental, Givental -Kim 94-96
Calogero ---
Ruijsenaars ----cotangent bundle to flags

Koroteev,Zeitlin,Pushkar,Smirnov, Sage 17'-21'

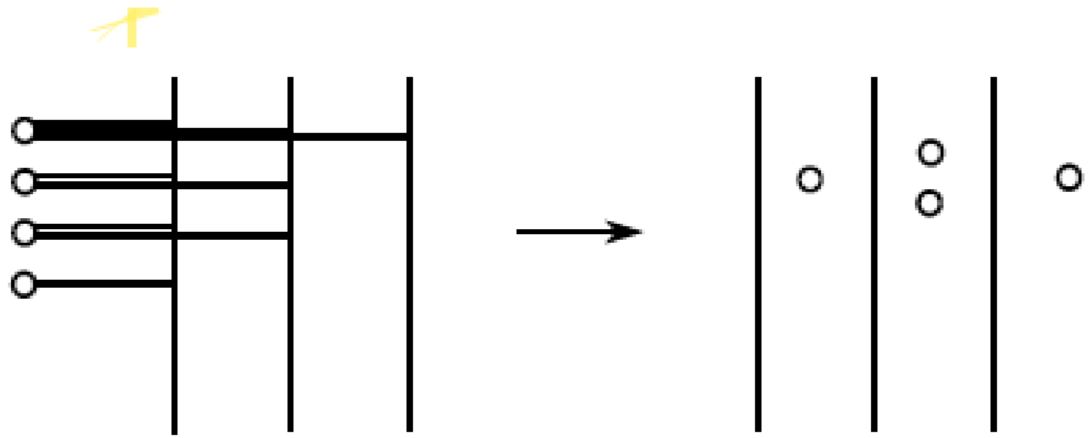
Spectral duality and SYM



Branes where the gauge theory lives on

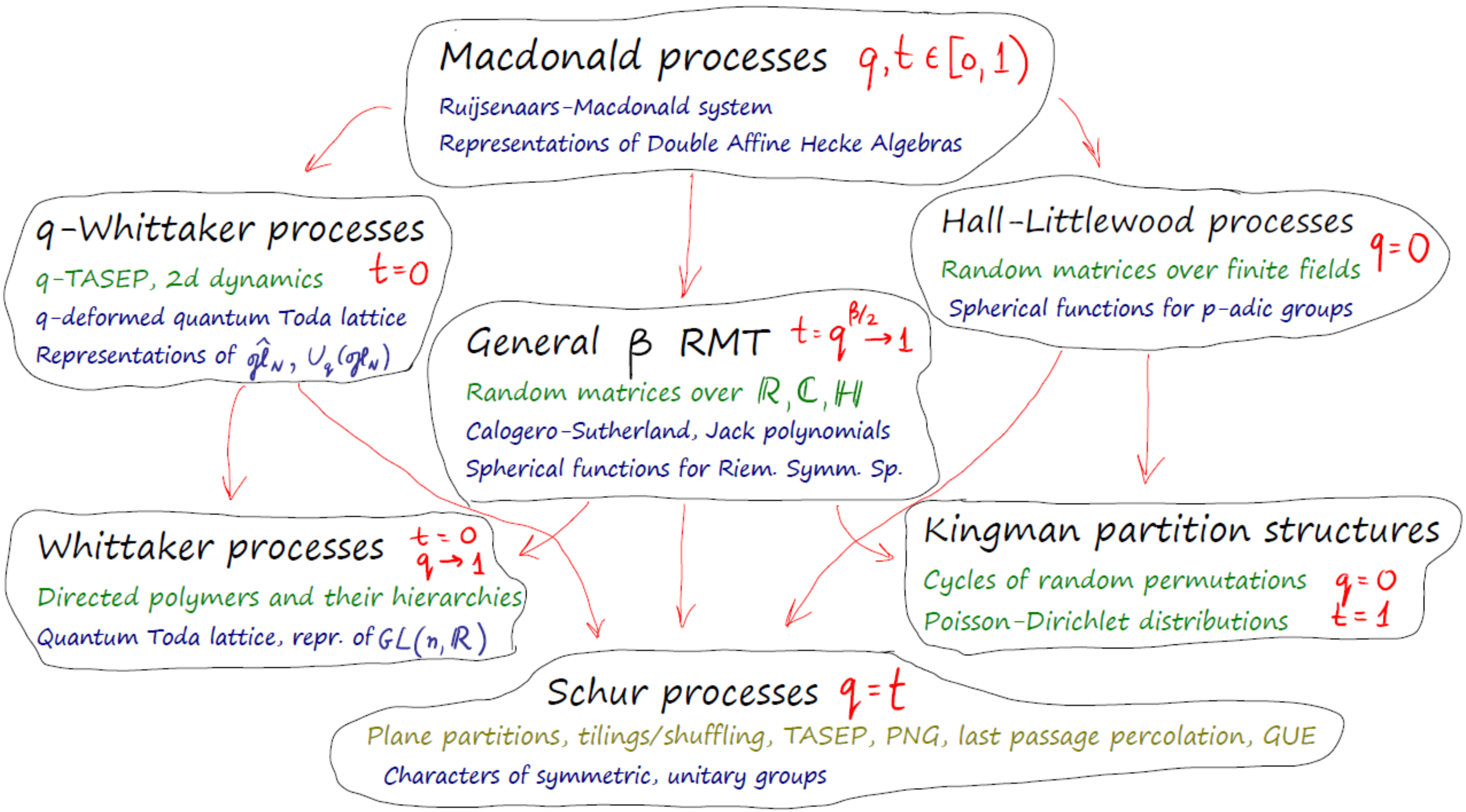
Spectrally dual models
90 angle rotation
Gukov,Mironov, A.G. 97'

