The DELL System

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Literature

[arXiv:2110.02157] Phys.Lett.B **826** (2022) 136919
Double Inozemtsev Limits of the Quantum DELL System
A. Gorsky, P. Koroteev, O. Koroteeva, S. Shakirov

[arXiv:1906.10354] Lett.Math.Phys. **110** (2020) 969
The Quantum DELL System
P. Koroteev, S. Shakirov

[arXiv:1805.00986] Commun.Math.Phys. **381** (2021) 175
A-type Quiver Varieties and ADHM Moduli Spaces
P. Koroteev

[arXiv:1412.6081] JHEP **05** (2015) 095
Defects and Quantum Seiberg-Witten Geometry
M. Bullimore, H. Kim, P. Koroteev

The Quantum Double Elliptic Model

[PK Shakirov]

Hamiltonians $\widehat{\mathcal{H}}_a = \widehat{\mathcal{O}}_0^{-1}\widehat{\mathcal{O}}_a$ **in involution** $[\widehat{\mathcal{H}}_a, \widehat{\mathcal{H}}_b] = 0$

$$\widehat{\mathcal{O}}(z) = \sum_{n\in\mathbb{Z}} \widehat{\mathcal{O}}_n\, z^n = \sum_{n_1,\ldots,n_N=-\infty}^{\infty} (-z)^{\sum n_i}\, w^{\sum \frac{n_i(n_i-1)}{2}} \prod_{i<j} \theta\big(t^{n_i-n_j}\widehat{x}_i/\widehat{x}_j|p\big)\ \widehat{p}_1^{n_1}\ldots\widehat{p}_N^{n_N}$$

Spectrum $H_i\mathcal{Z}(p,\mathbf{x}) = E_i(p,\mathbf{a})\mathcal{Z}(p,\mathbf{x})$

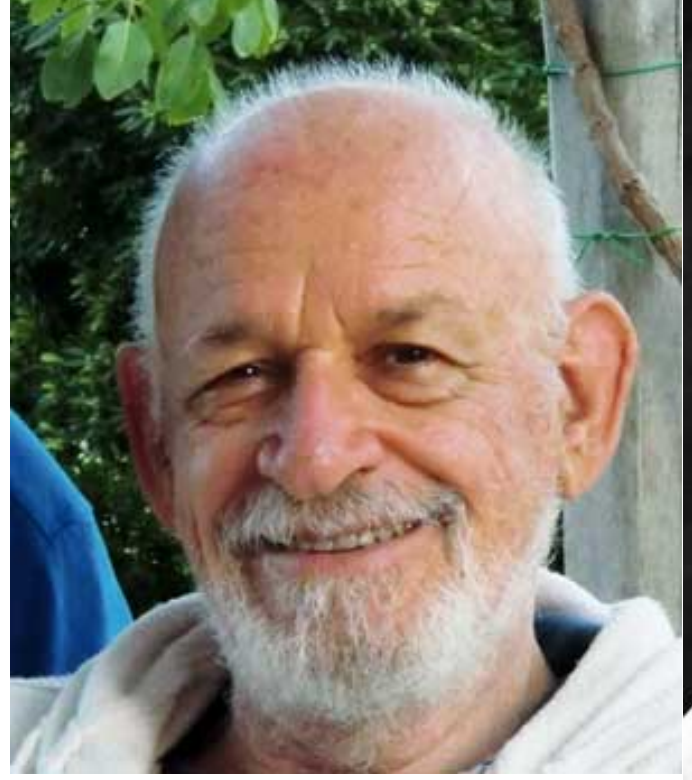
Eigenfunction $\mathscr{Z}(\mathbf{x}) = x_1 + x_2 + x_3$

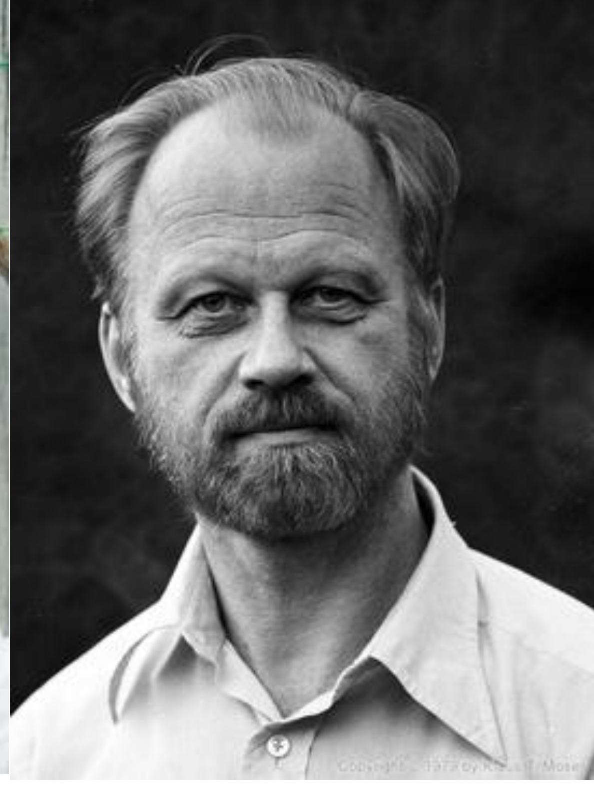
$$\begin{gathered}
+\mathbf{p}\frac{q(-1+t)}{(q-1)(qt-1)^3(qt+1)^2t}\cdot(q^4t^6x_1^3x_2+q^4t^6x_1^3x_3+2q^4t^6x_1^2x_2^2+3q^4t^6x_1^2x_2x_3+2q^4t^6x_1^2x_3^2+\\
q^4t^6x_1x_2^3+3q^4t^6x_1x_2^2x_3+3q^4t^6x_1x_2x_3^2+q^4t^6x_1x_3^3+q^4t^6x_2^3x_3+2q^4t^6x_2^2x_3^2+q^4t^6x_2x_3^3-\\
2q^4t^5x_1^2x_2x_3-2q^4t^5x_1x_2^2x_3-2q^4t^5x_1x_2x_3^2+2q^3t^6x_1^2x_2x_3+2q^3t^6x_1x_2^2x_3+2q^3t^6x_1x_2x_3^2-\\
q^4t^4x_1^2x_2^2-2q^4t^4x_1^2x_2x_3-q^4t^4x_1^2x_3^2-2q^4t^4x_1x_2^2x_3-2q^4t^4x_1x_2x_3^2-q^4t^4x_2^2x_3^2-q^3t^5x_1^3x_2-\\
q^3t^5x_1^3x_3-q^3t^5x_1^2x_2^2-q^3t^5x_1^2x_2x_3-q^3t^5x_1^2x_3^2-q^3t^5x_1x_2^3-q^3t^5x_1x_2^2x_3-q^3t^5x_1x_2x_3^2-\\
q^3t^5x_1x_3^3-q^3t^5x_2^3x_3-q^3t^5x_2^2x_3^2-q^3t^5x_2x_3^3-q^4t^3x_1^2x_2^2-q^4t^3x_1^2x_3^2-q^4t^3x_2^2x_3^2+2q^3t^4x_1^2x_2^2+\\
2q^3t^4x_1^2x_3^2+2q^3t^4x_2^2x_3^2-q^2t^5x_1^2x_2^2-q^2t^5x_1^2x_3^2-q^2t^5x_2^2x_3^2+q^4t^2x_1^2x_2x_3+q^4t^2x_1x_2^2x_3+\\
q^4t^2x_1x_2x_3^2-q^3t^3x_1^3x_2-q^3t^3x_1^3x_3-q^3t^3x_1^2x_2^2-5q^3t^3x_1^2x_2x_3-q^3t^3x_1^2x_3^2-q^3t^3x_1x_2^3-\\
5q^3t^3x_1x_2^2x_3-5q^3t^3x_1x_2x_3^2-q^3t^3x_1x_3^3-q^3t^3x_2^3x_3-q^3t^3x_2^2x_3^2-q^3t^3x_2x_3^3-q^2t^4x_1^3x_2-\\
q^2t^4x_1^3x_3-3q^2t^4x_1^2x_2^2-3q^2t^4x_1^2x_3^2-q^2t^4x_1x_2^3-q^2t^4x_1x_3^3-q^2t^4x_2^3x_3-3q^2t^4x_2^2x_3^2-q^2t^4x_2x_3^3-\\
2qt^5x_1^2x_2x_3-2qt^5x_1x_2^2x_3-2qt^5x_1x_2x_3^2+2q^3t^2x_1^2x_2x_3+2q^3t^2x_1x_2^2x_3+2q^3t^2x_1x_2x_3^2-\\
2qt^4x_1^2x_2x_3-2qt^4x_1x_2^2x_3-2qt^4x_1x_2x_3^2+2q^3tx_1^2x_2x_3+2q^3tx_1x_2^2x_3+2q^3tx_1x_2x_3^2+\\
q^2t^2x_1^3x_2+q^2t^2x_1^3x_3+3q^2t^2x_1^2x_2^2+3q^2t^2x_1^2x_3^2+q^2t^2x_1x_2^3+q^2t^2x_1x_3^3+q^2t^2x_2^3x_3+3q^2t^2x_2^2x_3^2+\\
q^2t^2x_2x_3^3+qt^3x_1^3x_2+qt^3x_1^3x_3+qt^3x_1^2x_2^2+5qt^3x_1^2x_2x_3+qt^3x_1^2x_3^2+qt^3x_1x_2^3+5qt^3x_1x_2^2x_3+\\
5qt^3x_1x_2x_3^2+qt^3x_1x_3^3+qt^3x_2^3x_3+qt^3x_2^2x_3^2+qt^3x_2x_3^3-t^4x_1^2x_2x_3-t^4x_1x_2^2x_3-t^4x_1x_2x_3^2+\\
q^2tx_1^2x_2^2+q^2tx_1^2x_3^2+q^2tx_2^2x_3^2-2qt^2x_1^2x_2^2-2qt^2x_1^2x_3^2-2qt^2x_2^2x_3^2+t^3x_1^2x_2^2+t^3x_1^2x_3^2+t^3x_2^2x_3^2+\\
qtx_1^3x_2+qtx_1^3x_3+qtx_1^2x_2^2+qtx_1^2x_2x_3+qtx_1^2x_3^2+qtx_1x_2^3+qtx_1x_2^2x_3+qtx_1x_2x_3^2+qtx_1x_3^3+\\
qtx_2^3x_3+qtx_2^2x_3^2+qtx_2x_3^3+t^2x_1^2x_2^2+2t^2x_1^2x_2x_3+t^2x_1^2x_3^2+2t^2x_1x_2^2x_3+2t^2x_1x_2x_3^2+t^2x_2^2x_3^2-\\
2qx_1^2x_2x_3-2qx_1x_2^2x_3-2qx_1x_2x_3^2+2tx_1^2x_2x_3+2tx_1x_2^2x_3+2tx_1x_2x_3^2-x_1^3x_2-x_1^3x_3-
\end{gathered}$$

$$2qx_1^2x_2x_3-2qx_1x_2^2x_3-2qx_1x_2x_3^2+2tx_1^2x_2x_3+2tx_1x_2^2x_3+2tx_1x_2x_3^2-x_1^3x_2-x_1^3x_3-$$

