Shlomo S. Razamat (Technion & IAS)

Three roads to the Van Diejen model, and beyond …

ZOOM, 8/3/2021

Workshop on Elliptic Integrable Systems

Based mainly on work with **Belal Nazzal** and **Anton Nedelin** (to appear and WIP)

Builds on previous work with **Gabi Zafrir (1906.05088)** and **Evyatar Sabag (1910.03603, 2006.03480)**

Happy families are all alike;

every unhappy family is unhappy in its own way.

Happy Families:

→ 6d (1,0) SCFTs

→ Elliptic Integrable models

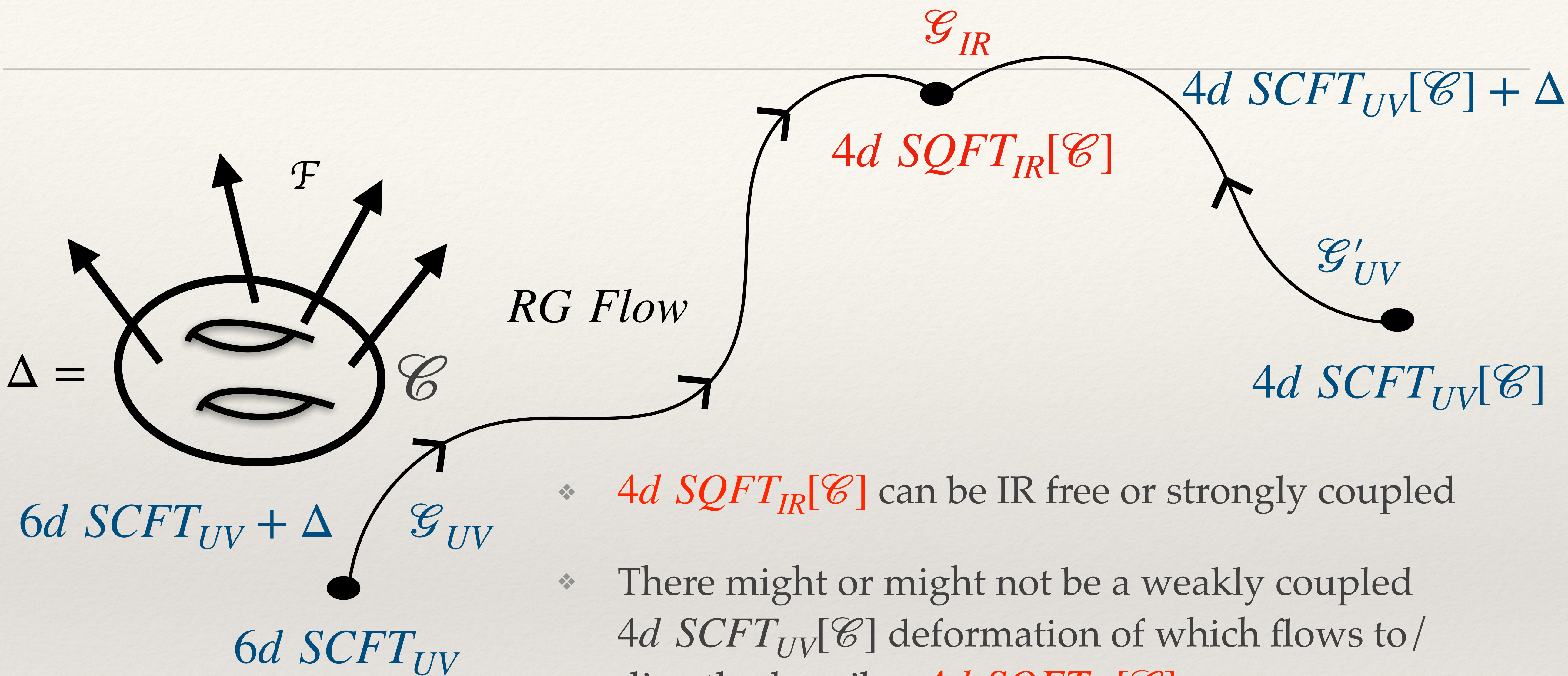
Outline

- General logic: From Indices and Surface Defects to $A\Delta O$
- Three roads from rank one E-string to the BC_1 van Diejen Model
  - The $A_{N=1}$ van Diejen Model
  - The $C_{N=1}$ van Diejen Model
  - The $(A_1)^{N=1}$ van Diejen Model
- The A_N generalization
- Comments

Part I: $A\Delta O$ from 6d SCFTs

$4d\ \mathcal{N} = 1$ SQFTs from 6d SCFTs

Gaiotto 2009 and many others



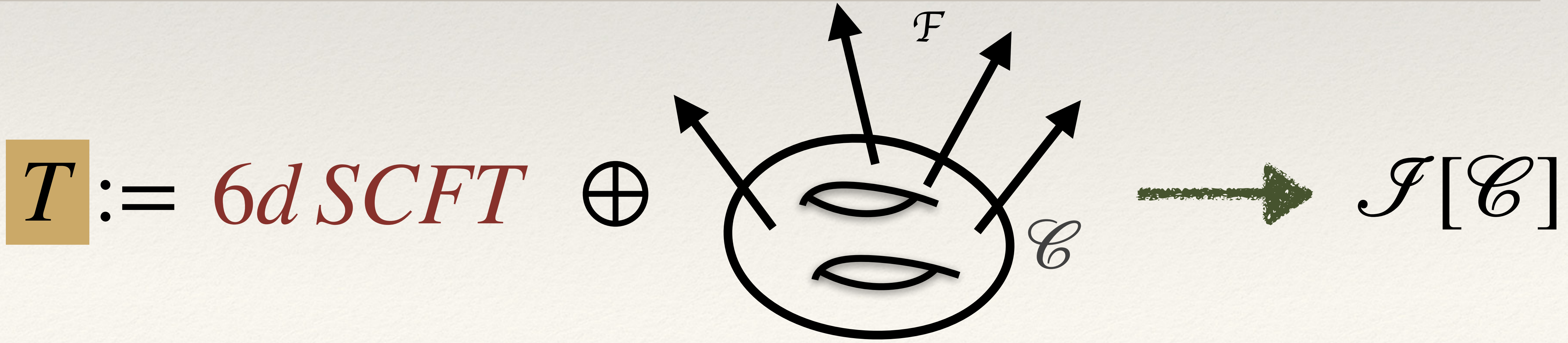
- $4d\ SQFT_{IR}[\mathscr{C}]$ can be IR free or strongly coupled
- There might or might not be a weakly coupled $4d\ SCFT_{UV}[\mathscr{C}]$ deformation of which flows to/ directly describes $4d\ SQFT_{IR}[\mathscr{C}]$
- In case such a flow in 4d exists many of its properties are encoded by the $6d\ SCFT_{UV}$ and geometry $\mathscr{C}$
- Many strong coupling phenomena follow from geometry

- Q: Given $6d\ SCFT_{UV}$ and $\mathscr{C}$ what is $4d\ SCFT_{UV}[\mathscr{C}]$?

4d Theories and indices

- Say the 4*d* $SQFT_{IR}[\mathscr{C}]$ has been derived
- We can compute various protected quantities for 4*d* $SQFT_{IR}[\mathscr{C}]$
- Such partition functions can be non-perturbatively computed and encode interesting information about the strongly coupled fixed point: invariants of continuous parameters
- Example of such a quantity is the supersymmetric index

$$\mathscr{I}[T](q,p,\{u\}) = \mathrm{Tr}(-1)^F q^{j_1-j_2+\frac{1}{2}R} p^{j_1+j_2+\frac{1}{2}R} \prod_{i=1}^{\mathrm{rank}\, G_F} u_i^{Q_i}$$



The various parameters of the index

- The parameters p and q are there for any $\mathcal{N} = 1$ SCFT: superconformal fugacities
- The parameters u are of two sorts:
- (a) Correspond to Cartan generators of the symmetry of 6d SCFT G_{6d} : internal
- (b) Correspond to Cartan generators of the symmetry associated to the puncture
- Different types of punctures:
- Maximal with symmetry G_{5d}
- Minimal with rank one symmetry $U(1)$ *or* $SU(2)$

Examples

- Take A_1 (2,0) SCFT on three punctured sphere with $\mathcal{N}=2$ preserving flux
- The theory is given by a tri-fundamental chiral superfirld

$$\mathcal{I}[T] = \Gamma_e(t^{\frac{1}{2}} x^{\pm 1} y^{\pm 1} z^{\pm 1})$$

$$\Gamma_e(z) := \prod_{i,j=0}^{\infty} \frac{1 - q^{i+1} p^{j+1} z^{-1}}{1 - q^i p^j z}$$

- Take rank 1 E-string (1,0) SCFT on three punctured sphere with certain flux $\mathcal{F}$
- The theory is $SU(3)$ $\mathcal{N}=1$ SQCD with $N_f = 6$

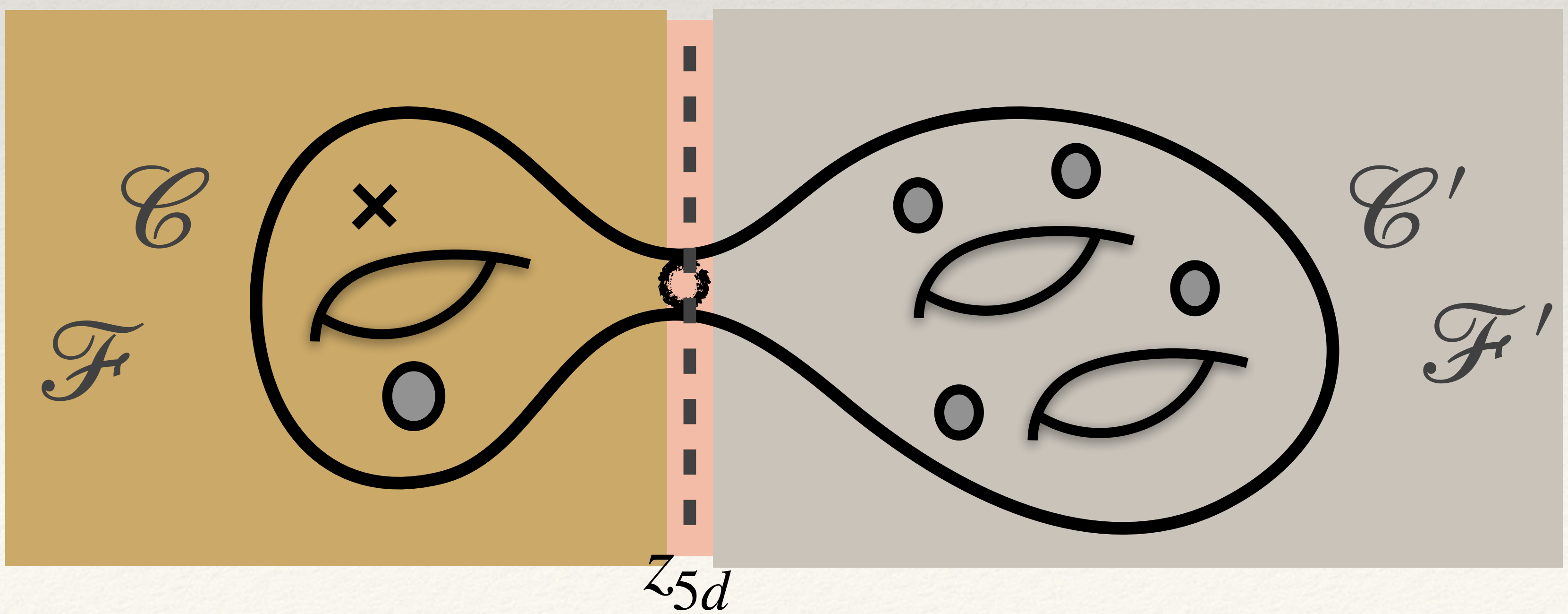
$$\mathcal{I}[T] = \frac{(q;q)^2 (p;p)^2}{6} \oint \frac{dt_1}{2\pi i t_1} \frac{dt_2}{2\pi i t_2} \frac{\prod_{i=1}^{3} \left[\Gamma_e((pq)^{1/6} u^6 t_i x^{\pm 1}) \Gamma_e((pq)^{1/6} v^6 t_i y^{\pm 1}) \Gamma_e((pq)^{1/6} w^6 t_i z^{\pm 1}) \prod_{j=1}^{6} \Gamma_e((pq)^{1/3} u^{-2} v^{-2} w^{-2} t_i^{-1} a_j) \right]}{\prod_{i \neq j}^{3} \Gamma_e(\frac{t_i}{t_j})}$$