QQ and QC duality in SYM



Gaiotto-Koroteev 13'

World of integrable probabilities



Probability ->Measure ->Process

Okounkov-Reshetikhin, 03'

$$\Psi_\lambda(y) \longrightarrow \Psi^+_\lambda(x)\Psi_\lambda(y) \longrightarrow \Psi_{\lambda_1}(x_1)\Psi_{\lambda_2/\lambda_1}(x_2)\dots\Psi_{\lambda_N/\lambda_{N-1}}(x_{N-1})\Psi_{\lambda_N}(x_N)$$

Dualities for the integrable probabilities as QQ duality

- There were observations that the Macdonald measure is related with the measure for the 6-vertex higher rank stochastic models
- The arguments of the Macdonald measure were identified with the local jump rates in 6-vertex models
- The non-symmetric Macdonald polynomial was derived from the higher rank 6-vertex model on the cylinder with the twists
- These results have no good explanations

Example of duality

Height function

$$(-1)^l E_{6v}\left(\prod_{i=1}^{l} \frac{Q^{h(M,N)} - Q^{i-1}}{1-Q^i}\right) = E_{MM}\left(e_l(q^{\lambda_1}t^{n-1}, q^{\lambda_2}t^{n-2}, \dots, q^{\lambda_n})\right),$$

Two vev's in different stochastic models coincide

$$E_{6v}\left(\prod_{i=1}^{l}(Q^{h(M,N)} - Q^{i-1})\right) =$$

$$Q^{\frac{l(l-1)}{2}} \oint \frac{dw_i}{2\pi i} \cdots \oint \frac{dw_l}{2\pi i} \prod_{1\le a<b\le l} \frac{w_a - w_b}{w_a - Qw_b} \times \prod_{i=1}^{l}\left(w_i^{-1} \prod_{x=1}^{M-1} \frac{1 - s_x \chi_x^{-1} w_i}{1 - s_x^{-1}\chi_x^{-1} w_i} \prod_{y=1}^{N} \frac{1 - Qu_y}{1 - Qu_y}\right).$$

Borodin, Petrov, Orr, Agarwall,Corwin, Bufetov 16'-19'

Example of duality

6-vertex family

Macdonald family

$$E_{q-TASEP}\Big(\prod_{j=1}^{k} q^{x_{n_i}+n_i}\Big) = E_{q-Whit}\Big(\prod_{j=1}^{k} q^{\lambda_{n_i}}\Big) = \frac{D_1^k \Pi(a)}{\Pi(a)},$$