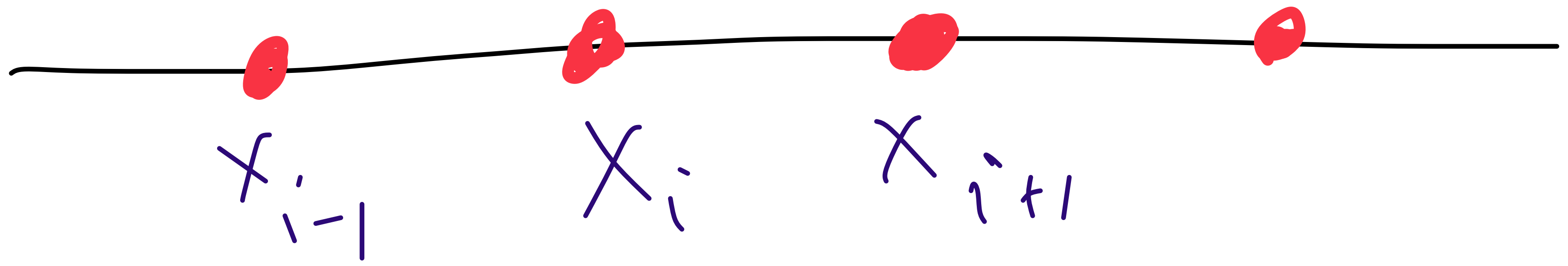
$$\begin{gathered}
2x_1^2x_2^2-3x_1^2x_2x_3-2x_1^2x_3^2-x_1x_2^3-3x_1x_2^2x_3-3x_1x_2x_3^2-x_1x_3^3-x_2^3x_3-2x_2^2x_3^2-x_2x_3^3)\\
+\mathbf{w}\ \mathbf{p}\frac{(-1+t)(t^3q-1)(-t+q)}{q(q-1)(qt-1)^3(qt+1)^2t^4}\cdot(q^6t^8x_1^3x_2+q^6t^8x_1^3x_3+q^6t^8x_1^2x_2^2+2q^6t^8x_1^2x_2x_3+\\
q^6t^8x_1^2x_3^2+q^6t^8x_1x_2^3+2q^6t^8x_1x_2^2x_3+2q^6t^8x_1x_2x_3^2+q^6t^8x_1x_3^3+q^6t^8x_2^3x_3+q^6t^8x_2^2x_3^2+q^6t^8x_2x_3^3-\\
q^6t^7x_1^2x_2^2-2q^6t^7x_1^2x_2x_3-q^6t^7x_1^2x_3^2-2q^6t^7x_1x_2^2x_3-2q^6t^7x_1x_2x_3^2-q^6t^7x_2^2x_3^2+q^5t^8x_1^2x_2^2+\\
2q^5t^8x_1^2x_2x_3+q^5t^8x_1^2x_3^2+2q^5t^8x_1x_2^2x_3+2q^5t^8x_1x_2x_3^2+q^5t^8x_2^2x_3^2-q^6t^6x_1^2x_2^2-q^6t^6x_1^2x_2x_3-\\
q^6t^6x_1^2x_3^2-q^6t^6x_1x_2^2x_3-q^6t^6x_1x_2x_3^2-q^6t^6x_2^2x_3^2-q^5t^7x_1^3x_2-q^5t^7x_1^3x_3+q^5t^7x_1^2x_2^2-q^5t^7x_1^2x_2x_3+\\
q^5t^7x_1^2x_3^2-q^5t^7x_1x_2^3-q^5t^7x_1x_2^2x_3-q^5t^7x_1x_2x_3^2-q^5t^7x_1x_3^3-q^5t^7x_2^3x_3+q^5t^7x_2^2x_3^2-q^5t^7x_2x_3^3+\\
q^4t^8x_1^2x_2x_3+q^4t^8x_1x_2^2x_3+q^4t^8x_1x_2x_3^2+q^6t^5x_1^2x_2^2+2q^6t^5x_1^2x_2x_3+q^6t^5x_1^2x_3^2+2q^6t^5x_1x_2^2x_3+\\
2q^6t^5x_1x_2x_3^2+q^6t^5x_2^2x_3^2+q^5t^6x_1^3x_2+q^5t^6x_1^3x_3+3q^5t^6x_1^2x_2^2+q^5t^6x_1^2x_2x_3+3q^5t^6x_1^2x_3^2+\\
q^5t^6x_1x_2^3+q^5t^6x_1x_2^2x_3+q^5t^6x_1x_2x_3^2+q^5t^6x_1x_3^3+q^5t^6x_2^3x_3+3q^5t^6x_2^2x_3^2+q^5t^6x_2x_3^3-\\
2q^4t^7x_1^2x_2^2-2q^4t^7x_1^2x_3^2-2q^4t^7x_2^2x_3^2-q^6t^4x_1^2x_2x_3-q^6t^4x_1x_2^2x_3-q^6t^4x_1x_2x_3^2-3q^5t^5x_1^2x_2^2-\\
4q^5t^5x_1^2x_2x_3-3q^5t^5x_1^2x_3^2-4q^5t^5x_1x_2^2x_3-4q^5t^5x_1x_2x_3^2-3q^5t^5x_2^2x_3^2-q^4t^6x_1^3x_2-q^4t^6x_1^3x_3+\\
q^4t^6x_1^2x_2^2+4q^4t^6x_1^2x_2x_3+q^4t^6x_1^2x_3^2-q^4t^6x_1x_2^3+4q^4t^6x_1x_2^2x_3+4q^4t^6x_1x_2x_3^2-q^4t^6x_1x_3^3-\\
q^4t^6x_2^3x_3+q^4t^6x_2^2x_3^2-q^4t^6x_2x_3^3-2q^3t^7x_1^2x_2x_3-2q^3t^7x_1x_2^2x_3-2q^3t^7x_1x_2x_3^2-2q^5t^4x_1^2x_2^2+
\end{gathered}$$

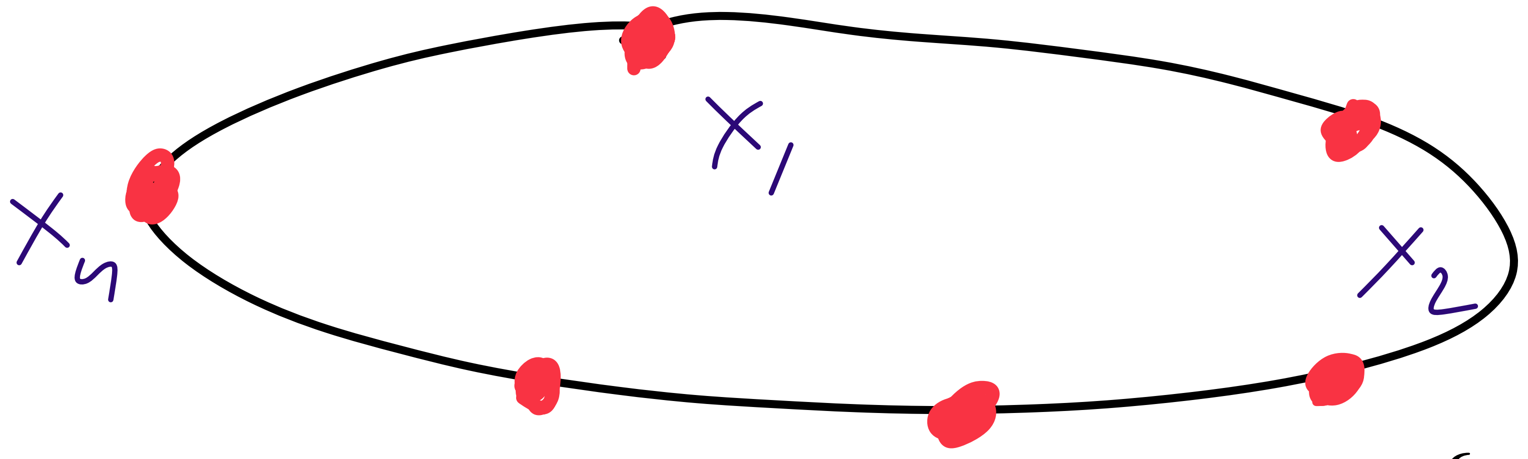
Integrable Many-Body Systems





Calogero in 1971 introduced a new integrable system. Moser in 1975 proved its integrability using Lax pair





$$H_{CM} = \sum_{i=1}^{n} \frac{p_i^2}{2m} + g^2 \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}$$

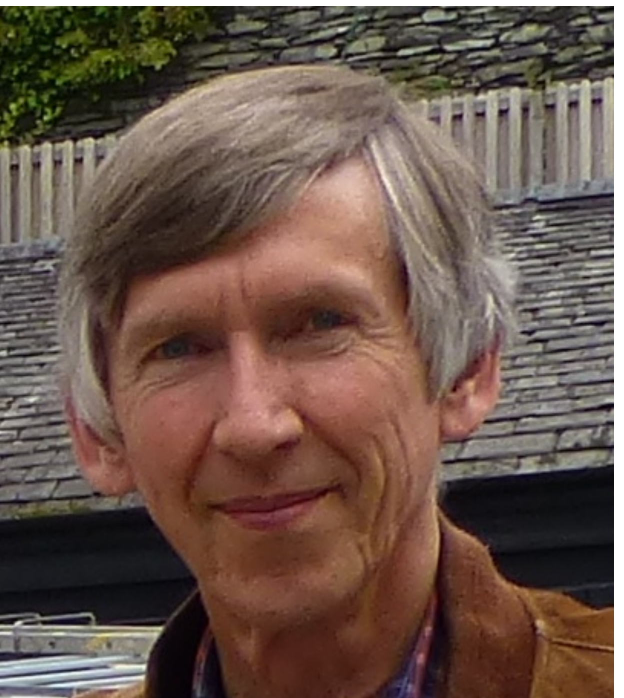
$$\{H_i, H_j\} = 0$$
$$i = 1, \dots, n$$

$$V(z) \simeq \frac{1}{z^2} \qquad \wp(x_j - x_i)$$

The **Calogero-Moser (CM)** system has several generalizations

rCM —> tCM —> eCM

$$V(z) \simeq \frac{1}{\sinh z^2}$$



Another relativistic generalization called **Ruijsenaars-Schneider (RS)** family

rRS —> tRS —> eRS

Geometrically described by Hamiltonian reduction of T*GL(n)

$$H_{CM} = \lim_{c \to \infty} H_{RS} - nmc^2$$

Algebraic Integrable Systems

- These are examples of complex algebraic integrable systems with n degrees of freedom whose phase space is a Lagrangian fibration of complex dimension $2n$ equipped with holomorphic symplectic 2-form $\Omega = \sum_{i=1}^{n} dp_i \wedge dx_i$ over a smooth base whose fibers are Abelian varieties (admit group law)
- There are n Poisson commuting Hamiltonians $H_1, \ldots, H_n$
- In *action-angle variables,* Hamiltonian evolution is linearized on the fibers which serve as level sets of the Hamiltonians

Hitchin Integrable System

[Donagi Witten]
[Gorsky Nekrasov]
[Nekrasov Pestun Shatashvili]

Seiberg-Witten solution of $\mathscr{N}=2$ gauge theories leads to Hitchin integrable system $(\mathscr{E},\varphi)$

Holomorphic G vector bundle over C_p with holomorphic section φ (Higgs field) of $K_{C_p}\otimes \mathrm{ad}(E)\otimes \mathscr{O}(p)$

$$A=\alpha_p\, d\vartheta+\cdots$$

$$\varphi=\frac{1}{2}(\beta_p+i\gamma_p)\frac{dz}{z}+\cdots$$

$$\begin{array}{c}\mathcal{E}\to\mathcal{M}_{\mathrm{vac}}(\mathbb{R}^3\times S^1)\\ \downarrow\\ \mathcal{B}=\mathcal{M}_{\mathrm{vac}}(\mathbb{R}^4)\end{array}$$