$$(a;b) := \prod_{i=0}^{\infty} (1 - b^i a)$$

Gluing indices

- Let us assume that we have derived theories corresponding to two surfaces $\mathscr{C}$ and $\mathscr{C}'$ with fluxes $\mathscr{F}$ and $\mathscr{F}'$ and have computed the corresponding indices
- We then can compute the index of the theory corresponding to a glued surface:

$$\mathcal{I}\left[\mathscr{C}\oplus\mathscr{C}',\mathscr{F}+\mathscr{F}'\right]=\oint\prod_{i=1}^{\text{rank}\,G_{5d}}\frac{dz_i}{2\pi i z_i}\Delta(z_{5d};u_{6d};q,p)\times$$
$$\mathcal{I}\left[\mathscr{C},\mathscr{F}\right](z_i,u_{6d},\cdots;q,p)\times\mathcal{I}\left[\mathscr{C}',\mathscr{F}'\right](z_i,u_{6d},\cdots;q,p)$$



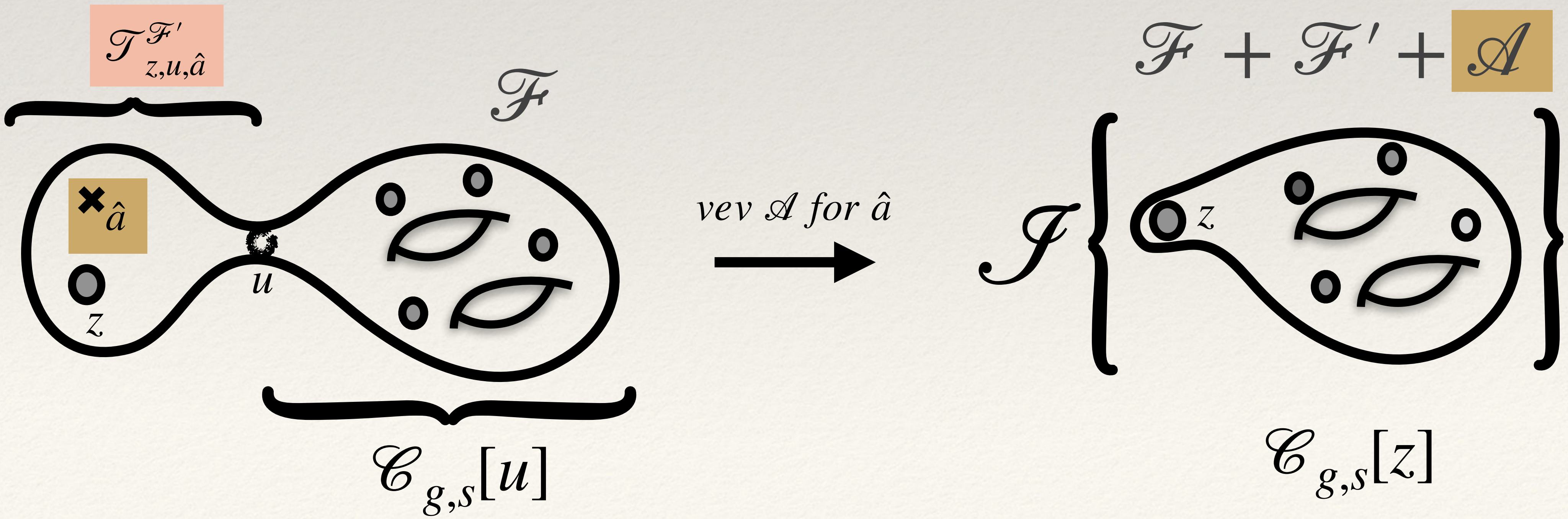
Analytic structure of indices

- The index is a meromorphic function of the various parameters: what are the poles and the residues?
- Take $\mathcal{O}$ to be an operator which can obtain a vacuum expectation value $\langle \mathcal{O} \rangle \neq 0$
- Then the claim is that $\text{Res}_{u \to u^*} \mathcal{I} = \mathcal{I}^{IR}$ where $\mathcal{O}$ contributes to the index with weight $u^{-1} \cdot u^*$
- Residues of indices encode the index of the theory obtain in the IR after turning on a vev
- The vev can be space time dependent if u^* involves p or/and q
- Such a vev will lead to a surface defect in the IR SCFT
- Residues of poles involving p or/and q encode indices in presence of surface defects

Gaiotto, Rastelli, SR 2012

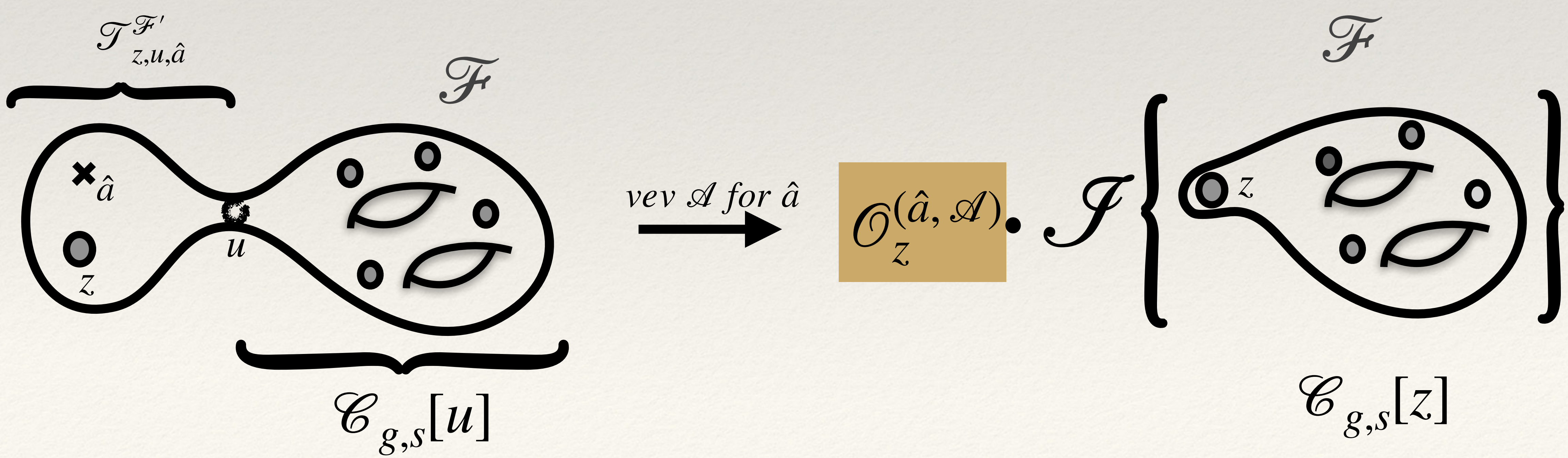
Flows between surfaces

- Let us then compute residues of indices of theories labeled by geometries and 6d SCFTs
- Assume we have derived a theory corresponding to a sphere with two maximal punctures, one minimal and some value of flux $\mathscr{F}'$: $\mathscr{T}^{\mathscr{F}'}_{z,u,\hat{a}}$
- Let us glue this theory to a generic one along a maximal puncture and give a constant vev to some operator $\mathscr{O}$ charged under the minimal puncture symmetry $\hat{a}$.
- Different choices of the operator we give the vev to lead to different theories in the IR
- The theory in the IR corresponds to the same surface but with the flux shifted by some amount $\mathscr{A}$ depending on the operator we give a vev to.



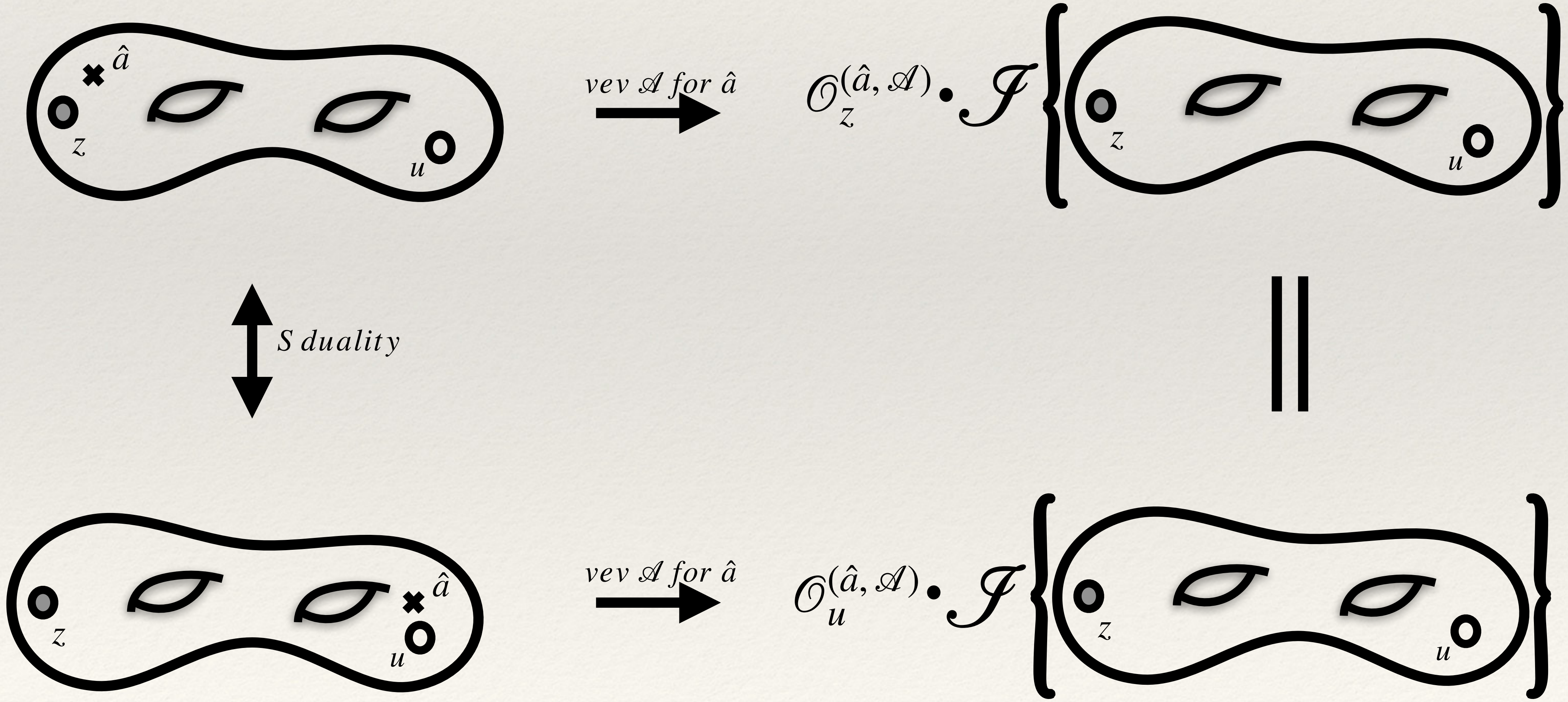
$A\Delta O$ from Indices

- Let us now assume that an operator $\mathscr{O}$ exists such that $\mathscr{A} + \mathscr{F}' = 0$
- Then with constant vev the theory in the IR is the same as the one we glued the three punctured sphere to: the gluing and the vev can be though as action of identity operator
- Now in this setup let us turn on a non constant vev for this operator
- The result turns out to be an $A\Delta O$ acting on the index of the theory we glued.