$$D_1^N P_{\lambda_N^N}(a) = q^{x_N} P_{\lambda_N^N}(a),$$

t=0 limit of Macdonald operator

λ_N^N Boundary variables in Gelfand-Tsetlin scheme

Inhomogeneous TASEP stochastic model

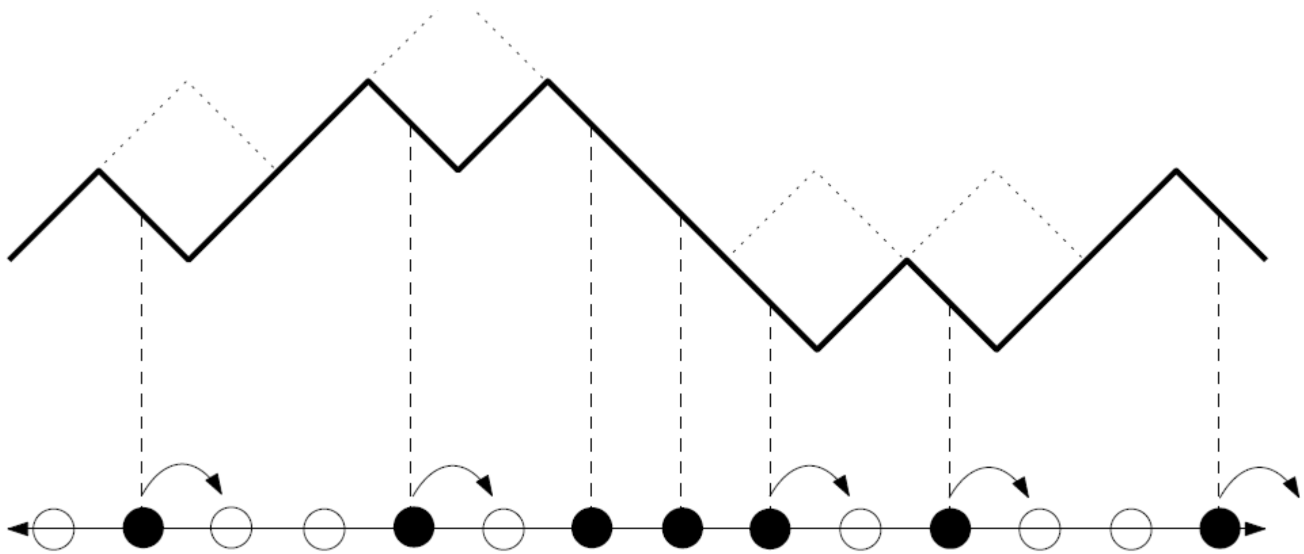


FIGURE 1. TASEP particle configuration and height function with possible right jumps of particles denoted by arrows and possible height function growth locations denoted by dotted wedges.

Jump rates are position dependent

Inhomogeneous TASEP stochastic model

$$M = \sum_{i=1}^{L} (s S_i^- S_{i+1}^+ + r S_i^+ S_{i+1}^- + \frac{1}{4} S_i^z S_{i+1}^z) - \frac{L}{4} .$$

$$\frac{dP}{dt} = MP ,$$

Markov operator for homogeneous ASEP

$$T(x|\vec{z}) = tr_0 \Big[R_{0,L}(\frac{x}{z_L}) R_{0,L-1}(\frac{x}{z_{L-1}}) \dots R_{0,1}(\frac{x}{z_1}) \Big] ,$$

Transfer-matrix for Inhomogeneous TASEP

$$P(t+1) = M(x|\vec{z}) P(t) ,$$

$$M(x|\vec{z}) = T(x|\vec{z}) T(z_1|\vec{z})^{-1} .$$

Discrete time evolution

$$M|S> = |S>$$

Analogue of the steady state for inhomogeneous stochastic models

Inhomogeneous TASEP model

$$|S> = v_1(z_1)v_2(z_2)\dots v_L(z_L)\,.$$

$$v(z) = \begin{pmatrix} z \\ 1 \end{pmatrix}\,,$$

$$R_{12}(z_2/z_1)v_1(z_1)v_2(z_2) = v_2(z_2)v_1(z_1)\,.$$

Relation from the Faddeev-Zamolodchikov algebra

$$R(u) = \frac{P}{u} + \sum_{\alpha>\beta}^{n} E_{\alpha\alpha} \otimes E_{\beta\beta}\,, \qquad P = \sum_{\alpha,\beta}^{n} E_{\alpha\beta} \otimes E_{\beta\alpha}\,,$$

$$R(u) = \begin{pmatrix} \frac{1}{u} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{u} & 0 \\ 0 & \frac{1}{u} & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{u} \end{pmatrix}$$