Hyperkähler (3d Coulomb branch)

Special Kähler (4d Coulomb branch) $\{a_1,\ldots,a_n\}$

The n-dimensional Abelian variety is parameterized by the period matrix $\tau_{ij}=\frac{\partial\mathcal{F}}{\partial a_i\partial a_j}$

Liouville tori can be found inside the Jacobians of the algebraic curve

$$\det(z-\varphi)=0$$

Coordinates

$$x_i=\sum_{j=1}^{N-1}\int_{P_0}^{P_j}\omega_i$$

The Abelian nature of Lagrangian fibers suggests that coordinates and momenta take values in

$$\mathbb{C},\ \mathbb{C}^\times,\ \mathcal{E}=\mathbb{C}^\times/q^{\mathbb{Z}}$$

Many-Body Systems of CM/RS type

p \ q	rational	trigonometric	elliptic
r	rational CMS	$\epsilon \to 0$ trigonometric CMS	$p \to 0$ elliptic CMS *quantum cohomology*
t	$R \to 0$ rational RS (dual trig. CMS)	$\epsilon \to 0$ $R \to 0$ trigonometric RS	$R \to 0$ $p \to 0$ elliptic RS *quantum K-theory*
e	dual elliptic CMS	$w \to 0$ dual elliptic RS	$p \to 0$ $w \to 0$ DELL *Elliptic Cohomology*

[Oblomkov]
[Etingof]

The Calogero-Moser Space

Let V be an N-dimensional vector space over $\mathbb{C}$. Let $\mathscr{M}'$ be the subset of $GL(V) \times GL(V) \times V \times V^*$ consisting of elements (M, T, u, v) such that

$$qMT - TM = u \otimes v^T$$

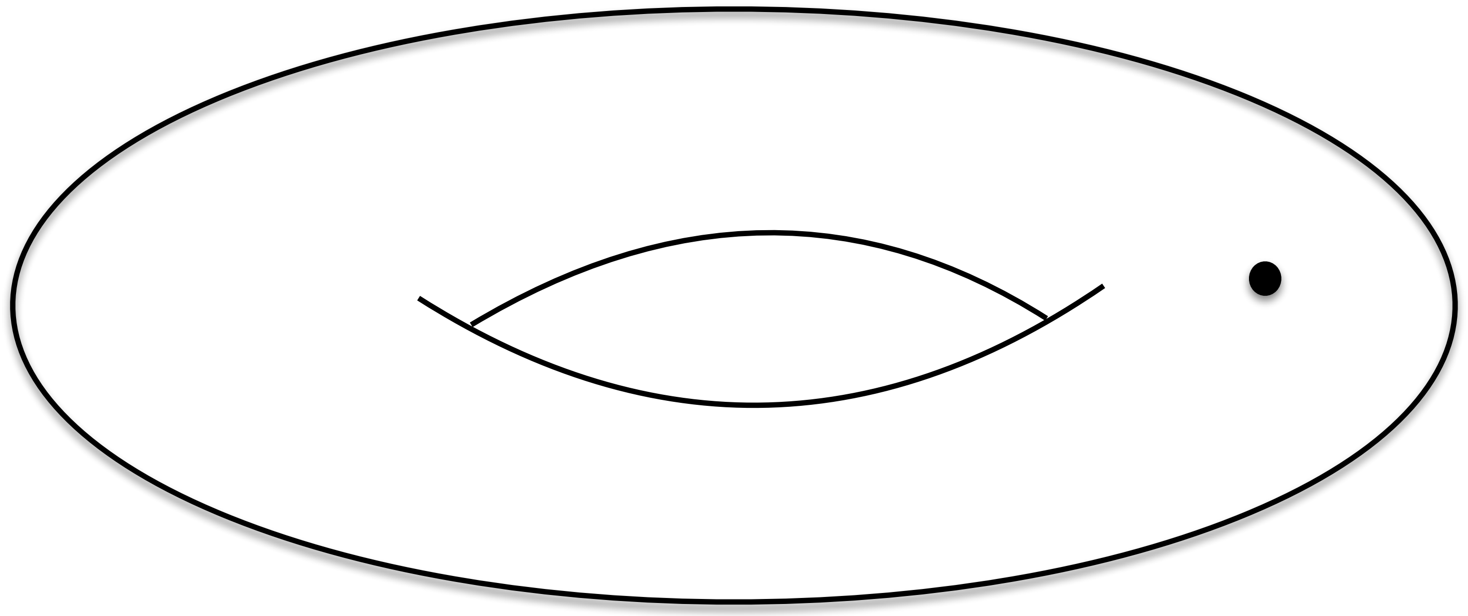
The group $GL(N; \mathbb{C}) = GL(V)$ acts on $\mathscr{M}'$ by conjugation

$$(M, T, u, v) \mapsto (gMg^{-1}, gTg^{-1}, gu, vg^{-1})$$

The quotient of $\mathscr{M}'$ by the action of $GL(V)$ is called **Calogero-Moser space $\mathscr{M}$**

$$\mathcal{M}_n = \{A, B, C\}/GL(n; \mathbb{C})$$

Also can be understood as moduli space of flat connections on punctured torus

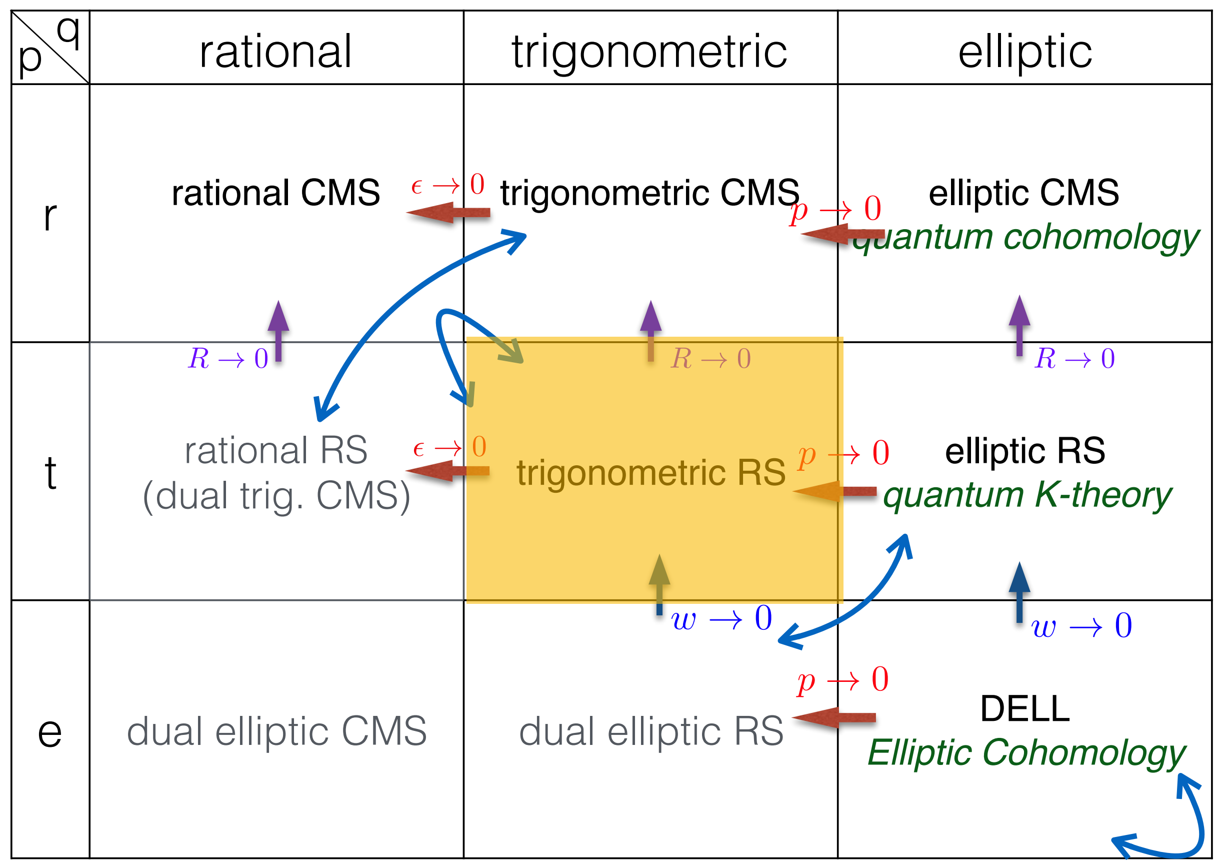


$$ABA^{-1}B^{-1} = C$$

$$C = \text{diag}(q, \ldots, q, q^{n-1})$$

Integrable Hamiltonians are ~TrT^k

Trigonometric RS Model



Flatness condition $\quad qMT - TM = u \otimes v^T$

In the basis where M is diagonal with eigenvalues $\xi_1, \ldots, \xi_n$ matrix T

$$T_{ij} = \frac{u_i v_j}{q\xi_i - \xi_j}$$

Define tRS momenta

$$p_i = -u_i v_i \frac{\prod_{k \neq i} (\xi_i - \xi_k)}{\prod_{k} (\xi_i - \xi_k q)}$$

The tRS Lax matrix reads

$$T_{ji} = \frac{\xi_j (1-q)}{\xi_j - \xi_i q} \prod_{k \neq j} \frac{\xi_j - \xi_k q}{\xi_j - \xi_k} p_j = \frac{\prod_{k \neq i} (\xi_j - \xi_k q)}{\prod_{k \neq j} (\xi_j - \xi_k)} p_j$$

Hamiltonians

$$\det \left(z - T(\xi_i, p_i, q)\right) = \sum_{k=0}^{L} (-1)^l H_k(\xi_i, p_i, q)^{n-k}$$

Eigenproblem

$$\sum_{\substack{\mathcal{I} \subset \{1,\ldots,L\} \\ |\mathcal{I}|=k}} \prod_{\substack{i \in \mathcal{I} \\ j \notin \mathcal{I}}} \frac{q\,\xi_i - \xi_j}{\xi_i - \xi_j} \prod_{m \in \mathcal{I}} p_m = e_k(a_i)$$

Two particles

$$H_1 = \frac{q\xi_1 - \xi_2}{\xi_1 - \xi_2} p_1 + \frac{q\xi_2 - \xi_1}{\xi_2 - \xi_1} p_2$$

$$H_2 = p_1 p_2$$

Quantum tRS Spectrum

Difference operators $p_i f(\xi_i) = f(q\xi_i)$ $p_i \xi_j = q^{\delta_{ij}} \xi_i p_j$

tRS eigenvalue problem

$$H_i(\xi, p) V(\xi, a) = e_i(a) V(\xi, a)$$

What is the geometric meaning of V?