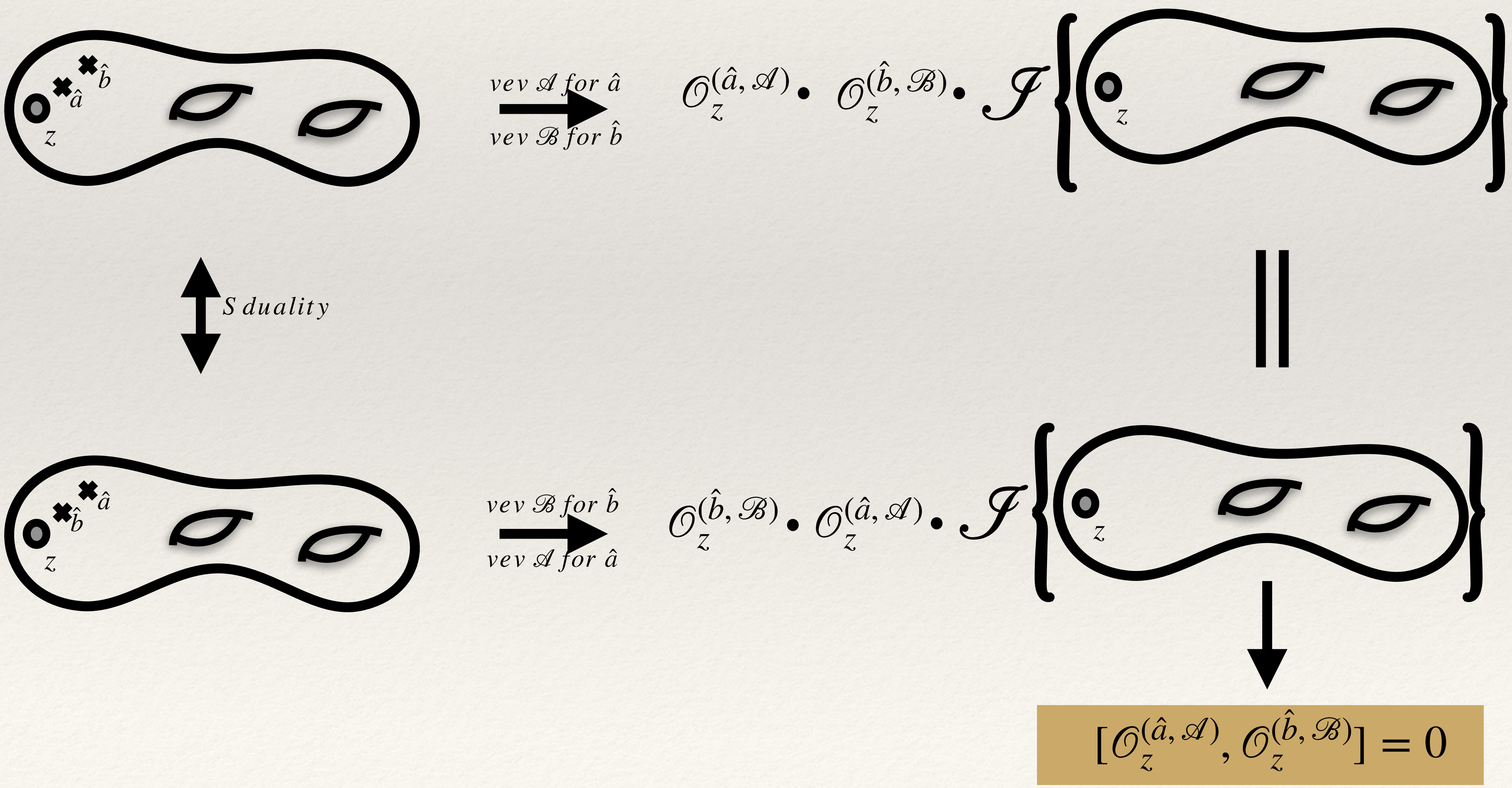
Kernel functions from indices

- As the index is independent of continuous parameters the $A\Delta O$ satisfy various properties
- We can construct the same surface in different ways leading to equivalent theories
- It does not matter in which duality frame we compute the index it is the same
- $\rightarrow$ The index is a Kernel function of the $A\Delta O$



Commutativity from Indices

- We can in general produce different $A\Delta O$ turning on different vevs
- These $A\Delta O$ introduce different types of surface defects
- It does not matter in which order we introduce the defects
- $\rightarrow$ The $A\Delta O$ derived in this way from a commuting set of operators



Summary Part I

- Given a derivation of 4d theories resulting from compactifications these need to satisfy various non trivial properties, such as dualities
- By manipulating the indices of these theories we can derive a set of $A\Delta O$s
- The dualities imply that these $A\Delta O$s have to be commuting and that the indices are Kernel functions
- Since the duality properties are conjectural if the above properties of $A\Delta O$s can be shown to hold true would be a highly non trivial check of these conjectures

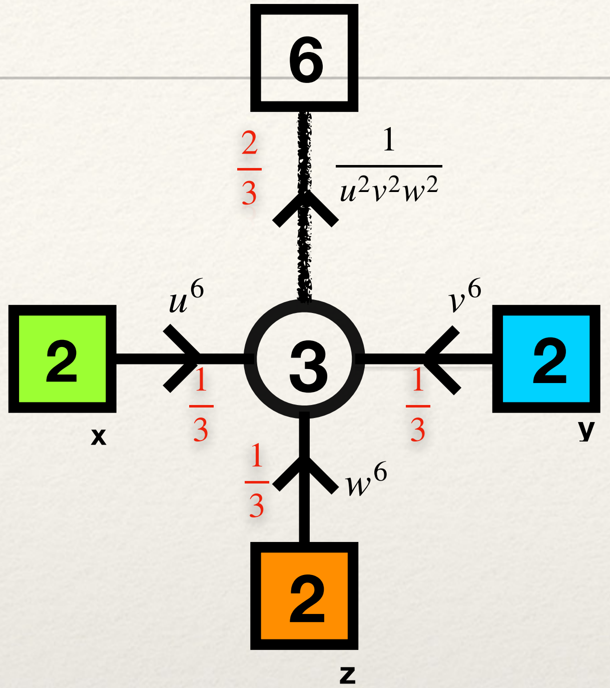
Part II: Three roads to the vD model

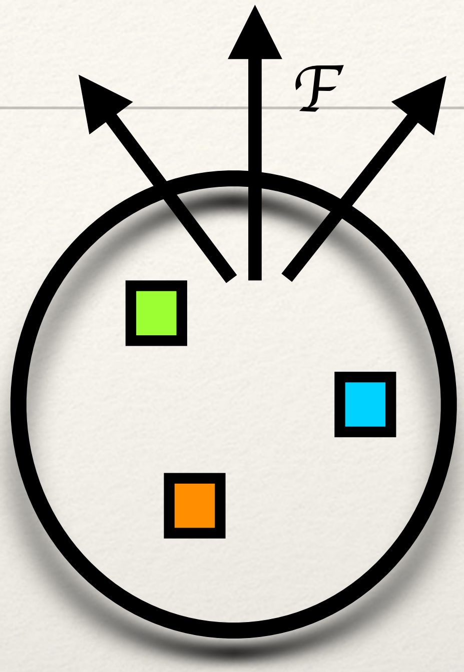
The setup and the result

- Let us apply this procedure to the 6d SCFT being rank one E-string theory
- The 6d symmetry is $G_{6d} = E_8$
- The maximal puncture and minimal are the same with symmetry $G_{5d} = SU(2)$
- There are known (at least) three rather different three punctured spheres for this compactification.
- These differ by values of flux and subtle details of the punctures.
- Each three punctured sphere will lead in principle to $A\Delta O$ operator
- The $A\Delta O$ will turn out to be all van Diejen $A\Delta O$ s shifted by a constant
- The three punctured spheres will be Kernel functions depending on three sets of parameters

E-string three punctured sphere I ($A_{N=1}$)

SR, Sabag 20





- $SU(3)$ $N_f = 6$ SQCD
- $SU(6) \times SU(6) \times U(1)_B \to$
- punct: $SU(2) \times SU(2) \times SU(2)$
- $SU(6) \times U(1)^3 \subset E_8$

``Moment Map" Operators:

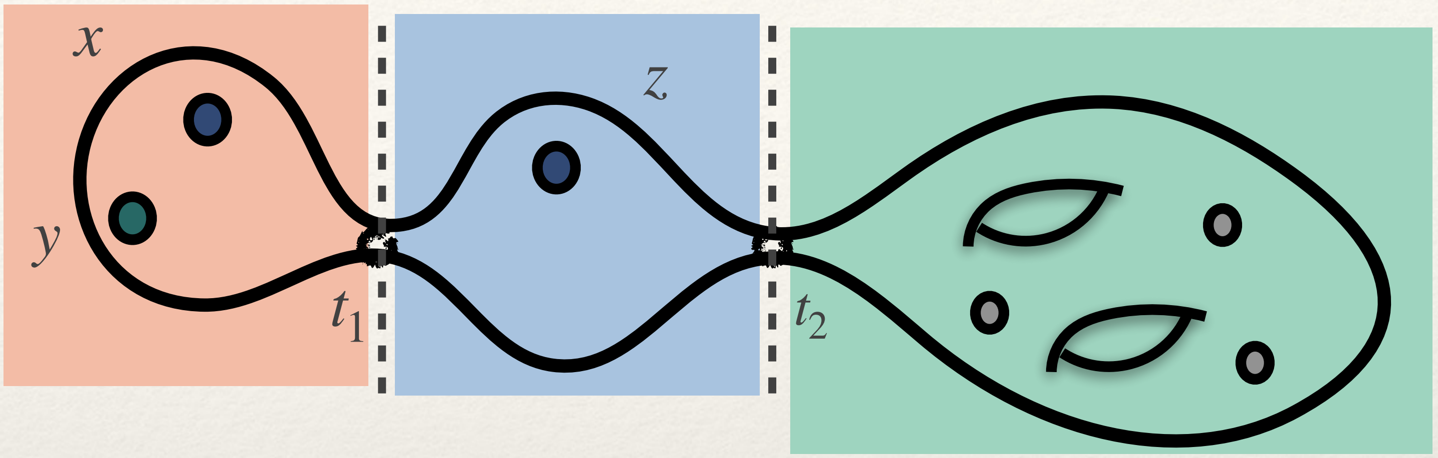
$$M_u : \mathbf{2}_x \otimes \left(\mathbf{6}_{\frac{u^4}{v^2w^2}} \oplus \mathbf{1}_{u^6v^{12}} \oplus \mathbf{1}_{u^6w^{12}} \right) \quad M_v : \mathbf{2}_y \otimes \left(\mathbf{6}_{\frac{v^4}{u^2w^2}} \oplus \mathbf{1}_{v^6u^{12}} \oplus \mathbf{1}_{v^6w^{12}} \right) \quad M_w : \mathbf{2}_z \otimes \left(\mathbf{6}_{\frac{w^4}{u^2v^2}} \oplus \mathbf{1}_{w^6u^{12}} \oplus \mathbf{1}_{w^6v^{12}} \right)$$

