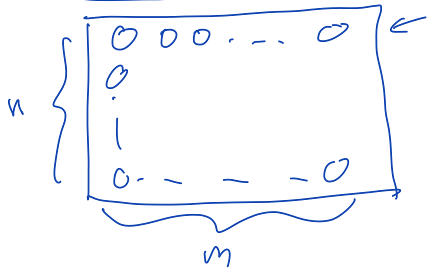
This week: Counting, Combinatorics, Probability



$n \cdot m$ $\qquad \underbrace{n+n+\dots+n}_{m \text{ times}} = n \cdot m$

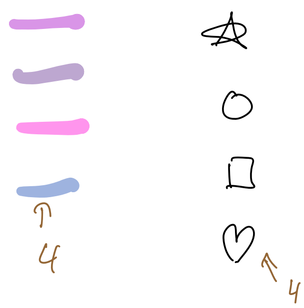
Product rule: Suppose a counting task can be broken into a sequence of counts. There are n_1 ways to complete the first task, n_2 ways to complete the second, so in total there are $n_1 \cdot n_2$ ways.

Ex: 26 letters, followed by integer ≤ 100.
How many items can be labelled this way

$\underset{26}{X}\ \underset{100 \text{ ways}}{100}$

Answer: $26 \times 100 = 2600$

Ex:



$4 \times 4 = 16$ different pictures.

Generalization: If task can be broken into k tasks with $n_1, n_2, \dots, n_k$ ways to complete each of them then the total number of ways is $n_1 \cdot n_2 \cdot \dots \cdot n_k$

Ex: License plates

NLLL NNN $\quad$ + 10 backgrounds

AWP

XIR

$26^3 \cdot 10^4 \cdot 10$

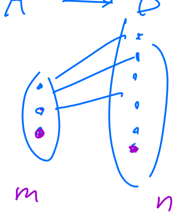
Ex: (later)

$A = \{a_1, \dots, a_m\}$

$B = \{b_1, \dots, b_n\}$

$f : A \to B$ $\qquad$ one-to-one

$m \leq n$



How many different 1-1 functions from A to B?

$n \cdot (n-1) \cdot \dots \cdot (n-m+1)$

Ex: S - finite set $\mathcal{P}(S)$ - power set $|\mathcal{P}(S)| = 2^{|S|}$

Binary representation

$S = \{a_1, a_2 \ldots a_n\}$ $S \leftrightarrow 111\ldots - 1$

$\cup$

$U = \{a_1, a_2\}$ $U \leftrightarrow 1100\ldots0$

In total there are 2^n bit strings for $n = |S|$

$V = \{a_5, a_7, a_n\}$ $V \leftrightarrow 0000101\,0\ldots01$

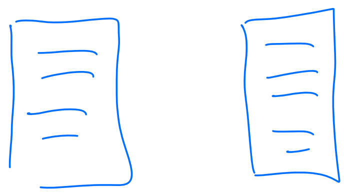
$$\left(\begin{array}{lcl} 1010 & \leftrightarrow & 2^3 + 2^1 = 10 \\ 1111 & \leftrightarrow & 2^3 + 2^2 + 2^1 + 2^0 = 13 \end{array} \right)$$

Product of sets $A_1 \times A_2 \times \ldots \times A_n$

$$|A_1 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$$

Sum rule:

Ex:



50 projects Number Th 25 project probab

$50 + 25 = 75$ projects

If a counting task can be completed either by n_1 ways or by n_2 ways and none of them are the same, then there are $n_1 + n_2$ ways to complete the task

Generalization to $n_1, n_2, \ldots n_k$ tasks which are independent yields $n_1 + n_2 + \ldots + n_k$ ways to complete the task.

$$|A_1 \cup A_2 \cup \ldots \cup A_k| = |A_1| + \ldots + |A_k|$$

$$A_i \cap A_j = \emptyset \quad i \neq j$$

Ex: Password: 6 – 8 characters, both numbers, letters (lower case)

How many pass words?

$P = P_6 + P_7 + P_8$

w/ 6 char

26 letters
10 numbers } 36

$P_6 = 36^6 - 26^6 - 10^6$

only letters

only numbers

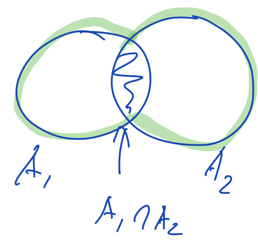
P A S S W O
$\sqcup\ \sqcup\ \sqcup\ \sqcup\ \sqcup\ \sqcup$

(inclusion – exclusion)

$P_1 \ D_2 \ 4 \ 5 \ 6 \quad P_7, P_8 \ldots$

Subtraction rule: If task 1 can be completed in n_1 ways, task 2 in n_2 ways then the total number of completing the task is $n_1 + n_2$ minus the number of common ways.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$



Ex: How many bit strings with 8 digits which begin either with 1 or end with 00?