Answering this question will help us to understand elliptic models

Before we answer this question notice the symmetry of the flatness condition

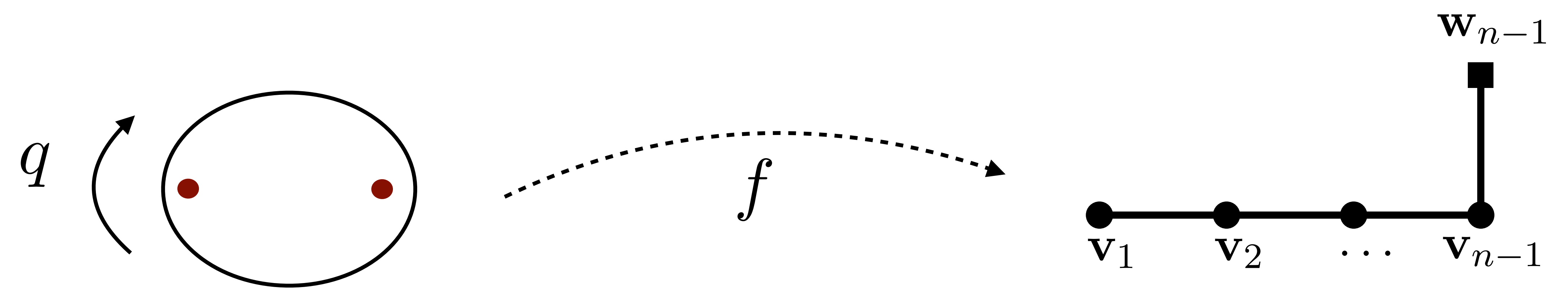
$$qMT - TM = u \otimes v^T$$

$$q \mapsto q^{-1}\,, \qquad M \mapsto T\,, \qquad T \mapsto M\,.$$

3d mirror symmetry

Enumerative AG/Integrable Systems

Quantum equivariant K-theory of Nakajima quiver varieties



$$A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$$

$$\mathrm{V}^{(\tau)}(\boldsymbol{z}) = \sum_{\boldsymbol{d}} \mathrm{ev}_{p_2,*}\big(\widehat{\mathcal{O}}^{\boldsymbol{d}}_{\mathrm{vir}} \otimes \tau|_{p_1}, \mathsf{QM}^{\boldsymbol{d}}_{\mathrm{nonsing}\,p_2}\big)\boldsymbol{z}^{\boldsymbol{d}} \in K_{\mathsf{T}\times\mathbb{C}_q^\times}(X)_{loc}[[\boldsymbol{z}]]$$

Saddle point limit yields Bethe equations for XXZ

Quantum classes satisfy interesting difference equations in equivariant parameters and Kahler parameters ***qKZ, Dynamical equation*** [Okounkov, Smirnov]

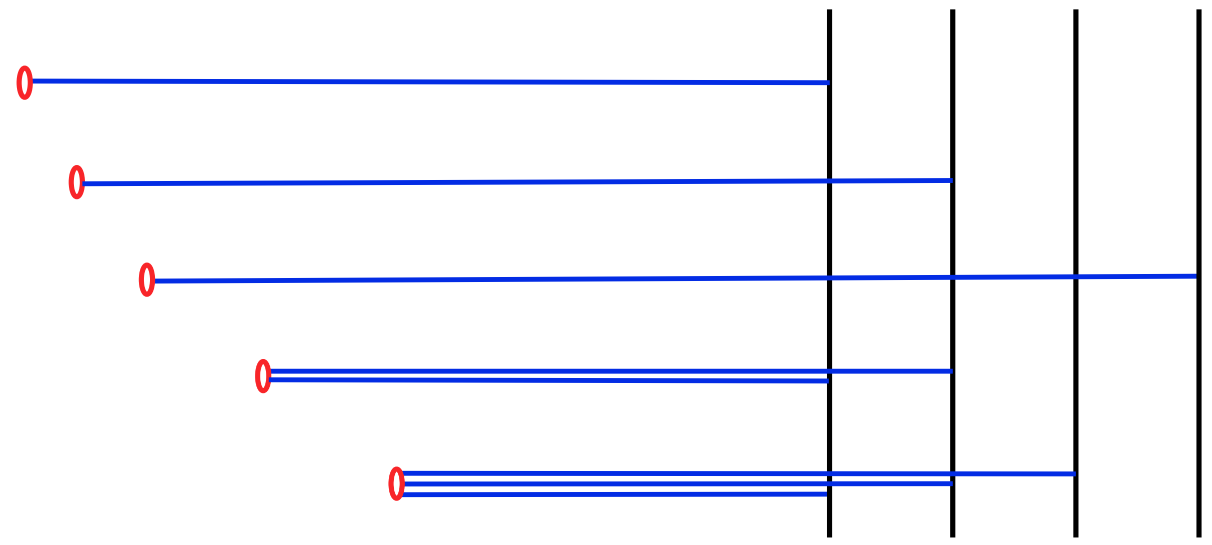
After symmetrization they can be rewritten as eigenvalue equations for ***trigonometric Ruijsenaars-Schneider (tRS)*** system [PK, Zeitlin] [PK]

mirror frame

$$T_r(\mathbf{a}) = \sum_{\substack{\mathcal{I}\subset\{1,\dots,n\}\\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I}\\ j\notin\mathcal{I}}} \frac{t\,a_i - a_j}{a_i - a_j} \prod_{i\in\mathcal{I}} p_i$$

$$T_r(\boldsymbol{a})\mathrm{V}(\boldsymbol{a},\vec{\zeta}) = S_r(\vec{\zeta},t)\mathrm{V}(\boldsymbol{a},\vec{\zeta})$$

In terms of string/gauge theory tRS eigenproblem is Ward identity [Gaiotto, PK] [Bullimore, Kim, PK]



Quantum tRS Spectrum

[PK]

Theorem 2.10. *Let $V_{\boldsymbol{p}}^{(1)}$ be the coefficient for the vertex function for X given in (2.[illegible]). Define*

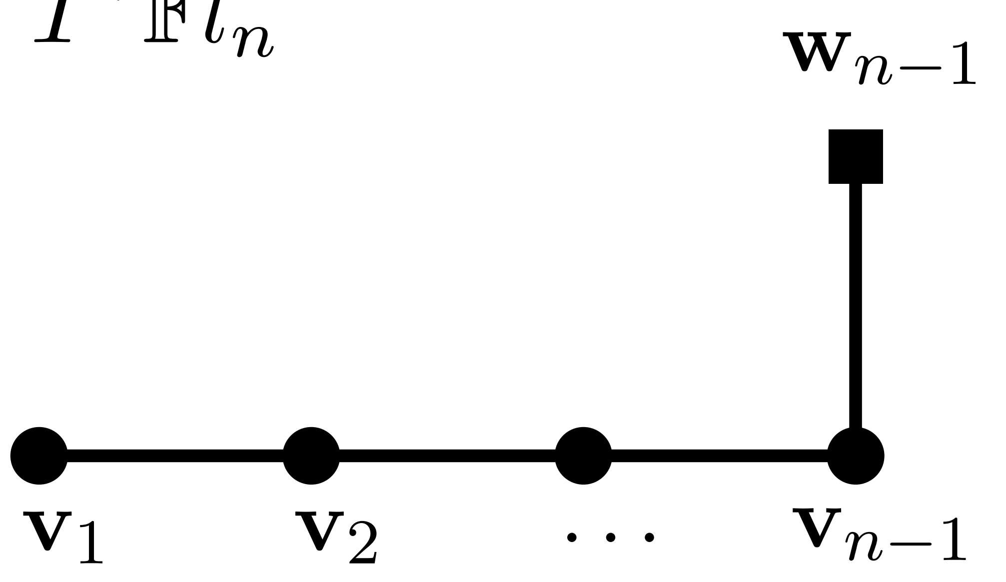
$$\mathsf{V}_{\boldsymbol{p}}^{(1)} = \prod_{i=1}^{n} \frac{\theta(\hbar^{i-n}\zeta_i, q)}{\theta(a_i\zeta_i, q)} \cdot V_{\boldsymbol{p}}^{(1)}\,, \tag{2.9}$$

where $\theta(x,q) = (x,q)_\infty(qx^{-1},q)_\infty$ is basic theta-function. Then $\mathsf{V}_{\boldsymbol{p}}$ are eigenfunctions for tRS difference operators (2.8) for all fixed points $\boldsymbol{p}$

$$T_r(\boldsymbol{\zeta})\mathsf{V}_{\boldsymbol{p}}^{(1)} = e_r(\mathbf{a})\mathsf{V}_{\boldsymbol{p}}^{(1)}\,, \qquad r = 1,\dots,n\,, \tag{2.10}$$

where e_r is elementary symmetric polynomial of degree r of $a_1,\dots,a_n$.

$T^*\mathbb{F}l_n$



tRS momenta $\qquad p_i = \dfrac{s_{i+1,1}\cdot\dots\cdot s_{i+1,i+1}}{s_{i,1}\cdot\dots\cdot s_{i,i}}$

Quantum multiplication by class

$$\widehat{\Lambda^i\mathscr{V}_i} \otimes \widehat{\Lambda^{i+1}\mathscr{V}^*_{i+1}}$$

Chern roots $s_{i,a}$ satisfy XXZ Bethe Ansatz equations

Macdonald Polynomials

Proposition 2.11. *Consider coefficient functions for K-theory of QM to X_n (2.9) for all fixed points of the maximal torus. Let λ be a partition of k elements of length n and $\lambda_1 \geq \cdots \geq \lambda_n$. Let*

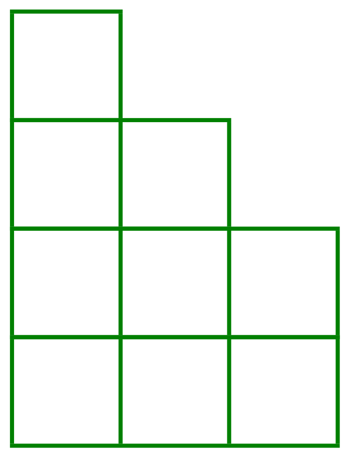
$$\frac{a_{i+1}}{a_i} = q^{\ell_i}\hbar\,, \quad \ell_i = \lambda_{i+1} - \lambda_i\,, \quad i = 1, \ldots, n-1\,. \tag{2.18}$$

Then there exists a fixed point $\boldsymbol{q}$ for which

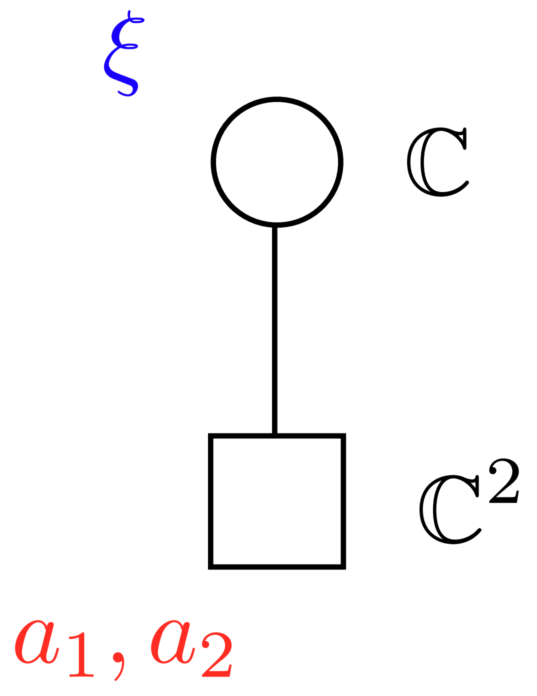
$$\mathsf{V}_{\boldsymbol{q}} = P_\lambda(\boldsymbol{\zeta}; q, \hbar)\,. \tag{2.19}$$

$$a_i = a q^{\lambda_i} \hbar^{i-n}\,, \quad i = 1, \ldots, n$$

$$T_1(\boldsymbol{\zeta}) P_\lambda(\boldsymbol{\zeta}; q, \hbar) = \left(\sum_{i=1}^{n} q^{\lambda_i} \hbar^{i-n} \right) P_\lambda(\boldsymbol{\zeta}; q, \hbar)$$