The index:

$$\mathcal{K}(x,y,z) = \frac{(q;q)^2(p;p)^2}{6} \oint \frac{dt_1}{2\pi i t_1} \frac{dt_2}{2\pi i t_2} \frac{\prod_{i=1}^{3} \left[\Gamma_e((pq)^{1/6}u^6 t_i x^{\pm 1}) \Gamma_e((pq)^{1/6}v^6 t_i y^{\pm 1}) \Gamma_e((pq)^{1/6}w^6 t_i z^{\pm 1}) \prod_{j=1}^{6} \Gamma_e((pq)^{1/3}u^{-2}v^{-2}w^{-2}t_i^{-1}a_j) \right]}{\prod_{i \neq j}^{3} \Gamma_e(\frac{t_i}{t_j})}$$

$A\Delta O$ from E-string three punctured sphere I

(I) Construct index $\mathcal{J}(x,y,z)$



$$\mathcal{J}(x,y,z) = \left[\frac{(q;q)(p;p)}{2}\right]^2 \oint \frac{dt_1}{2\pi i t_1}\frac{dt_2}{2\pi i t_2} \frac{\mathcal{K}(x,y,t_1)\cdot\overline{\mathcal{K}(t_1,z,t_2)}\cdot\mathcal{I}_{\mathcal{C}}(t_2)}{\Gamma_e(t_1^{\pm 2})\Gamma_e(t_2^{\pm 2})}$$

(II) Compute the residue:

$$\text{Res}_{z\to z^*}\text{Res}_{y\to y^*}\,\mathcal{J}(x,y,z) \sim \mathcal{D}_x^{(y^*,z^*)}\cdot \mathcal{I}_{\mathcal{C}}(x)$$

We can choose any component of $M_w : \mathbf{2}_y \otimes \left(\mathbf{6}_{\frac{w^4}{u^2v^2}} \oplus \mathbf{1}_{w^6u^{12}} \oplus \mathbf{1}_{w^6v^{12}}\right)$ **to give a vev to.**

For concreteness let us choose: $y^* = (qp)^{-\frac{1}{2}}u^{-12}w^{-6}q^{-1} \quad z^* = (qp)^{-\frac{1}{2}}u^{12}w^{6}$

The $A_{N=1}$ $A\Delta O$

- The residue computation is lengthy but in principle straightforward procedure
- (Analyze pinching of the integration contours and use various known integral identities)

$$\mathcal{D}_x^{(y^*,z^*)} \cdot \mathcal{I}_{\mathcal{C}}(x) = \frac{\theta_p((pq)^{\frac{1}{2}}u^{-6}w^{-12}x)\theta_p((pq)^{\frac{1}{2}}u^{-6}v^{-12}x)}{\theta_p(qx^2)\theta_p(x^2)}\prod_{j=1}^{6}\theta_p((pq)^{\frac{1}{2}}u^{-4}v^2w^2a_j^{-1}x)\,\mathcal{I}_{\mathcal{C}}(qx)+$$

$$\frac{\theta_p((pq)^{\frac{1}{2}}u^{-6}w^{-12}x^{-1})\theta_p((pq)^{\frac{1}{2}}u^{-6}v^{-12}x^{-1})}{\theta_p(qx^{-2})\theta_p(x^{-2})}\prod_{j=1}^{6}\theta_p((pq)^{\frac{1}{2}}u^{-4}v^2w^2a_j^{-1}x^{-1})\,\mathcal{I}_{\mathcal{C}}(q^{-1}x) + W^{(y^*,z^*)}(x)\,\mathcal{I}_{\mathcal{C}}(x)$$

$$W^{(y^*,z^*)}(x) = \left[\frac{\theta_p((pq)^{\frac{1}{2}}u^6w^{12}x)\theta_p((pq)^{\frac{1}{2}}v^{12}u^6x)}{\theta_p((pq)^{\frac{1}{2}}u^{18}qx)\theta_p(q^{-1}x^{-2})\theta_p(x^2)}\theta_p((pq)^{\frac{1}{2}}u^{18}x^{-1})\prod_{j=1}^{6}\theta_p((pq)^{\frac{1}{2}}u^4v^{-2}w^{-2}a_jx) + (x\to x^{-1})\right]$$

$$+\prod_{j=1}^{6}\theta_p(u^{-14}v^{-2}w^{-2}q^{-1}a_j)\frac{\theta_p(q^{-1}v^{12}u^{-12})\theta_p((pq)^{\frac{1}{2}}u^6w^{12}x^{\pm1})}{\theta_p(pq^2w^{12}u^{24})\theta_p((pq)^{-\frac{1}{2}}u^{-18}q^{-1}x^{\pm1})}$$

$$\theta_p(z) := \prod_{l=0}^{\infty}(1-zp^l)(1-z^{-1}p^{l+1})$$

The BC_1 van Diejen $A\Delta O$

- The BC_1 van Diejen operator is defined as follows: Use notations of Rains, Ruijsenaars 12

$$\mathcal{D}_x \cdot \mathcal{I}(x) = \frac{\prod_{j=1}^{8} \theta_p((pq)^{\frac{1}{2}} a_j\, x)}{\theta_p(qx^2)\theta_p(x^2)} \mathcal{I}(qx) + \frac{\prod_{j=1}^{8} \theta_p((pq)^{\frac{1}{2}} a_j\, x^{-1})}{\theta_p(qx^{-2})\theta_p(x^{-2})} \mathcal{I}(q^{-1}x) + W(x;a_i)\, \mathcal{I}(x)$$

$$\omega_0 = 1,\ \omega_1 = -1,\ \omega_2 = p^{\frac{1}{2}},\ \omega_3 = -p^{\frac{1}{2}}$$

$$p_0(a) = \prod_{i=1}^{8} \theta_p(p^{\frac{1}{2}} a_i)\,,\ p_1(a) = \prod_{i=1}^{8} \theta_p(-p^{\frac{1}{2}} a_i)\,,$$

$$p_2(a) = p \prod_{i=1}^{8} a_i^{-\frac{1}{2}} \theta_p(a_i)\,,\ p_3(a) = p \prod_{i=1}^{8} a_i^{\frac{1}{2}} \theta_p(-a_i^{-1})$$

$$\mathcal{E}_i(\xi;x) = \frac{\theta_p(q^{-\frac{1}{2}}\xi\omega_i^{-1}x)\theta_p(q^{-\frac{1}{2}}\xi\omega_i x^{-1})}{\theta_p(q^{-\frac{1}{2}}\omega_i^{-1}x)\theta_p(q^{-\frac{1}{2}}\omega_i x^{-1})}$$

$$W(x;a_i) = \frac{\sum_{j=0}^{3} p_j(a)(\mathcal{E}_j(\xi;x) - \mathcal{E}_j(\xi;\omega_j))}{2\theta_p(\xi)\theta_p(q^{-1}\xi)}$$

- The choice of ξ is inessential

The $A_{N=1}$ $A\Delta O$ and the van Diejen $A\Delta O$

- The operator we derived is precisely (up to conjugations) the BC_1 van Diejen $A\Delta O$
- The eight parameters of van Diejen are: $\left(\mathbf{6}_{\frac{v^2w^2}{u^4}} \oplus \mathbf{1}_{u^{-6}v^{-12}} \oplus \mathbf{1}_{u^{-6}w^{-12}}\right)$
- One can repeat the exercise with any component of the moment map and with any puncture, the computations might be different but the result is always the same (up to x-independent constant shift)
- All these operators are thus trivially commuting and $\mathcal{K}(x,y,z)$ is expected to be a Kernel function:

$$\mathcal{D}_x^{(y^*,z^*)} \cdot \mathcal{K}(x,y,z) = \mathcal{D}_y^{(y^*,z^*)} \cdot \mathcal{K}(x,y,z) = \mathcal{D}_z^{(y^*,z^*)} \cdot \mathcal{K}(x,y,z)$$

Proof?

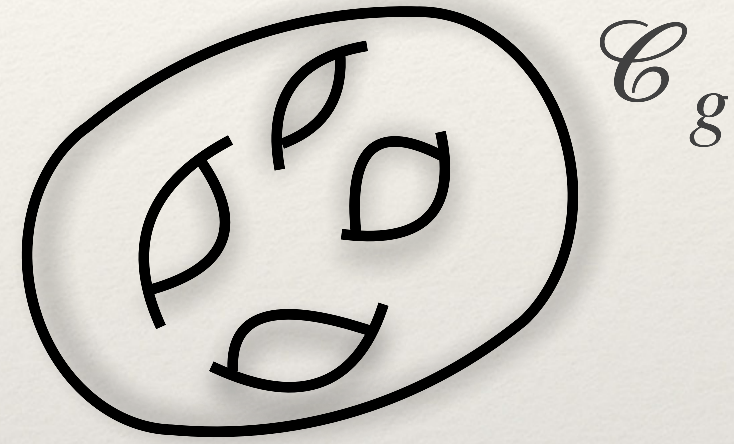
- The imbedding of $SU(6) \times U(1)^3$ in E_8 is

$$E_8 \;\to\; E_7 \times SU(2)_{u^6v^6w^6} \;\to\; SU(6) \times SU(3)_{u^8/(w^4v^4),v^8/(w^4u^4)} \times SU(2)_{u^6v^6w^6}$$

The E_8 structure

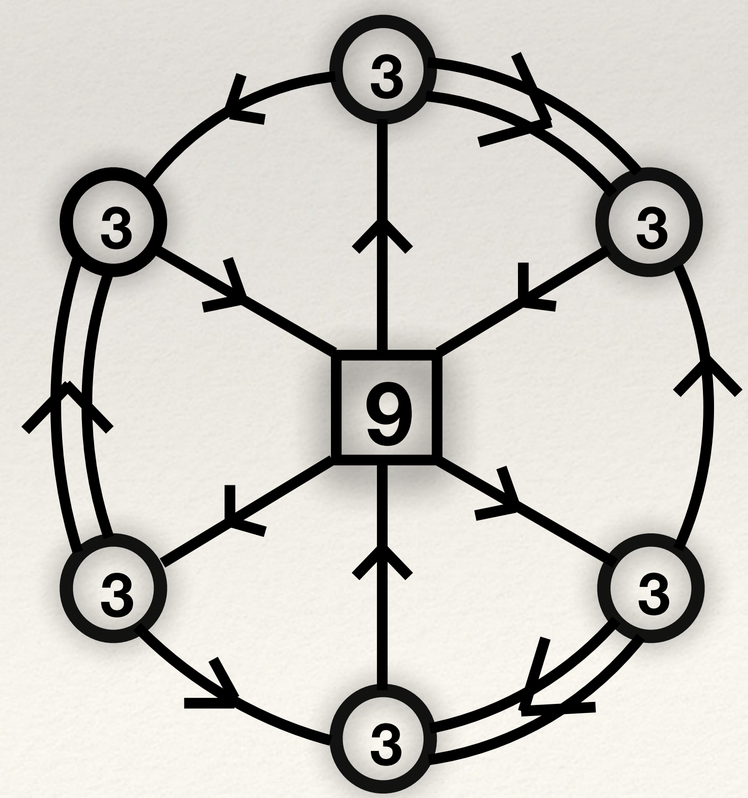
SR, Zafrir 19

- One can combine the three punctured sphere into closed Riemann surface of genus g with zero flux index of which should be invariant under the action of the Weyl group of E_8.