$$R_{12}(u)R_{21}(-u) = -\frac{1}{u^2}\,1_n \otimes 1_n\,.$$

Unitarity constraint

For SL(2) chain

Goldfish-(inhomogeneous multispecies)TASEP QC duality

Theorem 1 *Let us identify the inhomogeneity parameters z_i in the quantum rational* $\mathrm{GL}(n)$ *spin chain (7.4)-(7.11) on $N \geq n$ sites with the positions of particles q_i in the rational Calogero's Goldfish model (6.6)*

$$z_j = q_j\,, \quad j = 1, \ldots, N \tag{8.1}$$

and make a substitution of the eigenvalues (7.8)[8]

$$\dot{q}_j = h_j(\{q\}, \{\mu\}) \tag{8.2}$$

into the classical higher integrals of motion $H_k^{\mathrm{R}}(\{\dot{q}\}, \{q\})$ (6.6). If, for any $N \geq M_1 \geq M_2 \geq \ldots \geq M_{n-1} \geq 0$, the Bethe roots $\{\mu_a^i\}$ satisfy the Bethe equations (7.9)-(7.11), i.e., $h_j(\{q\}, \{\mu\})$ belong to the spectrum of the rational five-vertex model, then the conserved quantities of the classical rational Calogero's Goldfish system are equal to

$$H_k^{\mathrm{R}}\Big(\{h(\{q\},\{\mu\})\},\{q\}\Big) = \begin{cases} V_1^{N-M_1} V_2^{M_1-M_2} \ldots V_{n-1}^{M_{n-2}-M_{n-1}} V_n^{M_{n-1}}, & k = N\,, \\ V_{i+1}^{M_i-M_{i+1}} V_{i+2}^{M_{i+1}-M_{i+2}} \ldots V_n^{M_{n-1}}, & k = M_i\,, \\ 0, & \text{otherwise}\,, \end{cases} \tag{8.3}$$

$k = 1, \ldots, N$, so that only n of N classical Hamiltonians are non-zero.

Trigonometric Goldfish-TASEP QC duality

Theorem 2 *Identify the inhomogeneity parameters z_i in the quantum trigonometric* GL(n) *spin chain (7.16)-(7.23) on $N \geq n$ sites with the positions of particles q_i in the trigonometric Calogero's Goldfish model particles*

$$z_j = e^{q_j}\,, \quad j = 1\,, \ldots\,, N \tag{8.34}$$

and make a substitution of the eigenvalues (7.20)[10]

$$\dot{q}_j = h_j(\{z\}, \{\mu\}) \tag{8.35}$$

into the classical higher integrals of motion $H_k^{\mathrm{T}}(\{\dot{q}\}, \{q\})$ (6.14). If, for any $N \geq M_1 \geq M_2 \geq \ldots \geq M_{n-1} \geq 0$, the Bethe roots $\{\mu_a^i\}$ satisfy the Bethe equations (7.21)-(7.23), i.e., $h_j(\{q\}, \{\mu\})$ belong to the spectrum of the trigonometric five-vertex model, then the conserved quantities of the classical trigonometric Calogero's Goldfish system are equal to

$$H_k^{\mathrm{T}}\Big(\{h(\{z\}, \{\mu\})\}, \{z\}\Big) = V_1^{k-[M_1]_k} V_2^{[M_1]_k-[M_2]_k} .. V_{n-1}^{[M_{n-2}]_k-[M_{n-1}]_k} V_n^{[M_{n-1}]_k}\,, \tag{8.36}$$

where

$$[M]_k = \begin{cases} M, & M \leq k, \\ k, & M > k. \end{cases} \tag{8.37}$$

QQ Goldfish -TASEP duality rational case

$$e^{\hbar\partial_i}\,|\Psi\rangle = \hat{K}_i^{\hbar}\,|\Psi\rangle\ ,$$

where

$$\hat{K}_i^{\hbar} = \tilde{R}_{i,i-1}\left(q_i - q_{i-1} + \hbar\right)\ldots\tilde{R}_{i,1}\left(q_i - q_1 + \hbar\right)V^{(i)}\tilde{R}_{i,n}\left(q_i - q_N\right)\ldots\tilde{R}_{i,i+1}\left(q_i - q_{i+1}\right)\ ,$$

$$\hat{H}_k^{\mathrm{R}}\,\langle\Omega|\Psi\rangle = \sum_{I_k}\left[\prod_{j\in I_k}\prod_{l\notin I_k}\frac{1}{q_j - q_l}\right]\prod_{n=1}^{k}e^{\hbar\partial_{i_n}}\langle\Omega|\Psi\rangle$$

$$= (-1)^{\frac{k(k-1)}{2}}\sum_{I_k}\langle\Omega|\hat{H}_{i_1}\ldots\hat{H}_{i_k}|\Psi\rangle\prod_{m<r}^{k}(q_{i_m} - q_{i_r})^2 = \lambda_k\,\langle\Omega|\Psi\rangle\ ,$$