$\lambda =$ 

Example: $T^*\mathbb{P}^1$



$$H_1 = \frac{q\xi_1 - \xi_2}{\xi_1 - \xi_2} p_1 + \frac{q\xi_2 - \xi_1}{\xi_2 - \xi_1} p_2$$

$$H_2 = p_1 p_2$$

$$V^{(1)}_{\mathbf{p}} = \sum_{d>0} \left(\frac{\zeta_1}{\zeta_2}\right)^d \prod_{i=1}^{2} \frac{\left(\frac{q}{\hbar}\frac{a_{\mathbf{p}}}{a_i}; q\right)_d}{\left(\frac{a_{\mathbf{p}}}{a_i}; q\right)_d} = {}_2\phi_1\left(\hbar, \hbar\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}, q\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}; q; \frac{q}{\hbar}\frac{\zeta_1}{\zeta_2}\right)$$

Macdonald polynomials

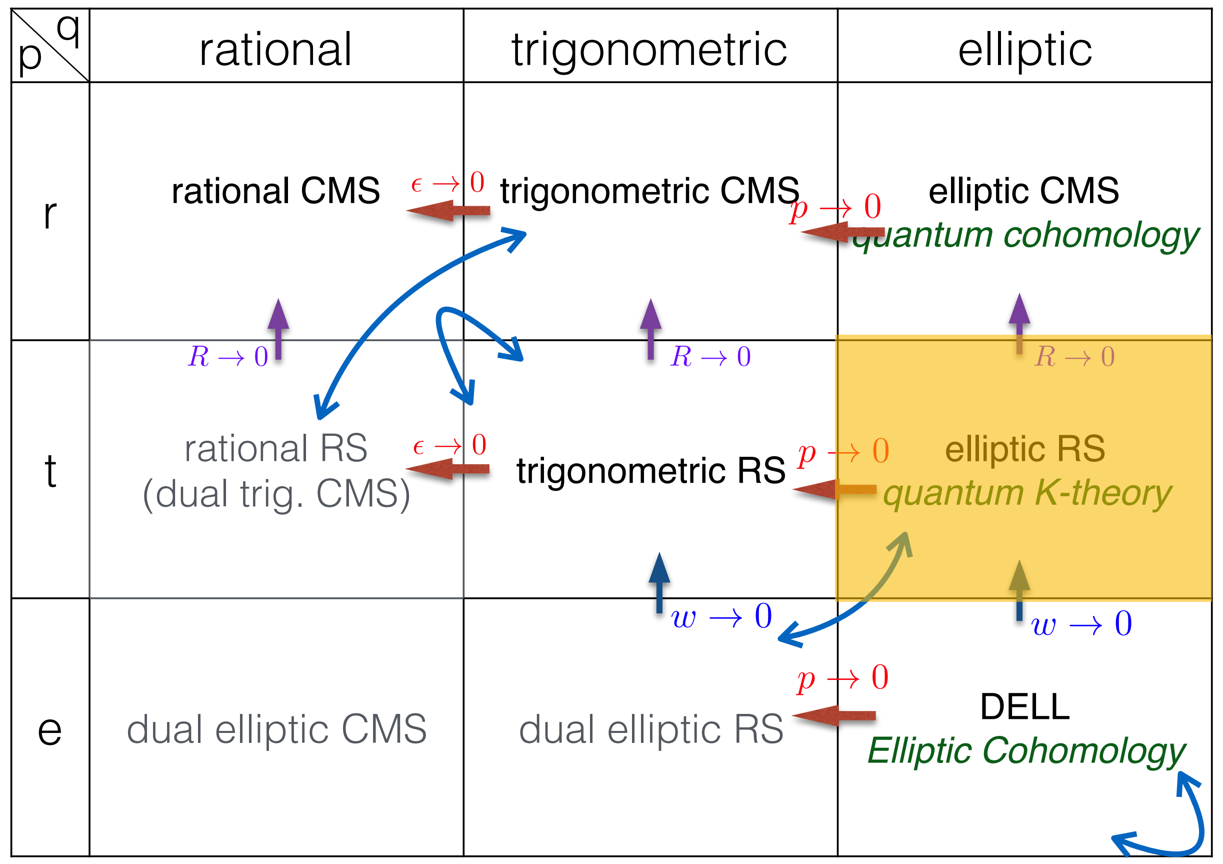
$$a_1 = a q^{\lambda_1}\hbar^{-1}, a_2 = a q^{\lambda_2}$$

$$\mathrm{V}_{\Box} = \zeta_1 + \zeta_2,$$

$$\mathrm{V}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} = \zeta_1^2 + \zeta_2^2 + \frac{(q+1)(\hbar-1)}{q\hbar - 1}\zeta_1\zeta_2,$$

$$\mathrm{V}_{\begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}} = \zeta_1^3 + \zeta_2^3 + \frac{\left(q^2+q+1\right)(\hbar-1)}{q^2\hbar - 1}\zeta_2\zeta_1^2 + \frac{\left(q^2+q+1\right)(\hbar-1)}{q^2\hbar - 1}\zeta_2^2\zeta_1$$

Elliptic RS Model



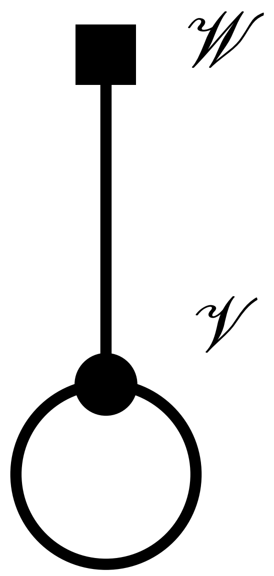
$$E_r(\boldsymbol{\zeta}) = \sum_{\substack{\mathcal{I}\subset\{1,\dots,n\}\\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I}\\ j\notin\mathcal{I}}} \frac{\theta_1(\hbar\zeta_i/\zeta_j|\mathfrak{p})}{\theta_1(\hbar\zeta_i/\zeta_j|\mathfrak{p})} \prod_{i\in\mathcal{I}} p_k$$

$$\mathcal{H}_k \mathcal{Z}^{RS}(\boldsymbol{a},\boldsymbol{x}) = \lambda_k(\boldsymbol{a})\mathcal{Z}^{RS}(\boldsymbol{a},\boldsymbol{x})$$

Conjecture 5.1. *The solution of (5.2) is given by the K-theoretic holomorphic equivariant Euler characteristic of the affine Laumon space*

$$\mathcal{Z} = \sum_{\boldsymbol{d}} \vec{\mathfrak{q}}^{\boldsymbol{d}} \int_{\mathcal{L}_{\boldsymbol{d}}} 1\,, \tag{5.3}$$

where $\vec{\mathfrak{q}} = (\mathfrak{q}_1,\dots,\mathfrak{q}_n)$ *is a string of* $\mathbb{C}^\times$*-valued coordinates on the maximal torus of* $\mathcal{L}^{\mathrm{aff}}_{\boldsymbol{d}}$*. The eigenvalues* $\mathscr{E}_r$ *are equivariant Chern characters of bundles* $\Lambda^r\mathscr{W}$*, where* $\mathscr{W}$ *is the constant bundle of the corresponding ADHM space. In other words they have the following form*



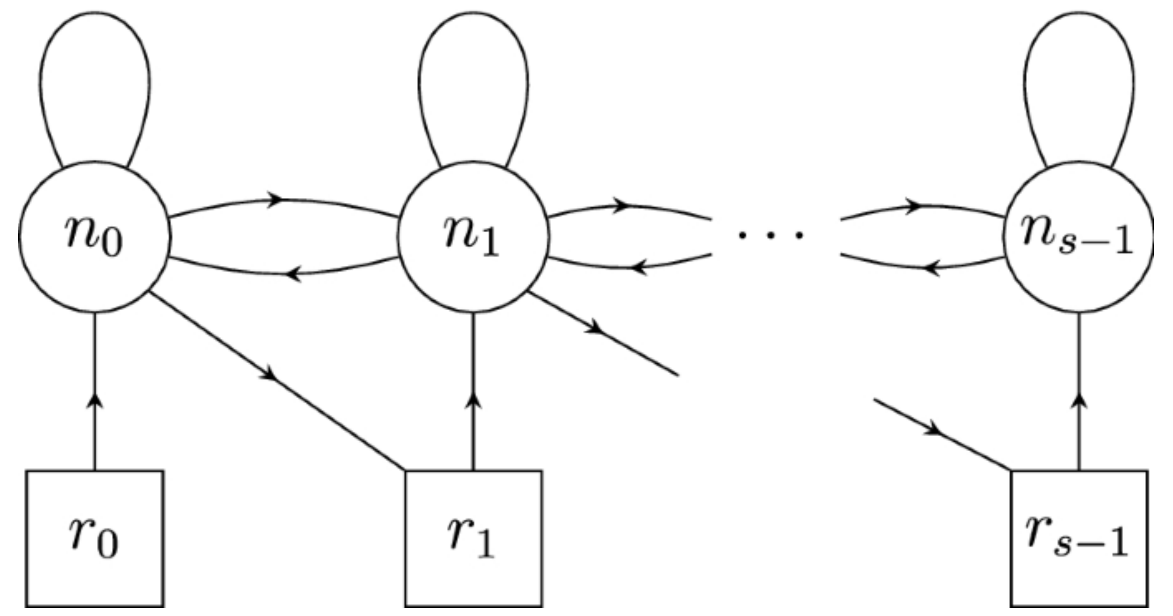
$$\mathscr{E}_r = e_r + \sum_{l=1}^{\infty} \mathfrak{p}^l \mathcal{E}_r^{(l)}\,, \tag{5.4}$$

where e_r *are symmetric functions of the equivariant parameters* $a_1,\dots,a_N$*.*

eRS Spectrum

Euler characteristic of ***affine Laumon space*** (representation space of a chain-saw quiver)

$$\mathcal{Z} = \sum_{\vec{\mu}} \prod_{l=1}^{n} \mathfrak{q}_l^{k_l(\vec{\mu})}\, z_{\vec{\mu}}(\vec{a}, \hbar, q) \qquad\qquad \mathfrak{p} = \mathfrak{q}_1 \cdot \dots \cdot \mathfrak{q}_n$$



Theorem 1.1. *Let $\boldsymbol{x} = (x_1, \dots x_N)$ be the position vector of the eRS model and $\mathcal{Z}^{RS}(\boldsymbol{a}, \boldsymbol{x}) = \lim_{w\to 0} \mathcal{Z}_{inst}^{6d/4d}(w, p, \boldsymbol{x})$ is its wavefunction. Then the following equality holds*

$$\mathcal{H}_k \mathcal{Z}^{RS}(\boldsymbol{a}, \boldsymbol{x}) = \lambda_k(\boldsymbol{a}) \mathcal{Z}^{RS}(\boldsymbol{a}, \boldsymbol{x}), \qquad k = 1, \dots, N-1\,. \tag{1.6}$$