Define: $$\mathcal{T}(x,y) = \left[\frac{(q;q)(p;p)}{2}\right]^2 \oint \frac{dt_1}{2\pi i t_1}\frac{dt_2}{2\pi i t_2}\frac{\mathcal{K}(x,t_2,t_1)\cdot\overline{\mathcal{K}(t_1,t_2,y)}}{\Gamma_e(t_1^{\pm 2})\Gamma_e(t_2^{\pm 2})}$$

Then: $$\mathcal{I}_{\mathcal{C}_g} = \left[\frac{(q;q)(p;p)}{2}\right]^{g-1} \oint \prod_{j=1}^{g-1}\frac{dt_j}{2\pi i t_j}\frac{\mathcal{T}(t_j,t_{j+1})}{\Gamma_e(t_j^{\pm 2})}$$

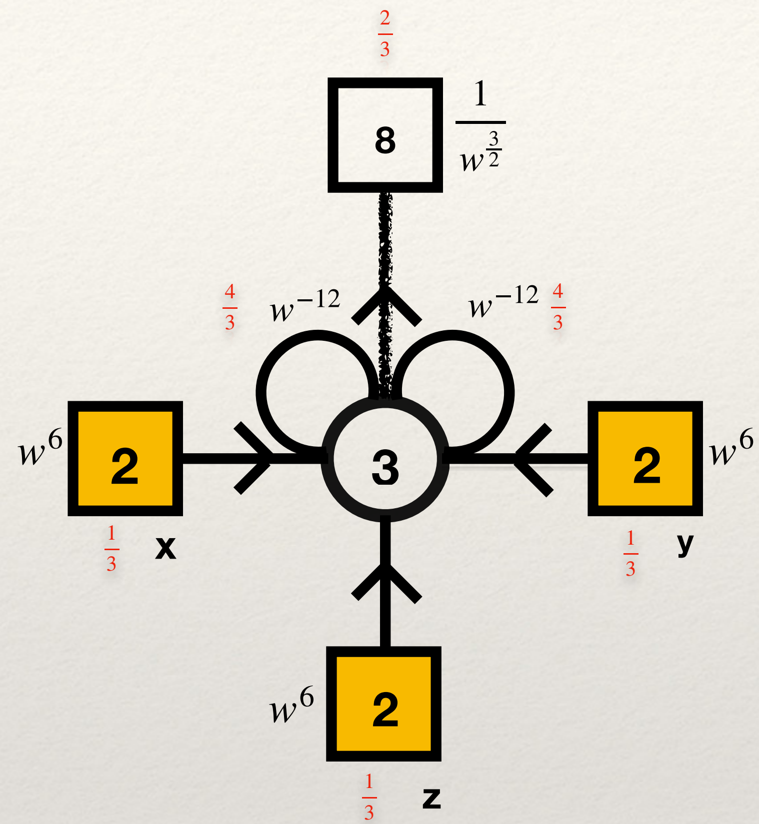


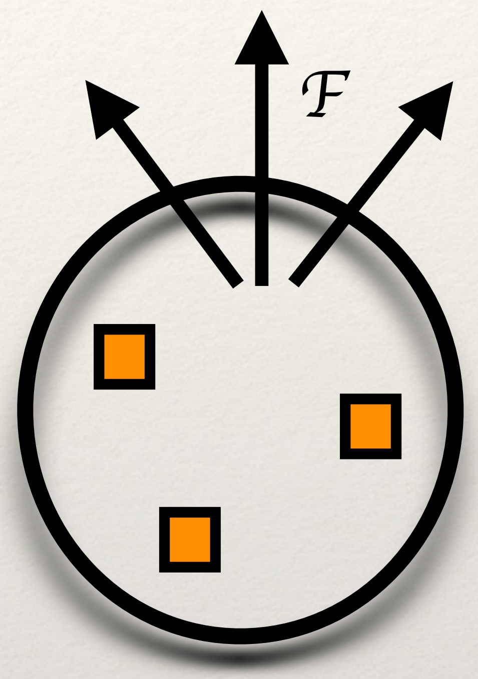
is invariant under the Weyl group of E_8 acting on $\{u, v, w, a_i\}$

Proof?

E-string three punctured sphere II ($C_{N=1}$)

- $SU(3)\ N_f = 8$ SQCD with W
- $SU(8) \times SU(8) \times U(1)_B \to$
- punct: $SU(2) \times SU(2) \times SU(2)$
- $SU(8) \times U(1) \subset E_8$





``Moment Map" Operators:

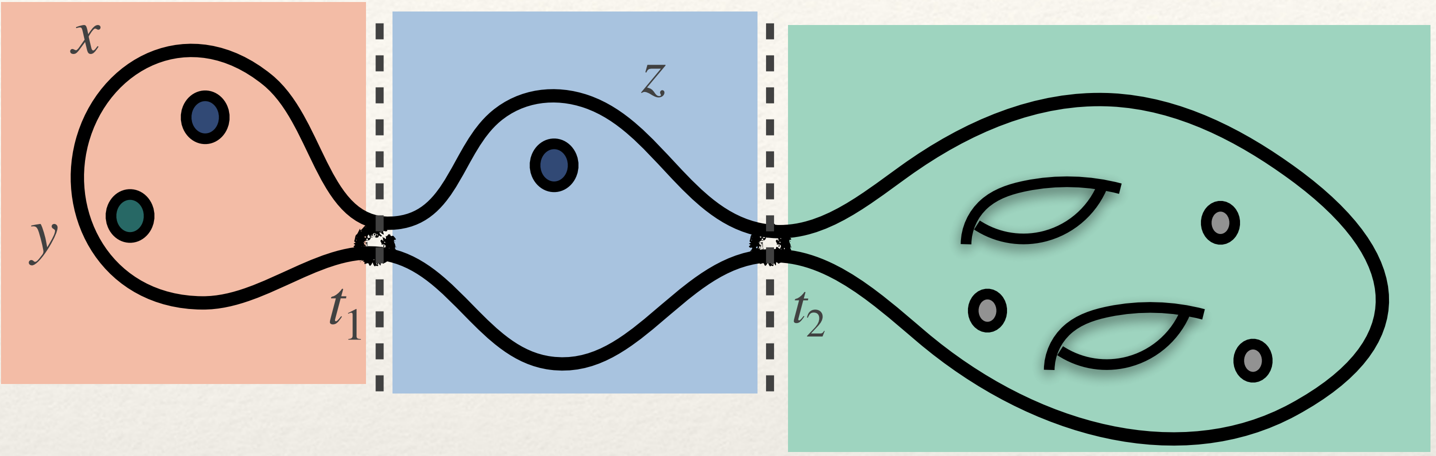
$$M : \mathbf{2}_{x,y,z} \otimes \left(\mathbf{8}_{w^{\frac{9}{2}}}\right)$$

The index: $\mathcal{K}(x,y,z) =$

$$\frac{(q;q)^2(p;p)^2}{6} \oint \frac{dt_1}{2\pi i t_1} \frac{dt_2}{2\pi i t_2} \frac{\prod_{i=1}^{3} \left[\Gamma_e((pq)^{1/6} w^6 t_i x^{\pm 1}) \Gamma_e((pq)^{1/6} w^6 t_i y^{\pm 1}) \Gamma_e((pq)^{1/6} w^6 t_i z^{\pm 1}) \Gamma_e((pq)^{2/3} w^{-12} t_i)^2 \prod_{j=1}^{8} \Gamma_e((pq)^{1/3} w^{-\frac{3}{2}} t_i^{-1} a_j) \right]}{\prod_{i \neq j}^{3} \Gamma_e(\frac{t_i}{t_j})}$$

$A\Delta O$ from E-string three punctured sphere II

(I) Construct index $\mathcal{J}(x,y,z)$



$$\mathcal{J}(x,y,z) = \left[\frac{(q;q)(p;p)}{2}\right]^2 \oint \frac{dt_1}{2\pi i t_1}\frac{dt_2}{2\pi i t_2}\frac{\mathcal{K}(x,y,t_1)\cdot\overline{\mathcal{K}(t_1,z,t_2)}\cdot\mathcal{I}_{\mathcal{C}}(t_2)}{\Gamma_e(t_1^{\pm 2})\Gamma_e(t_2^{\pm 2})}$$

(II) Compute the residue:

$$\mathrm{Res}_{z\to z^*}\,\mathrm{Res}_{y\to y^*}\,\mathcal{J}(x,y,z) \sim \mathcal{D}_x^{(y^*,z^*)}\cdot\mathcal{I}_{\mathcal{C}}(x)$$

We can choose any component of $M : \mathbf{2}_y \otimes \left(\mathbf{8}_{w^{\frac{9}{2}}}\right)$ **to give a vev to.**

For concreteness let us choose: $y^* = (qp)^{-\frac{1}{2}}w^{-\frac{9}{2}}a_1^{-1}q^{-1} \quad z^* = (qp)^{-\frac{1}{2}}w^{\frac{9}{2}}a_1$

The $C_{N=1}$ $A\Delta O$

- The residue computation is lengthy but in principle straightforward procedure
- (Analyze pinching of the integration contours and use various known integral identities)

$$\mathcal{D}_x^{(y^*,z^*)} \cdot \mathcal{I}_{\mathscr{C}}(x) = \frac{\prod_{j=1}^{8} \theta_p((pq)^{\frac{1}{2}} w^{-\frac{9}{2}} a_j^{-1} x)}{\theta_p(qx^2)\theta_p(x^2)} \mathcal{I}_{\mathscr{C}}(qx) + \frac{\prod_{j=1}^{8} \theta_p((pq)^{\frac{1}{2}} w^{-\frac{9}{2}} a_j^{-1} x^{-1})}{\theta_p(qx^{-2})\theta_p(x^{-2})} \mathcal{I}_{\mathscr{C}}(q^{-1}x) + W^{(y^*,z^*)} \mathcal{I}_{\mathscr{C}}(x)$$

$$W^{(y^*,z^*)}(x) = \left[\frac{\theta_p((pq)^{\frac{1}{2}} w^{18} x)\theta_p((pq)^{\frac{1}{2}} w^{\frac{9}{2}} a_1 x^{\pm 1}) \prod_{i=2}^{8} \theta_p((pq)^{\frac{1}{2}} w^{-\frac{9}{2}} a_i^{-1} x)}{\theta_p((pq)^{\frac{1}{2}} w^{-18} x)\theta_p((pq)^{\frac{1}{2}} q w^{\frac{9}{2}} a_1 x)\theta_p(x^2)\theta_p(q^{-1}x^{-2})} + \{x \to x^{-1}\}\right] +$$

$$\frac{\theta_p(q^{-1} w^{\frac{27}{2}} a_1^{-1}) \prod_{i=2}^{8} \theta_p(q^{-1} w^{-9} a_1^{-1} a_i^{-1}) \theta_p((pq)^{\frac{1}{2}} w^{\frac{9}{2}} a_1 x^{\pm 1})}{\theta_p(q^{-2} w^{-9} a_1^{-2})\theta_p(q^{-1} w^{-\frac{45}{2}} a_1^{-1})\theta_p((pq)^{-\frac{1}{2}} q^{-1} w^{-\frac{9}{2}} a_1^{-1} x^{\pm 1})} + \frac{\prod_{i=2}^{8} \theta_p(w^{\frac{27}{2}} a_i^{-1}) \theta_p((pq)^{\frac{1}{2}} w^{\frac{9}{2}} a_1 x^{\pm 1})}{\theta_p(q w^{\frac{45}{2}} a_1)\theta_p((pq)^{-\frac{1}{2}} w^{18} x^{\pm 1})}$$