For general multiplicities of twists

$$\lambda_k = \begin{cases} V_1^{M_1}V_2^{M_2}\ldots V_{n-1}^{M_{n-1}}V_n^{M_n}, & k = N, \\ V_{i+1}^{M_{i+1}}V_{i+2}^{M_{i+2}}\ldots V_n^{M_n}, & k = N - \sum_{l=1}^{i} M_l, \\ 0, & \text{otherwise.} \end{cases}$$

QQ Goldfish-TASEP duality trigonometric case

$$\hat{T}_i^q f(z_1, ..., z_i, .., z_N) = f(z_1, ..., qz_i, .., z_N),$$

$$\hat{T}_i^q \left|\Psi\right\rangle = \hat{K}_i^q \left|\Psi\right\rangle,$$

$$\hat{K}_i^q = \tilde{R}_{i,i-1}\left(qz_i/z_{i-1}\right)...\tilde{R}_{i,1}\left(qz_i/z_1\right)V^{(i)}\tilde{R}_{i,n}\left(z_i/z_N\right)...\tilde{R}_{i,i+1}\left(z_i/z_{i+1}\right)$$

Trigonometric qKZ

$$\langle J| = \sum_{i_1,\ldots,i_N=1}^{n} \langle e_{i_1}| \otimes ... \otimes \langle e_{i_N}|$$

$$\prod_{j=1}^{k} \hat{T}_j^q \langle J|\Psi\rangle = \langle J|\hat{K}_{i_1}^1 ... \hat{K}_{i_k}^1|\Psi\rangle, \quad \text{for} \quad i_l \neq i_m.$$

$$\hat{H}_k^{\mathrm{T}} \langle J|\Psi\rangle = (-1)^{\frac{k(k-1)}{2}} \sum_{I_k} \prod_{l<r} \frac{(z_{i_l} - z_{i_r})^2}{z_{i_l} z_{i_r}} \langle J|H_{i_1}..H_{i_k}|\Psi\rangle = \lambda_k \langle J|\Psi\rangle$$

Wave function of trigonometric goldfish

Goldfish model and spectral duality with open Toda spin chain

$$\partial_i |\Psi\rangle = \left(-\sum_{j>i}^{N} e^{x_i - x_j} T_{ij} + \sum_{j<i}^{N} e^{x_j - x_i} T_{ji} + y_i \right) |\Psi\rangle$$

KZ for nil-Hecke algebra

$$T_i^2 = 0, \quad T_i T_j = T_j T_i \ \text{ for } \ |i-j| > 1,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$

$$y_i y_j = y_j y_i,$$

$$y_j T_i = T_i y_j \ \text{ for } \ j \neq i, i+1,$$

$$T_i y_i - y_{i+1} T_i = y_i T_i - T_i y_{i+1} = k \in \mathbb{C}.$$

This KZ upon the Matsuo-Cherednik projection yields the wave function of open Toda. The dual KZ yields the wave function of the Goldfish model. Hence the QQ duality fits with the spectral duality for the TASEP-Goldfish

Ruijsenaars 90

General comments

- Stringy interpretation.Spectral duality — version of T-duality (mirror symmetry). QC and QQ duality- so called 3d mirror symmetry (version of Langlands, Gaiotto-Koroteev)
- Elliptic versions of these dualities are known only for few examples.No complete picture yet.

  (Mironov,Morozov,Zenkevich, Koroteev,Shakirov, Zotov,Grekov)
- Useful applications for SUSY Yang-Mills theory which I have not touched at all.

Conclusion

- The «mysterious» dualities between two families of stochastic models are just the counterpart of the QQ dualities for integrable many-body systems
- The stochastic processes suggest the interesting generalization of the QQ duality for the integrable models
- New duality between Goldfish model and periodic inhomogeneous higher rank TASEP

Thank you for attention!