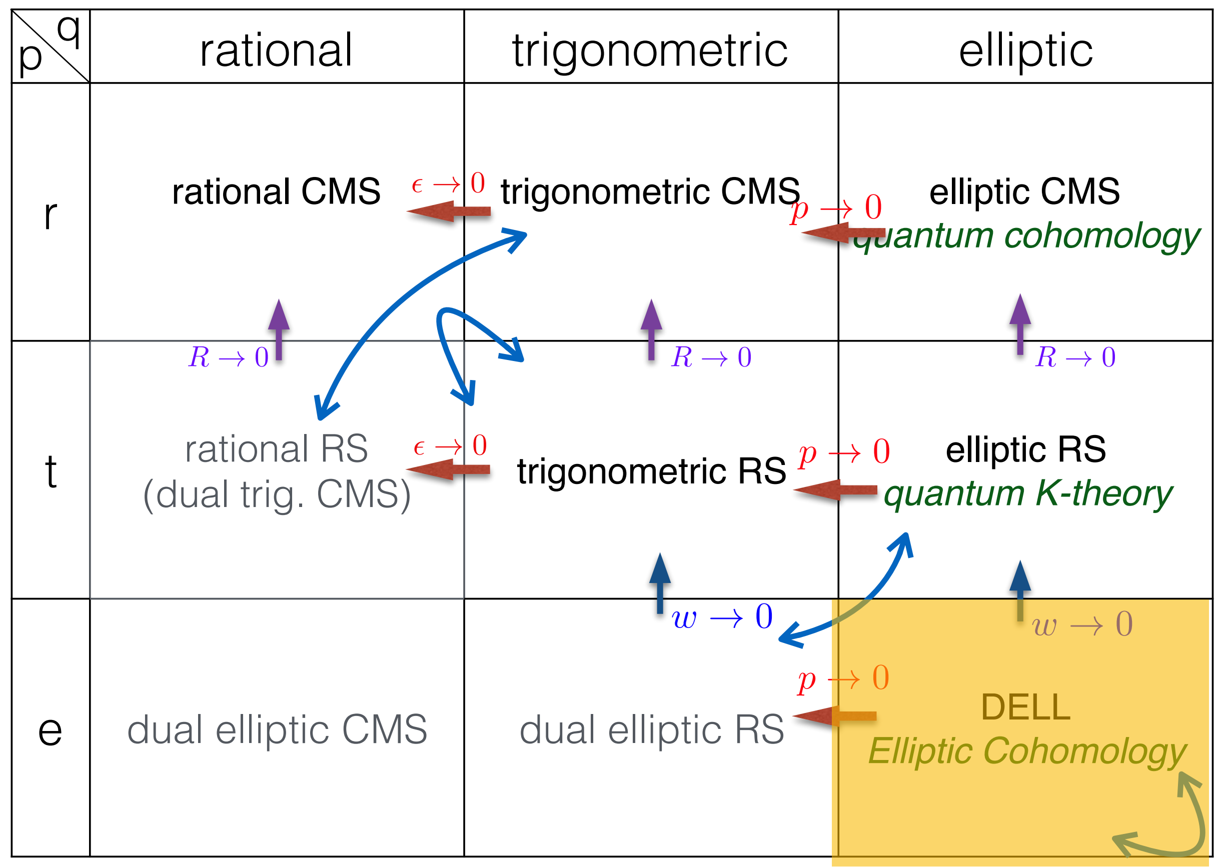
where the eigenvalues read

$$\lambda_k(\boldsymbol{a}) = \prod_{n=0}^{k-1} \frac{\theta(t^{N-n})}{\theta(t^{n+1})} \cdot \frac{\mathcal{Z}^{RS}(\boldsymbol{a}, t^{\vec{\rho}} q^{\vec{\omega_k}})}{\mathcal{Z}^{RS}(\boldsymbol{a}, t^{\vec{\rho}})}, \qquad k = 1, \dots, N-1 \tag{1.7}$$

where $\vec{\omega_k}$ is the k-th fundamental weight of representation of $SU(N)$ and $\vec{\rho} = \big((N-1)/2, (N-3)/2, \dots, (3-N)/2, (1-N)/2\big)$ is the $SU(N)$ Weyl vector.

[PK Shakirov]

The DELL



N-particle DELL Hamiltonians

$$\widehat{\mathcal{H}}_a = \widehat{\mathcal{O}}_0^{-1}\widehat{\mathcal{O}}_a$$

$$\widehat{\mathcal{O}}(z) = \sum_{n\in\mathbb{Z}} \widehat{\mathcal{O}}_n\, z^n = \sum_{n_1,\ldots,n_N=-\infty}^{\infty} (-z)^{\sum n_i}\, w^{\sum \frac{n_i(n_i-1)}{2}} \prod_{i<j}\theta\left(t^{n_i-n_j}\widehat{x}_i/\widehat{x}_j|p\right)\, \widehat{p}_1^{n_1}\ldots\widehat{p}_N^{n_N}$$

Quasiperiodicity

$$\widehat{\mathcal{O}}(wz) = \left((-1)^N z^{-N} t^{-N(N-1)} q^{-J}\right)\, \widehat{\mathcal{O}}(z), \qquad J = j_1 + \ldots + j_N$$

So that there are only $N-1$ independent operators

$$\widehat{\mathcal{O}}(z) = \sum_{a=0}^{N-1} \widehat{\mathcal{O}}_a\, \theta_a^{(N)}(z)$$

$$\theta_a^{(N)}(z) = z^a \sum_{n\in\mathbb{Z}} w^{Nn(n-1)/2+an}\, \left((-z)^N t^{N(N-1)} q^J\right)^n$$

Generator of all DELL Hamiltonians

$$\mathcal{L}(z) := \widehat{\mathcal{O}}_0^{-1}\widehat{\mathcal{O}}(z) = \sum_{a=0}^{\infty} z^n\, \widehat{\mathcal{O}}_0^{-1}\widehat{\mathcal{O}}_a = 1 + z\,\widehat{\mathcal{H}}_1 + z^2\,\widehat{\mathcal{H}}_2 + \ldots$$

Commutativity and Spectrum

Conjecture 1: DELL Hamiltonians commute $[\widehat{\mathcal{H}}_a, \widehat{\mathcal{H}}_b] = 0$

Conjecture 2: *Let $\boldsymbol{x} = (x_1, \ldots, x_N)$ be the position vector, $\mathscr{Z}(p, \boldsymbol{x})$ be a properly normalized equivariant elliptic genus of the affine Laumon space in the limit $\epsilon_2 \to 0$. Then there exists a function $\lambda(z, \boldsymbol{a}, w, p)$ such that*

$$\widehat{\mathcal{O}}(z)\mathscr{Z}(p, \boldsymbol{x}) = \lambda(z, \boldsymbol{a}, w, p)\ \widehat{\mathcal{O}}_0\mathscr{Z}(p, \boldsymbol{x})\,.$$

In particular, by expanding the currents in z, including $\lambda(z, \boldsymbol{a}, w, p) = \sum_n \lambda_n(\boldsymbol{a}, w, p) z^n$

we get $\widehat{\mathcal{H}}_n \mathcal{Z}^{6d/4d}_{inst}(w, p, \boldsymbol{x}) = \lambda_n(\boldsymbol{a}, w, p)\mathcal{Z}^{6d/4d}_{inst}(w, p, \boldsymbol{x})$

Both conjectures have been verified till certain order in z expansion

Two-Particle DELL

$$\widehat{\mathcal{H}} = \widehat{\mathcal{O}}_0^{-1}\widehat{\mathcal{O}}_1 \qquad \widehat{\mathcal{O}}_0 = \Theta\left[\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & 0 \end{array}\right]\left(\frac{\widehat{x}_2}{\widehat{x}_1}, \frac{\widehat{p}_2}{\widehat{p}_1}\right), \qquad \widehat{\mathcal{O}}_1 = \Theta\left[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{array}\right]\left(\frac{\widehat{x}_2}{\widehat{x}_1}, \frac{\widehat{p}_2}{\widehat{p}_1}\right)$$

Operators $\mathcal{O}$ can be combined into genus-2 theta functions

$$\Theta\left[\begin{array}{cc} a & b \\ 0 & 0 \end{array}\right]\left(q \quad gp \left| \begin{array}{cc} \tau_1 & gm \\ gm & \tau_2 \end{array}\right.\right) = \sum_{k_1,k_2=-\infty}^{\infty} p^{(k_1+a)^2-a^2}\, w^{(k_2+b)^2-b^2}\, t^{2(k_1+a)(k_2+b)}\, \left(-Q^2\right)^{k_1+a}\, \left(P^2\right)^{k_2+b}$$

Parameters $\quad q = e^{\pi i g}, \quad t = e^{\pi i m g}, \quad p = e^{\pi i \tau_1}, \quad w = e^{\pi i \tau_2}, \quad P = e^{\pi i g p}, \quad Q = e^{\pi i q}$

Limits $\quad w \to 0$

eRS model

$$\mathcal{O}(z) = \theta\left(\frac{x_1}{x_2}\middle|p\right) - z\left(\theta\left(\frac{tx_1}{x_2}\middle|p\right)p_1 + \theta\left(\frac{x_1}{tx_2}\middle|p\right)p_2\right) + z^2\theta\left(\frac{x_1}{x_2}\middle|p\right)p_1p_2$$

$$\mathcal{H}_1 = \frac{\mathcal{O}_1}{\mathcal{O}_0} = \frac{\theta\left(t\frac{x_1}{x_2}\middle|p\right)}{\theta\left(\frac{x_1}{x_2}\middle|p\right)}p_1 + \frac{\theta\left(t\frac{x_2}{x_1}\middle|p\right)}{\theta\left(\frac{x_2}{x_1}\middle|p\right)}p_2, \qquad \mathcal{H}_2 = \frac{\mathcal{O}_2}{\mathcal{O}_0} = p_1p_2$$

$p \to 0$

Dual eRS model

$$\begin{aligned}\mathcal{O}(z) = &\sum_{n_1\in\mathbb{Z}} w^{n_1^2}\left(1 - t^{2n_1}\frac{x_1}{x_2}\right)\left(\frac{p_1}{p_2}\right)^{n_1} \\ &- z\sum_{n_1\in\mathbb{Z}} w^{n_1^2-n_1}\left(\left(1 - t^{2n_1-1}\frac{x_1}{x_2}\right)\left(\frac{p_1}{p_2}\right)^{n_1}p_2 + \left(1 - t^{-2n_1+1}\frac{x_2}{x_1}\right)\left(\frac{p_2}{p_1}\right)^{n_1}p_1\right) \\ &+ z^2\sum_{n_1\in\mathbb{Z}} w^{n_1^2}\left(1 - t^{2n_1}\frac{x_1}{x_2}\right)\left(\frac{p_1}{p_2}\right)^{n_1}p_1p_2 + \ldots, \quad = \theta(zp_1|w)\theta(zp_2|w) - \frac{x_1}{x_2}\,\theta(ztp_1|w)\theta\left(\frac{z}{t}p_2|w\right)\end{aligned}$$

Classical DELL

[Braden Hollowood]
[Braden Marshakov Mironov Morozov]
[Braden Gorsky Odessky Rubtsov]