The $C_{N=1}$ $A\Delta O$ and the van Diejen $A\Delta O$

- This operator is (up to conjugation) precisely the BC_1 van Diejen $A\Delta O$
- The eight parameters of van Diejen are: $\left(\mathbf{8}_{w^{-\frac{9}{2}}}\right)$
- One can repeat the exercise with any component of the moment map, the result is always the same (up to x-independent constant shift)
- All these operators are thus trivially commuting and $\mathscr{K}(x, y, z)$ is expected to be a Kernel function:

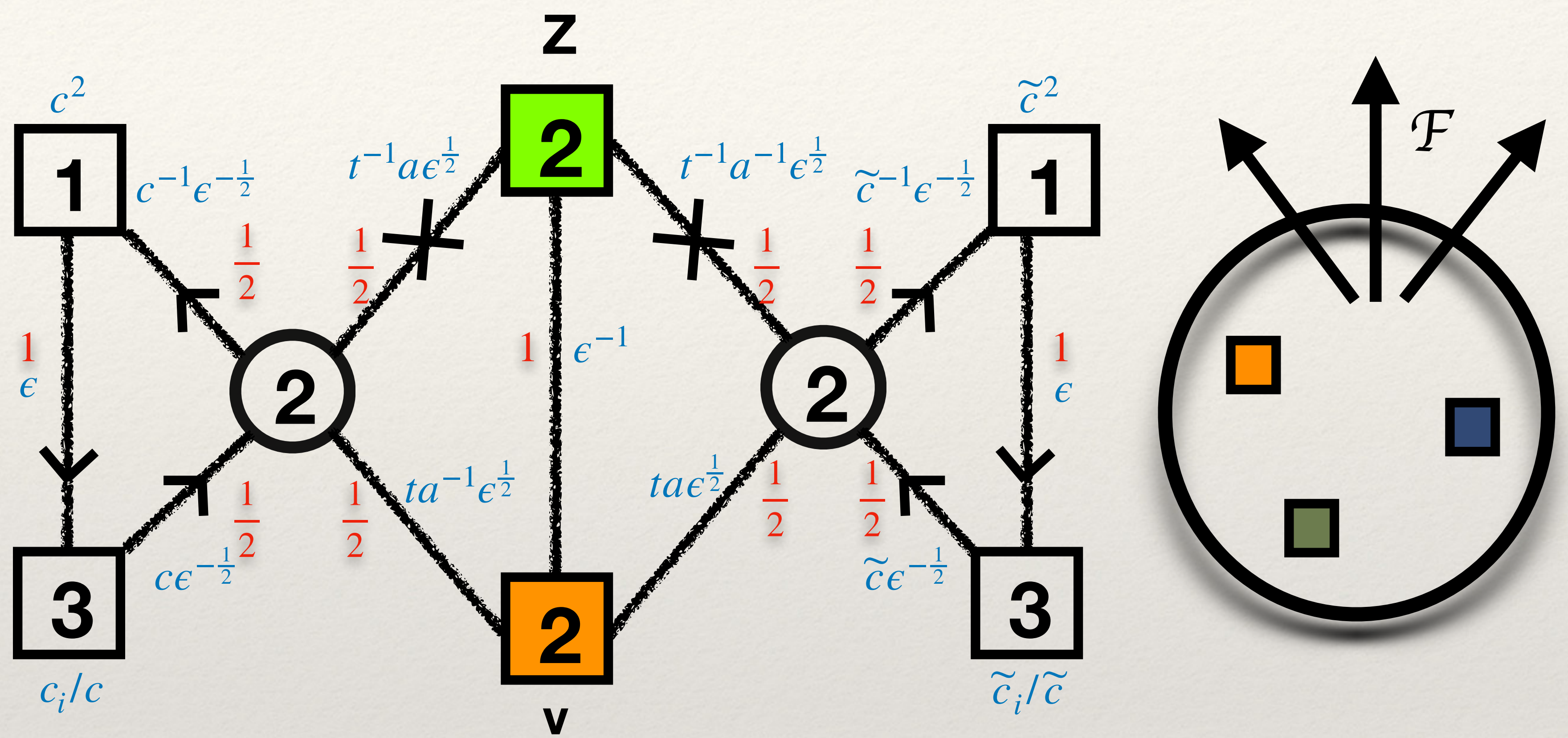
$$\mathscr{D}_x^{(y*,z*)} \cdot \mathscr{K}(x, y, z) = \mathscr{D}_y^{(y*,z*)} \cdot \mathscr{K}(x, y, z) = \mathscr{D}_z^{(y*,z*)} \cdot \mathscr{K}(x, y, z)$$

Proof?

E-string three punctured sphere III $((A_1)^{N=1})$

SR, Sabag 19

- $(SU(2)\, N_f = 4)^2$ SQCD with W and gauge singlets
- punct: $SU(2) \times SU(2) \times SU(2)$
- $SU(3)^2 \times U(1)^4 \subset E_8$



``Moment Map'' Operators:

M_v : $\{ta^{-1}c_1, ta^{-1}c_2, ta^{-1}c_3, ta^{-1}c_4, ta\widetilde{c}_1, ta\widetilde{c}_2, ta\widetilde{c}_3, ta\widetilde{c}_4\}$

M_z; : $\{t^{-1}ac_1, t^{-1}ac_2, t^{-1}ac_3, t^{-1}ac_4, t^{-1}a^{-1}\widetilde{c}_1, t^{-1}a^{-1}\widetilde{c}_2, t^{-1}a^{-1}\widetilde{c}_3, t^{-1}a^{-1}\widetilde{c}_4\}$

M_ϵ : $\{c_1^{-1}c_2^{-1}, c_1^{-1}c_3^{-1}, c_2^{-1}c_3^{-1}, \widetilde{c}_1^{-1}\widetilde{c}_2^{-1}, \widetilde{c}_1^{-1}\widetilde{c}_3^{-1}, \widetilde{c}_2^{-1}\widetilde{c}_3^{-1}, t^2a^{-2}, t^2a^2\}$

$A\Delta O$ from E-string three punctured sphere III

$$\mathcal{K}(\epsilon, v, z) = \frac{(q;q)^2(p;p)^2}{4}\oint\frac{dy_1}{2\pi i y_1}\oint\frac{dy_2}{2\pi i y_2}\frac{\prod_{i=1}^4\Gamma_e\left((pq)^{\frac14}\epsilon^{-\frac12}c_i y_1^{\pm1}\right)\Gamma_e\left((pq)^{\frac14}\epsilon^{-\frac12}y_2^{\pm1}\widetilde{c}_i\right)}{\Gamma_e\left(y_1^{\pm2}\right)\Gamma_e\left(y_2^{\pm2}\right)}\times$$

$$\Gamma_e\left((pq)^{1/4}t^{-1}a\epsilon^{1/2}y_1^{\pm1}z^{\pm1}\right)\Gamma_e\left((pq)^{1/4}t^{-1}a^{-1}\epsilon^{1/2}z^{\pm1}y_2^{\pm1}\right)\Gamma_e\left(\sqrt{pq}\epsilon^{-1}z^{\pm1}v^{\pm1}\right)\times$$

$$\Gamma_e\left((pq)^{1/4}ta^{-1}\epsilon^{1/2}y_1^{\pm1}v^{\pm1}\right)\Gamma_e\left((pq)^{1/4}ta\epsilon^{1/2}v^{\pm1}y_2^{\pm1}\right)\prod_{i=1}^3\Gamma_e\left(\sqrt{pq}\epsilon c_i c_4\right)\Gamma_e\left(\sqrt{pq}\epsilon\widetilde{c}_i\widetilde{c}_4\right)\Gamma_e\left(\sqrt{pq}t^2a^{\pm2}\epsilon^{-1}\right)$$

(I) Construct index $\mathcal{J}(x,y,z)$

$$\mathcal{J}(x,y,z) = \left[\frac{(q;q)(p;p)}{2}\right]^2\oint\frac{dt_1}{2\pi i t_1}\frac{dt_2}{2\pi i t_2}\frac{\mathcal{K}(x,y,t_1)\cdot\overline{\mathcal{K}(t_1,z,t_2)}\cdot\mathcal{I}_{\mathcal{C}}(t_2)}{\Gamma_e(t_1^{\pm2})\Gamma_e(t_2^{\pm2})}$$

(II) Compute the residue: $\mathrm{Res}_{z\to z^*}\mathrm{Res}_{y\to y^*}\,\mathcal{J}(x,y,z)\ \sim\ \mathcal{D}_x^{(y^*,z^*)}\cdot\mathcal{I}_{\mathcal{C}}(x)$

We can choose any component of $M_\epsilon : \mathbf{2}_\epsilon\otimes\left(c_i^{-1}c_j^{-1}, \widetilde{c}_i^{-1}\widetilde{c}_j^{-1}, t^2a^{-2}, t^2a^2\right)$ **to give a vev to.**

For concreteness let us choose: $y^* = (qp)^{\frac12}t^2a^2q \qquad z^* = (qp)^{\frac12}t^{-2}a^{-2}$

The $(A_1)^{N=1}$ $A\Delta O$

- The residue computation is lengthy but in principle straightforward procedure
- (Analyze pinching of the integration contours and use various known integral identities)

$$\mathcal{D}_v^{(y^*,z^*)} \cdot \mathcal{I}_{\mathcal{C}}(v) = \frac{\prod_{i=1}^4 \theta_p\left((pq)^{\frac{1}{2}}t^{-1}ac_i^{-1}v\right)\theta_p\left((pq)^{\frac{1}{2}}t^{-1}a^{-1}\tilde{c}_i^{-1}v\right)}{\theta_p(v^2)\theta_p(qv^2)}\mathcal{I}_{\mathcal{C}}(qv) + W^{(y^*,z^*)}(v)\mathcal{I}_{\mathcal{C}}(v) + (v \to v^{-1})$$

$$W^{(y^*,z^*)}(v) = \frac{\theta_p(q^{-1}a^{-4})\theta_p(q^{-1}t^{-4}a^{-4}v^2)\prod_{i=1}^4 \theta_p((pq)^{\frac{1}{2}}ta^{-1}c_iv)\theta_p((pq)^{\frac{1}{2}}ta\tilde{c}_iv)}{\theta_p(q^{-2}t^{-4}a^{-4})\theta_p(a^{-4}v^2)\theta_p(v^2)\theta_p(q^{-1}v^{-2})} + \frac{\theta_p(q^{-1}t^{-4})\prod_{i=1}^4 \theta_p((pq)^{\frac{1}{2}}ta^3c_iv^{-1})\theta_p((pq)^{\frac{1}{2}}ta\tilde{c}_iv)}{\theta_p(v^2)\theta_p(a^4v^{-2})\theta_p(q^{-2}t^{-4}a^{-4})}$$

The $(A_1)^{N=1}$ $A\Delta O$ and the van Diejen $A\Delta O$

- This operator is (up to conjugation) precisely the BC_1 van Diejen $A\Delta O$
- The eight parameters of van Diejen are: $\{t^{-1}ac_i^{-1}, t^{-1}a^{-1}\tilde{c}_i^{-1}\}$
- One can repeat the exercise with any component of the moment map, the result is always the same (up to v-independent constant shift)
- All these operators are thus trivially commuting and $\mathscr{K}(x, y, z)$ is expected to be a Kernel function:

$$\mathscr{D}_x^{(y^*,z^*)} \cdot \mathscr{K}(x,y,z) = \mathscr{D}_y^{(y^*,z^*)} \cdot \mathscr{K}(x,y,z) = \mathscr{D}_z^{(y^*,z^*)} \cdot \mathscr{K}(x,y,z)$$

Proof?