Spectral curve

$$\sum_{n=0}^{\infty} m^n \left[\partial_\xi^n \theta(e^\xi|p)\right] \cdot \left[\partial_\zeta^n H(\zeta, w, \mathbf{a})\right] = 0$$

$$H(\zeta, w, \mathbf{a}) = \prod_{i=1}^{N} \theta\left(e^\zeta/a_i|w\right)$$

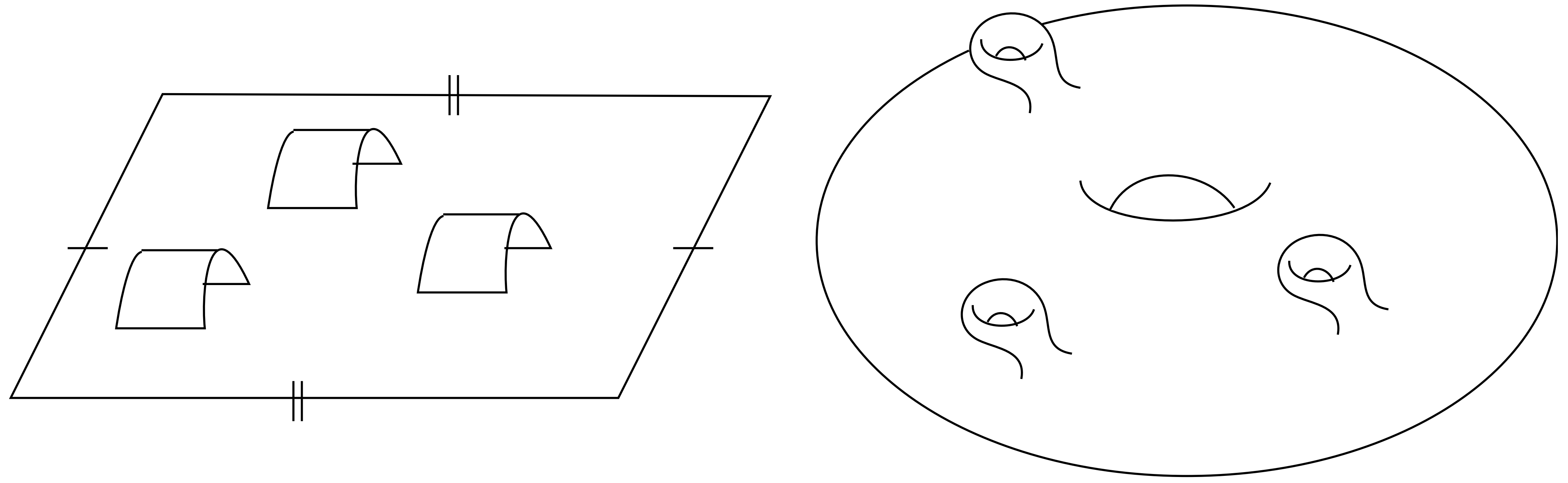
$$\Theta\left[\begin{array}{ccc} \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \cdots & \frac{1}{2} \end{array}\right]\left(z, Nm(x-a_1)), \ldots, Nm(x-a_N)|\widehat{\Pi}\right) = 0$$

$$\Theta\left[\begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array}\right](\mathbf{Z}|\Pi) = \sum_{\mathbf{m}\in\mathbb{Z}^g} \exp\left(\pi i(\mathbf{m}+\mathbf{a})\cdot\Pi\cdot(\mathbf{m}+\mathbf{a}) + 2\pi i(\mathbf{Z}+\mathbf{b})\cdot(\mathbf{m}+\mathbf{a})\right)$$

$(N+1)\times(N+1)$ period matrix

$$\widehat{\Pi} = \begin{pmatrix} \tau_1 & gm & gm & \cdots & gm \\ gm & \tau_2 & 0 & \cdots & 0 \\ gm & 0 & \tau_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ gm & 0 & \cdots & 0 & \tau_2 \end{pmatrix}$$

$$\oint_{A_i} \omega_j = \delta_{ij}\,, \qquad \oint_{B_i} \omega_j = \Pi_{ij}$$

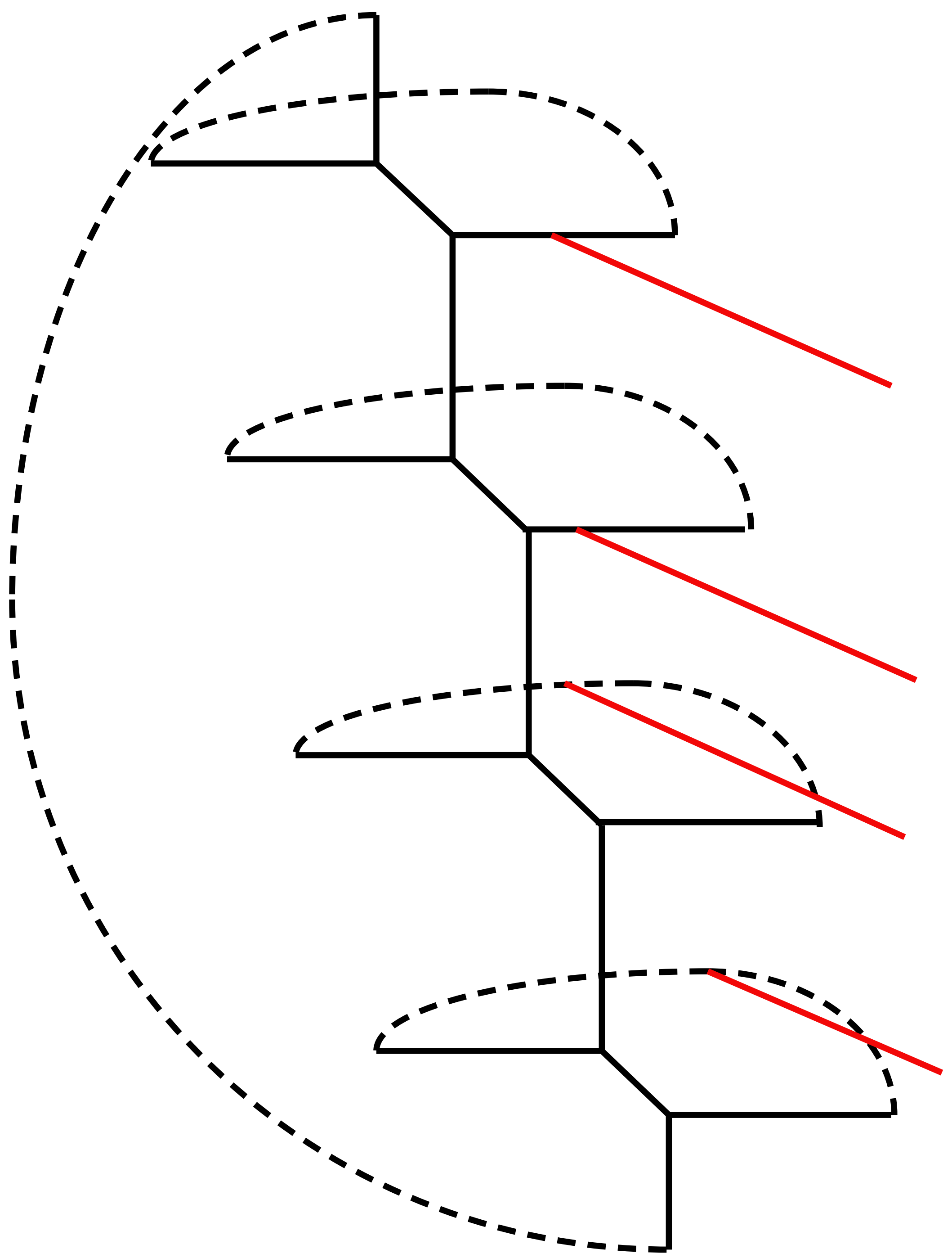


The Seiberg-Witten curve Σ for 6d $(1,0)^*$ theory compactified on T^2_w

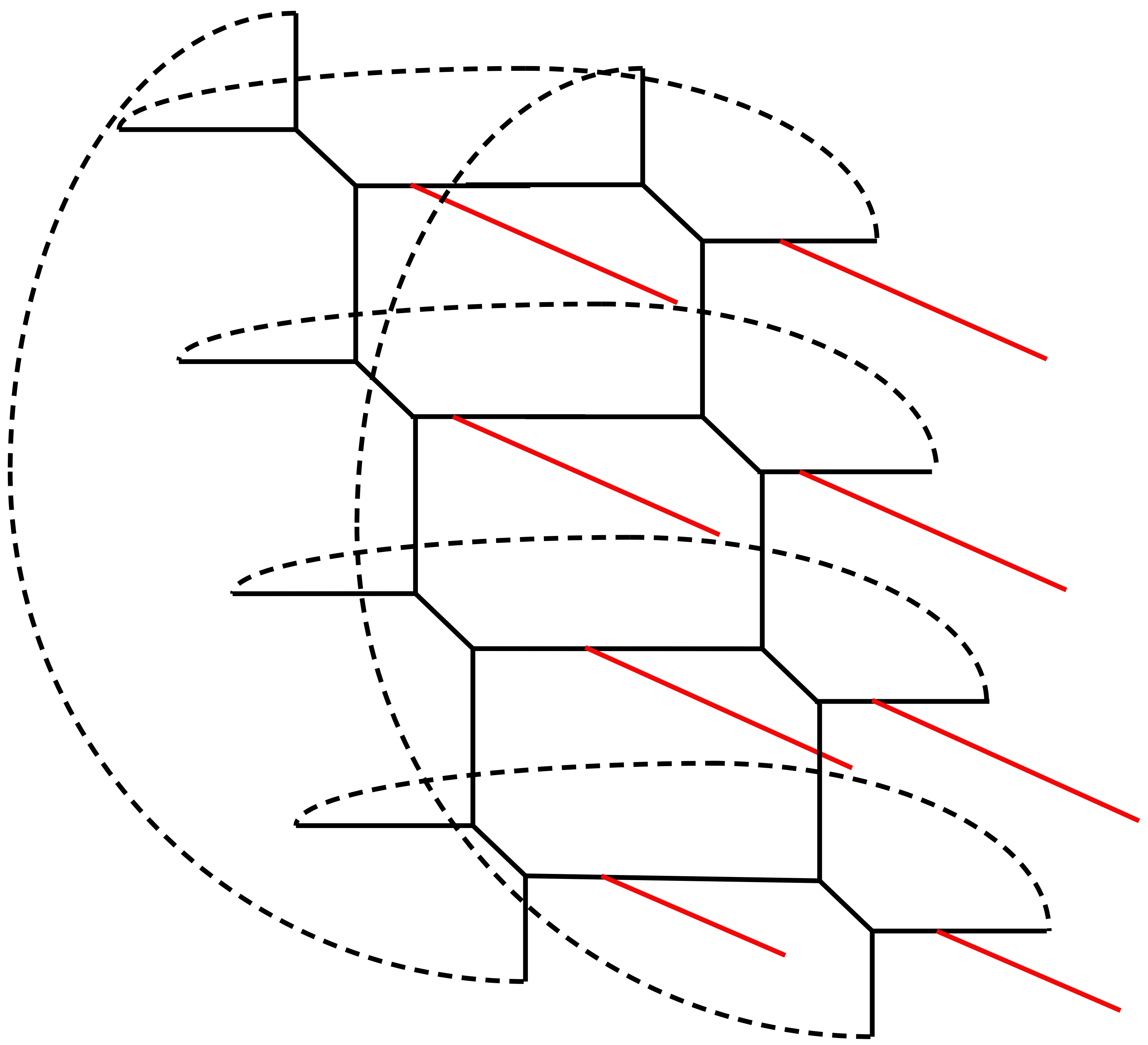
When $m \to 0$ `small' tori decouple

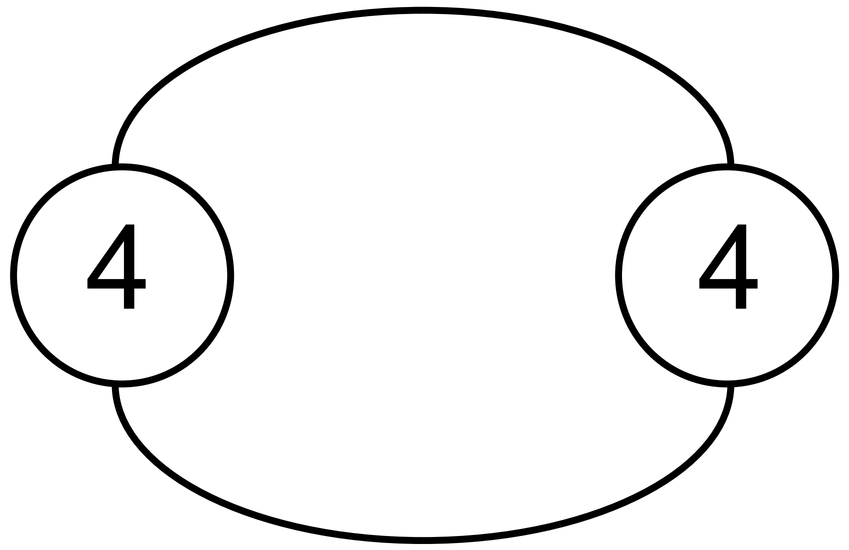
Toric Diagrams

DELL



Spin DELL





$$\sum_{a=1}^{N} \sum_{\mathbf{n}_a \in \mathbb{Z}^N} (-z)^{\sum n_{a,i}} \; w^{\sum \frac{n_{a,i}(n_{a,i}-1)}{2}} \; \prod_{a=1}^{N} \prod_{i<j} \theta\left(t^{n_{a,i}-n_{a,j}} \frac{x_{a,i}}{x_{a,j}} \Big| p \right) \; p_{a,1}^{n_{a,1}} \dots p_{a,N}^{n_{a,N}}$$