Summary Part II

- We obtain three different Kernel functions for the BC_1 van Diejen model
- The fact that we get the same $A\Delta O$ for the three different three-punctured spheres is a consistency check on the physics arguments
- The fact that all these operators (trivially) commute is yet another check
- Proving the Kernel property will be a rather non trivial check of the physics
- We have a second copy of commuting operators by exchanging $q \leftrightarrow p$
- In principle there are residues with higher powers of q, we do not expect these to give new operators but rather polynomials of the basic van Diejen operator.

Part III: and beyond …

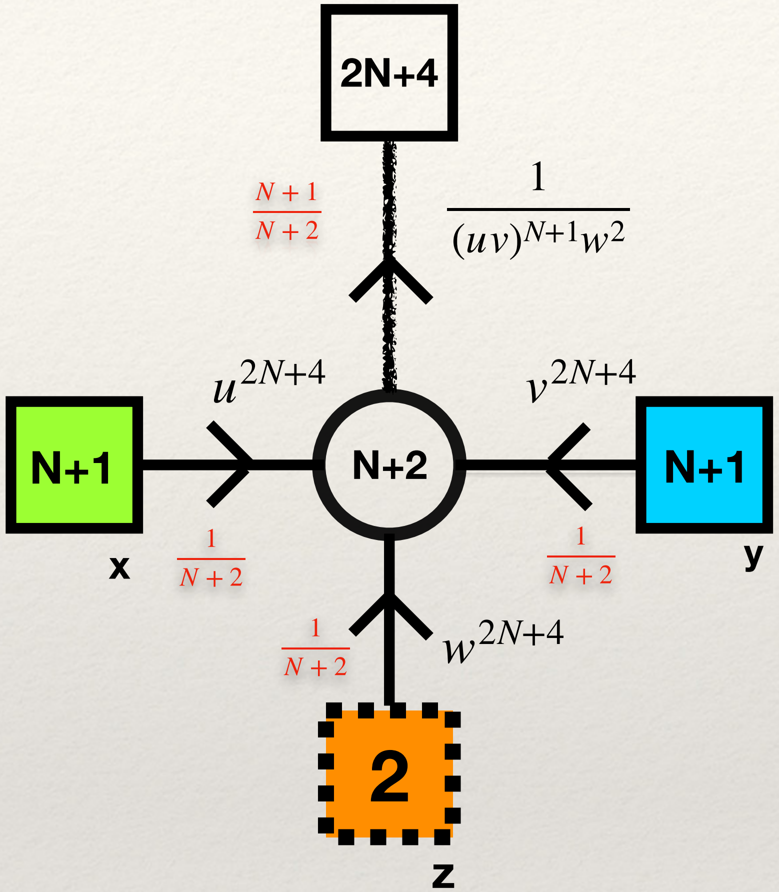
Minimal D conformal matter

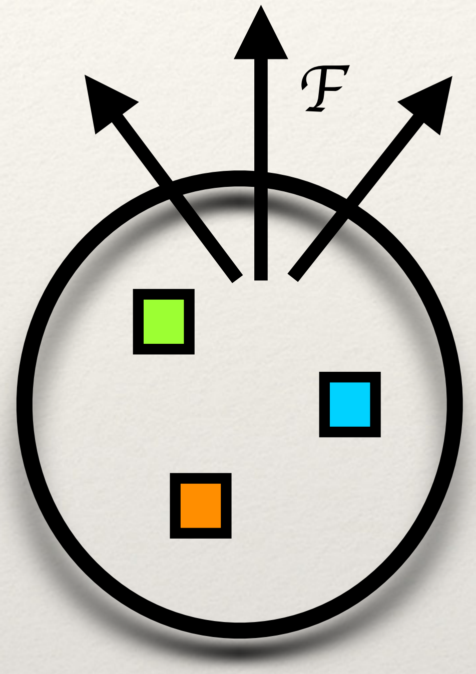
- E-string is the theory residing on a single M5 brane probing D_4 singularity
- Let us consider the theory residing on a single M5 brane probing D_{N+3} singularity
- A non-trivial 6d SCFT, $G_{6d} = SO(4N + 12)$
- *(N>0, Enhances to E_8 for $N = 1$)*
- Minimal (D_{N+3}, D_{N+3}) conformal matter Del Zotto, Heckman, Tomasiello, Vafa 2014
- Three different possibilities for G_{5d}:
- $G_{5d} =: SU(N+1)\ ,\ USp(2N)\ ,\ SU(2)^N$ Hayashi, Kim, Taki, Lee, Yagi 2015
- *Each one of the constructions generalises to these groups*

Minimal D_{N+3}-cm three punctured sphere (A_N)

SR, Sabag 20

- $SU(N+2)\ N_f = 2N+4$ SQCD
- $SU(2N+4) \times SU(2N+4) \times U(1)_B \;\to$
- punct:
  $SU(N+1) \times SU(N+1) \times SU(2)$
- $SU(2N+4) \times U(1)^3 \subset SO(4N+12)$





``Moment Map'' Operators:

$$M_u : \mathbf{N+1}^x \otimes \left(\mathbf{2N+4}_{u^{N+3}v^{-(N+1)}w^{-2}} \oplus \mathbf{1}_{(uv^{N+1})^{2N+4}}\right) \oplus \overline{\mathbf{N+1}}^x \otimes \mathbf{1}_{(u^N w^2)^{2N+4}}$$

$$M_v : \mathbf{N+1}^y \otimes \left(\mathbf{2N+4}_{v^{N+3}u^{-(N+1)}w^{-2}} \oplus \mathbf{1}_{(vu^{N+1})^{2N+4}}\right) \oplus \overline{\mathbf{N+1}}^y \otimes \mathbf{1}_{(v^N w^2)^{2N+4}}$$

$$M_w : \quad \mathbf{2}^z \otimes \left(\mathbf{2N+4}_{(uv)^{-(N+1)}w^{2N+2}} \oplus \mathbf{1}_{(wv^{N+1})^{2N+4}} \oplus \mathbf{1}_{(wu^{N+1})^{2N+4}}\right)$$

$A\Delta O$ from E-string three punctured sphere III

The index:

$$\mathcal{K}(x_i, y_j, z) = \frac{(q;q)^{N+1}(p;p)^{N+1}}{(N+2)!} \oint \prod_{j=1}^{N+1} \frac{dy_j}{2\pi i y_j} \frac{\prod_{i=1}^{N+2} \prod_{l=1}^{2N+4} \Gamma_e((pq)^{\frac{N+1}{2(N+2)}}(uv)^{-N-1} w^{-2} t_i^{-1} a_l)}{\prod_{i\neq j} \Gamma\left(y_i/y_j\right)} \times$$

$$\prod_{i=1}^{N+2} \prod_{j=1}^{N+1} \Gamma_e((pq)^{\frac{1}{2(N+2)}} u^{2N+4} t_i x_j) \Gamma_e((pq)^{\frac{1}{2(N+2)}} v^{2N+4} t_i y_j) \Gamma_e((pq)^{\frac{1}{2(N+2)}} w^{2N+4} t_i z^{\pm 1})$$

(I) Construct index $\mathcal{J}(x, y, z)$

$$\mathcal{J}(x_l, y, z) = \left[\frac{(q;q)^N (p;p)^N}{(N+1)!}\right]^2 \oint \prod_{i,j=1}^{N+1} \frac{dt_i^1}{2\pi i t_i^1} \frac{dt_j^2}{2\pi i t_j^2} \frac{\mathcal{K}(x_l, y, t_i^1) \cdot \overline{\mathcal{K}(t_i^1, z, t_j^2)} \cdot \mathcal{I}_{\mathscr{C}}(t_j^2)}{\prod_{i\neq j} \Gamma_e(t_i^1/t_j^1) \Gamma_e(t_i^2/t_j^2)}$$

(II) Compute the residue: $\text{Res}_{z\to z^*} \text{Res}_{y\to y^*} \mathcal{J}(x_i, y, z) \sim \mathcal{D}_x^{(y^*, z^*)} \cdot \mathcal{I}_{\mathscr{C}}(x_i)$

We can choose any component of $\left(\mathbf{2N+4}_{(uv)^{-(N+1)} w^{2N+2}} \oplus \mathbf{1}_{(wv^{N+1})^{2N+4}} \oplus \mathbf{1}_{(wu^{N+1})^{2N+4}}\right)$ **to give a vev to.**

For concreteness let us choose: $y^* = (qp)^{\frac{1}{2}} (wu^{N+1})^{-2N-4} q^{-1} \quad z^* = (qp)^{\frac{1}{2}} (wu^{N+1})^{2N+4}$

The A_N $A\Delta O$

$$\mathcal{D}_x^{(y^*,z^*)} \cdot \mathcal{I}_{\mathcal{C}}(x_i) = \left(\sum_{l\neq m}^{N+1} A_{lm}^{(y^*,z^*)}(x)\Delta_{lm} + W_x^{(y^*,z^*)}(x) \right) \mathcal{I}_{\mathcal{C}}(x_i)$$

$$\Delta_{lm}(x)f(x) \equiv f\left(x_l \to q^{-1}x_l, x_m \to qx_m\right)$$

$$A_{lm}^{(y^*,z^*)}(x_i) := \frac{\theta_p((qp)^{\frac{1}{2}}u^{-2N-4}v^{-(N+1)(2N+4)}x_l^{-1})\theta_p((qp)^{\frac{1}{2}}w^{-4N-8}u^{-N(2N+4)}x_m)}{\theta_p(q\frac{x_m}{x_l})\theta_p(\frac{x_m}{x_l})} \times$$

$$\prod_{j=1}^{2N+4} \theta_p((qp)^{\frac{1}{2}}u^{-N-3}v^{N+1}w^2x_l^{-1}a_j^{-1}) \prod_{i\neq m\neq l}^{N+1} \frac{\theta_p((qp)^{\frac{1}{2}}w^{4N+8}u^{N(2N+4)}x_i^{-1})\theta_p((qp)^{\frac{1}{2}}u^{(N+2)(2N+4)}x_i)}{\theta_p(\frac{x_i}{x_l})\theta_p(\frac{x_m}{x_i})}$$

$$W^{(y^*,z^*)}(x_i) = \prod_{j=1}^{2N+4} \theta_p(u^{-(N+1)(2N+5)}v^{-N-1}w^{-2}q^{-1}a_j)\frac{\theta_p(q^{-1}\left(vu^{-1}\right)^{(N+1)(2N+4)})}{\theta_p(pq^2w^{4N+8}u^{2(N+1)(2N+4)})} \times \prod_{i=1}^{N+1} \frac{\theta_p((qp)^{\frac{1}{2}}w^{4N+8}u^{N(2N+4)}x_i^{-1})}{\theta_p(qp)^{-\frac{1}{2}}u^{-(N+2)(2N+4)}q^{-1}x_i^{-1})} +$$

$$\sum_{m=1}^{N+1} \frac{\theta_p((qp)^{\frac{1}{2}}u^{2N+4}v^{(N+1)(2N+4)}x_m)}{\theta_p((qp)^{\frac{1}{2}}u^{(N+2)(2N+4)}qx_m)} \times \prod_{i\neq m} \frac{\theta_p((qp)^{\frac{1}{2}}w^{4N+8}u^{N(2N+4)}x_i^{-1})\theta_p((qp)^{\frac{1}{2}}u^{(N+2)(2N+4)}x_i)}{\theta_p(q^{-1}\frac{x_i}{x_m})\theta_p(\frac{x_m}{x_i})} \prod_{j=1}^{2N+4} \theta_p((qp)^{\frac{1}{2}}u^{N+3}v^{-N-1}w^{-2}x_m a_j)$$

Properties of the A_N $A\Delta O$s

- This operator is an A_N generalization of the BC_1 van Diejen $A\Delta O$
- The $A\Delta O$ depends on $2N+6$ parameters
- One can repeat the exercise with any component of the moment map, the result is a set of similarly looking but different operators
- All these operators are commuting and $\mathcal{K}(x_i, y_j, z)$ is expected to be a Kernel function:

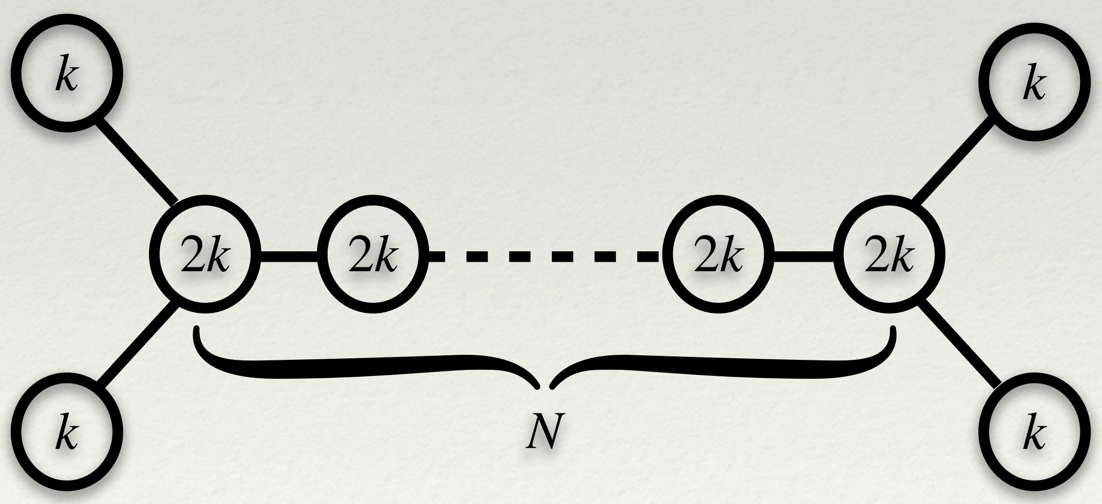
$$\mathcal{D}_x^{(y^*,z^*)} \cdot \mathcal{K}(x_i, y_j, z) = \mathcal{D}_y^{(y^*,z^*)} \cdot \mathcal{K}(x_i, y_j, z)$$

Proof?

Part IV: Comments

Generalizations to C_N and $(A_1)^N$ and more

- In a similar manner one can define C_N and $(A_1)^N$ generalisations
- *(We have not computed the operators yet.)*
- In fact the $(A_1)^N$ has a further generalization to $(A_{k-1})^4 \times (A_{2k-1})^N$
- This corresponds to non-minimal (D_{N+3}, D_{N+3}) conformal matter
- In turn this generalized to $G = ADE$ conformal matter with $G_{6d} = G \times G$

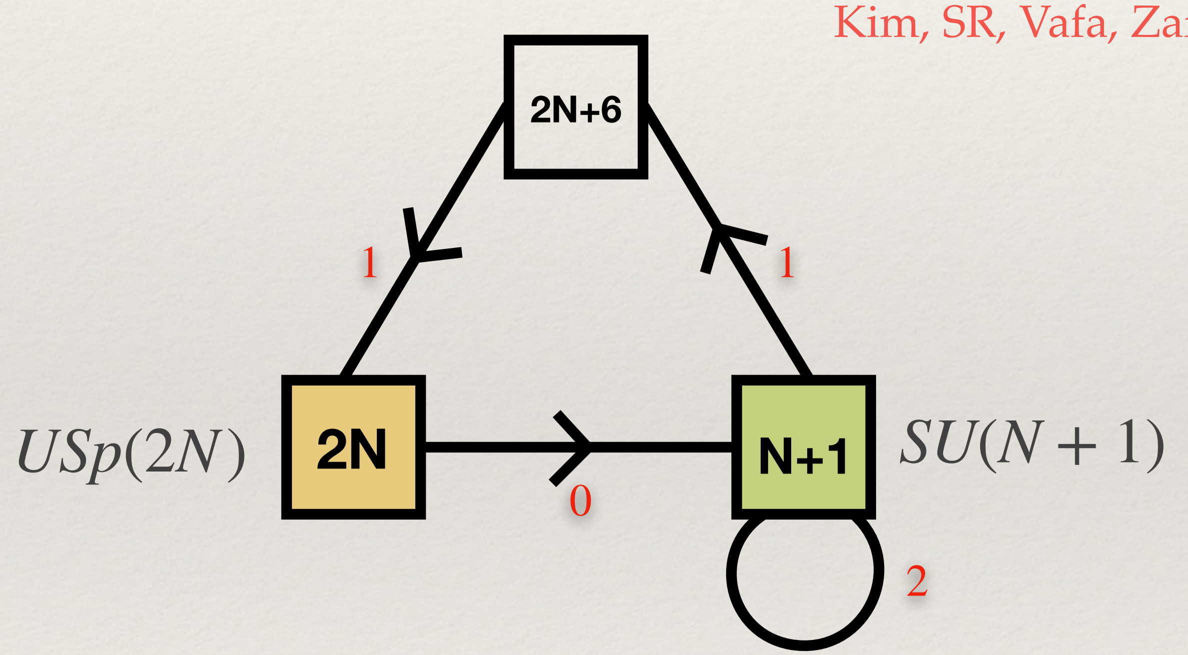


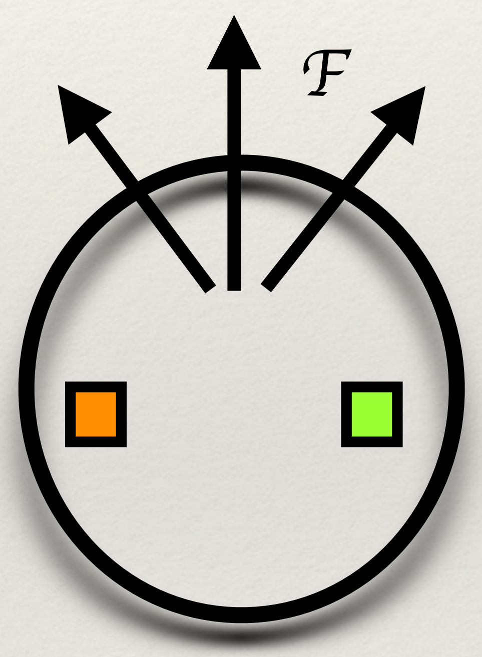
- *One can construct Kernel functions mixing the various types of parameters*

Example of a mixed Kernel functions

- A joint Kernel function for A_N and C_N $A\Delta O$s

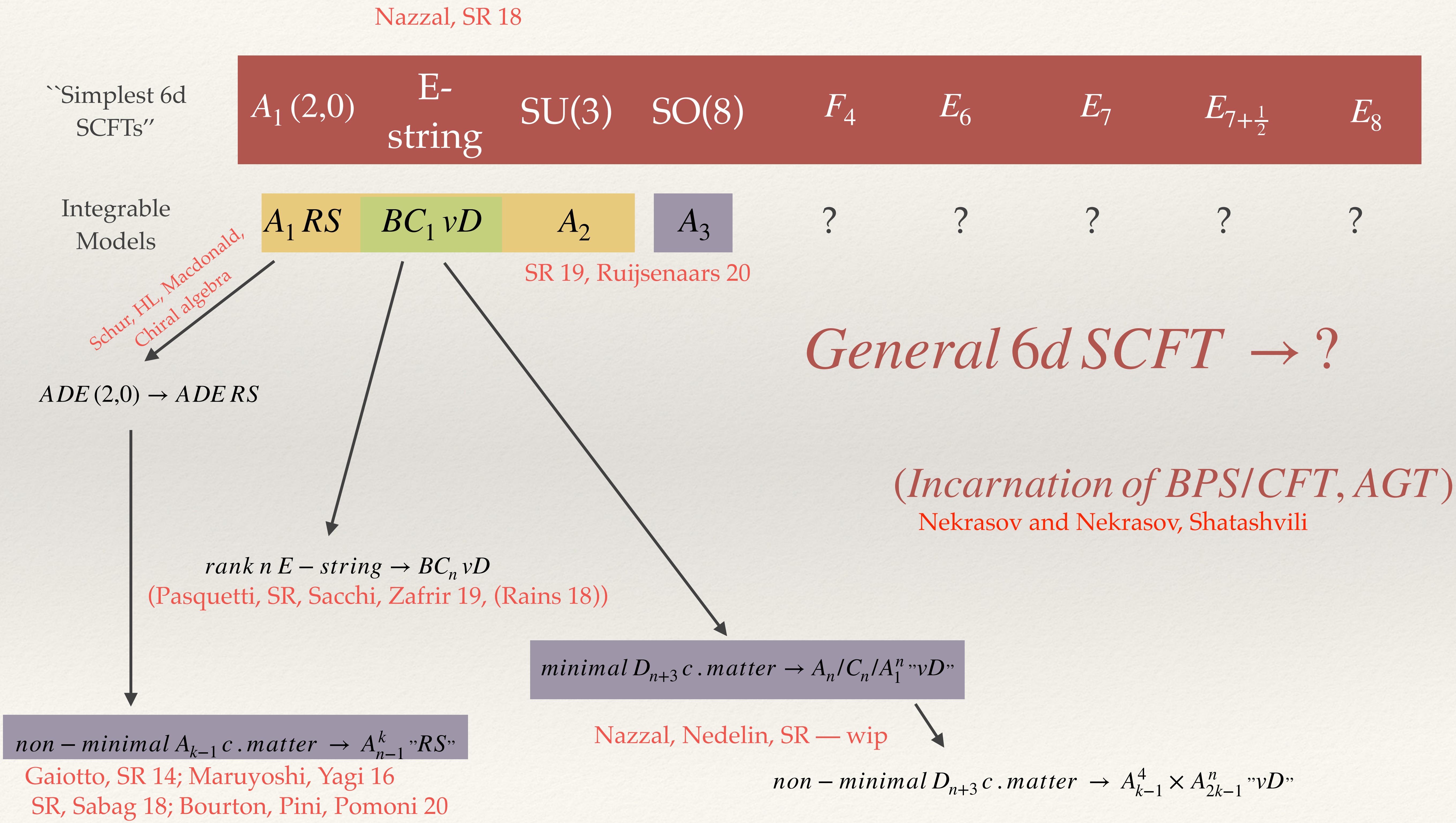
Kim, SR, Vafa, Zafrir 18





$$\mathcal{K}(z_i, y_j) = \left(\prod_{i=1}^{N}\prod_{j=1}^{N+1}\Gamma_e(t^{-\frac{3+N}{4}}z_i^{\pm 1}y_j)\right)\left(\prod_{j=1}^{N+1}\prod_{l=1}^{2N+6}\Gamma_e((qp)^{\frac{1}{2}}t^{\frac{1}{2}}a_l y_j^{-1})\right)\left(\prod_{j=1}^{N}\prod_{l=1}^{2N+6}\Gamma_e((qp)^{\frac{1}{2}}t^{\frac{N+1}{4}}a_l^{-1}z_j^{\pm 1})\right)\prod_{i\neq j}\Gamma_e(qpt^{\frac{N+3}{2}}z_i^{-1}z_j^{-1})$$

Integrable models vs 6d SCFTs



Outlook

- Proving the conjectures
- What is the integrable model corresponding to a general 6d SCFT?
- To a given SCFT can associate different models, how many?
- Can we map the classification of 6d SCFTs and / or 5d gauge theories with 6d UV completion to ``classification'' of elliptic integrable models?

Thank You!!