Spectrum

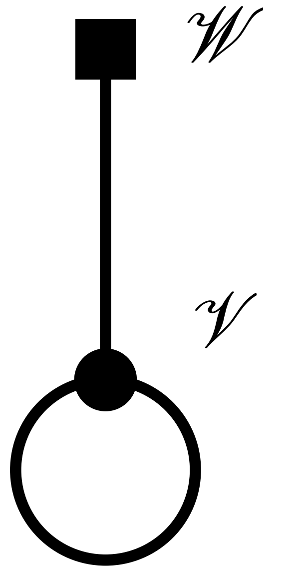
Eigenfunctions are computed as equivariant Euler characteristic of the instanton moduli spaces (ADHM)

Torsion free sheaves on $\mathbb{C}^2$ with framing at infinity

$$\chi_1 = -(1-q_1)(1-q_2)\mathscr{V}\mathscr{V}^* + q_1 q_2 \mathscr{W}\mathscr{V}^* + \mathscr{W}^*\mathscr{V}$$

Maximal torus $\quad T = \mathbb{C}^\times_{q_1} \times \mathbb{C}^\times_{q_2} \times \mathbb{T}(GL(k;\mathbb{C}))$

$$\chi_2 = (1 - Q_m q_1^{-1} q_2^{-1})\chi_1$$



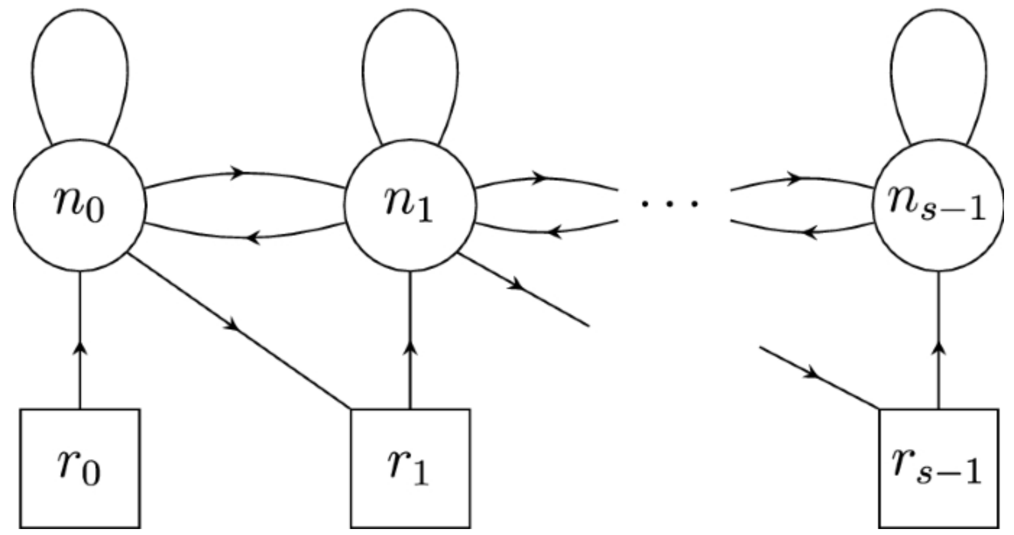
Compute T-equivariant character on $T^*\mathscr{M}_{k,n}$ in the presence of ramification

$$\oint A^a = 2\pi m_a, \qquad a = 1, \dots, N$$

$$m^a = (\underbrace{m_1, \dots, m_1}_{n_1}, \underbrace{m_2, \dots, m_2}_{n_2}, \dots, \underbrace{m_s, \dots, m_s}_{n_s})$$

Instanton moduli space decomposes

$$\mathcal{M}_{k,N} \to \bigoplus_{k_1,\dots,k_s} \mathcal{M}_{k_1,\dots,k_s,N}$$



$$\mathcal{Z}^{6d/4d}_{\text{inst}} = \sum_{\boldsymbol{\lambda}} x_1^{k_1(\boldsymbol{\lambda})} \dots x_s^{k_s(\boldsymbol{\lambda})} \mathcal{Z}_{\rho,\boldsymbol{\lambda}}$$

$$\mathcal{Z}_{\rho,\boldsymbol{\lambda}} = \prod_\alpha (\theta(\omega_\alpha | w))^{-n_\alpha}$$

Spectrum cont'd

Turn off one of the equivariant parameters $$\mathscr{Z}(\mathbf{x}, \mathbf{a}, q_1, Q_m, p, w) = \lim_{\epsilon_2 \to 0} \left(\frac{\mathcal{Z}^{6d/4d}_{\text{inst}}}{\mathcal{Z}} \right)$$ Then $$\widehat{\mathcal{O}}(z)\mathscr{Z} = \lambda(z)\ \widehat{\mathcal{O}}_0\mathscr{Z}$$

Macdonald function	Gauge theory	Relation
index $\mathbf{j} = (j_1, \ldots, j_n)$	Coulomb parameters $\mathbf{a}$	$a_k = q^{-2j_k} t^{2N-2k}$
argument $\mathbf{x} = (x_1, \ldots, x_n)$	Defect Kähler classes $\mathbf{x}$	–
parameter q	Equivariant parameter q_1	$q_1 = q^{-2}$
parameter t	Mass of the adjoint Q_m	$Q_m = t^2 q^{-2}$
parameter p	Instanton parameter p	–
parameter w	6d modular parameter w	–

Elliptic Macdonald Functions

As with q-hypergeometric series a truncation is expected for DELL, only now the truncated functions are no longer polynomials

$$\mathcal{P}_{\mathbf{j}}(\mathbf{x}) = \mathscr{Z}(\mathbf{x}, \mathbf{j}, q, t, p, w) \qquad a_k = q^{-2j_k} t^{2N-2k}$$

$$\widehat{\mathcal{O}}(z)\mathcal{P}_{\mathbf{j}}(\mathbf{x}) = \lambda_{\mathbf{j}}(z)\ \widehat{\mathcal{O}}_0\mathcal{P}_{\mathbf{j}}(\mathbf{x})$$

Can rewrite as

$$\left(\frac{d}{dz}\ \lambda_{\mathbf{j}}^{-1}(z)\ \widehat{\mathcal{H}}(z) \right)\ \mathcal{P}_{\mathbf{j}}(\mathbf{x}) = 0$$

N=2 example $\ (a_1, a_2) = \left(qt^{-1/2}, t^{-1/2}\right)$

$$\mathscr{Z}(\mathbf{x}) = (x_1 + x_2)\left[1 + p\frac{q(t-1)\left(qt^2-1\right)\left((x_1+x_2)^2\left(q^2t-1\right) - (q-1)(2qt+q+t+2)\right)}{x_1x_2(q-t)\left(q^2t-1\right)^2}\right.$$
$$\left.\cdot\left(1 + w\frac{(q-t)^2(qt-1)(qt+1)^2}{(q-1)qt^3} + O(w^2)\right) + \dots\right],$$

$$\lambda_1 = -t^{-1/2} - t^{1/2}q - p\frac{(t-1)(qt+1)(q-t)^2}{t^{3/2}(q^2t-1)} + w\frac{(qt+1)(q^2t^2+1)^2}{t^{3/2}q}$$

$$+ pw\frac{(t-1)(qt+1)(t^4q^4 - t^3q^3 + q^2t^3 + 2q^3t + 2q^2t^2 + 2t^3q + q^2t + qt - t + 1)(q-t)^2}{t^{7/2}q^2(q^2t-1)} + \dots$$

Elliptic Macdonald Polynomials

Take limit $p \to 0$

$$\mathscr{Z}(\mathbf{x}, j_{\mathbf{p}}) = \sum_{d_{i,j} \in C_{\mathbf{p}}} \prod_{i=1}^{n-1} \left(\frac{x_i}{x_{i+1}}\right)^{d_i} \prod_{j,k=1}^{i} \frac{\theta\left(q\frac{x_{i,j}}{x_{i,k}}\right)_{d_{i,j}-d_{i,k}}}{\theta\left(t\frac{x_{i,j}}{x_{i,k}}\right)_{d_{i,j}-d_{i,k}}} \cdot \prod_{j=1}^{i} \prod_{k=1}^{i+1} \frac{\theta\left(t\frac{x_{i+1,k}}{x_{i,j}}\right)_{d_{i,j}-d_{i+1,k}}}{\theta\left(q\frac{x_{i+1,k}}{x_{i,j}}\right)_{d_{i,j}-d_{i+1,k}}}$$

Elliptic hypergeometric series with

$$\theta(x|w)_d = \prod_{l=0}^{d-1} \theta(q^l x|w) = \frac{\theta(q^d x|w)}{\theta(x|w)}$$

They describe equivariant elliptic cohomology of $T^*\mathbb{F}l_N$

Truncate as before

$$\mathcal{P}_j(x) := \mathcal{P}_j(x; q, t, p) = \sum_{\ell=0}^{j} x^{j-2\ell} \prod_{i=0}^{\ell-1} \frac{[j-i]}{[j-i+\beta-1]} \frac{[i+\beta]}{[i+1]}$$

where

$$t = q^\beta \qquad [n] = \frac{\theta(q^n|p)}{\theta(q|p)}$$

Orthogonality

$$\frac{1}{2!} \oint_{|x|=1} \frac{dx}{x} \; \mathcal{P}_j(x) \; \mathcal{P}^\star_{j'}(x) = \delta_{j,j'}$$

$$\mathcal{P}^\star_j(x; q, t, 0) = \frac{\Gamma_q(x^2)\Gamma_q(x^{-2})}{\Gamma_q(tx^2)\Gamma_q(tx^{-2})} \; \frac{P_j(x)}{||P_j||^2}$$

What’s Next

- Inosemtsev limit from DELL
- Elliptic opers (quantum/classical duality)
- Large number of particles — double elliptic hydrodynamics?
- Proof of conjectures
- Add you own!