1 _ _ _ _ _ _ 0 0

A_1: $1\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}_{2^7}$ $\qquad 2^7$

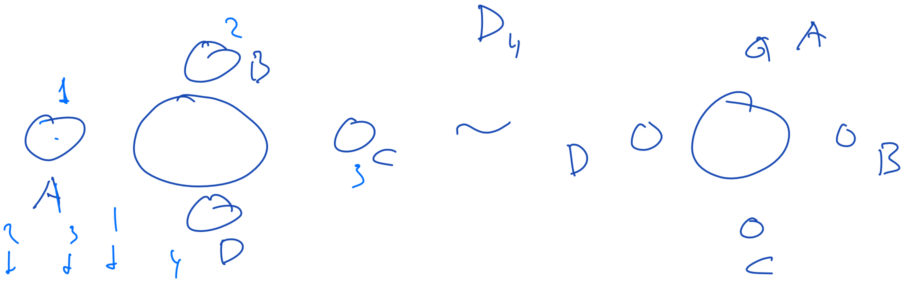
A_2: $\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{2^6}\ 0\ 0$ $\qquad + \ 2^6$

$A_1 \cap A_2$: $1\ \underbrace{\_\ \_\ \_\ \_\ \_}_{2^5}\ 0\ 0$ $\qquad - \ 2^5$

$2^7 + 2^6 - 2^5 = 128 + 64 - 32$
$= 160$

The Division rule:

sit 4 people on 4 chairs



$4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$

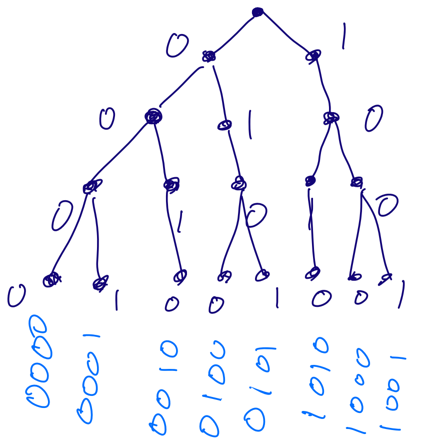
$\# = \frac{24}{4} = 6$ due to cyclic symmetry.

There are $\frac{n}{d}$ ways to complete the task with n ways such that for every way there are d equivalent ways.

Tree diagrams Ex: Find a bit string without consecutive 1s

$\to 1010010$

start

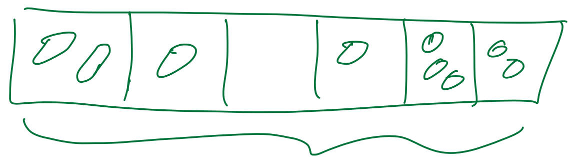


0 1

00 01 10

000 001 010 101 100

The pigeonhole principle



k boxes

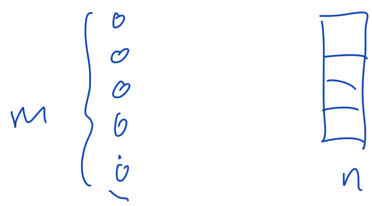
$k+1$ or more pigeons

If $k \in \mathbb{Z}_+$, there are $k+1$ objects placed in k boxes then at least one box contains two or more objects

Proof: Assume false – each box has at most one element. Then the total number of elements will be at most k, which is a contradiction.

$f : A \to B$ $m > n$ cannot be 1-1 by pigeonhole principle

$m \quad n$



Generalized pigeonhole principle: If N objects are placed into k boxes then there is at least one that contains $\lceil \frac{N}{k} \rceil$

Proof: Assume false. The total number of objects is less than

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N$$

which is a contradiction (we have N objects)

Ex: $\underbrace{NXX}_{\text{area code}} - NXX - XXXX$ $X: 0, \dots, 9$

$N: 1, \dots, 9$

$25 \cdot 10^6$ phones. How many area codes do we need?

$N\ X\ X - XXXX$ $\quad 9 \cdot 10^6$

$\uparrow \quad \uparrow$

$9 \quad 10$

Need at least $\left\lceil \frac{25}{9} \right\rceil = 3$ area codes

$300 \cdot 10^6$

Ex: Show that among $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers

$n=3$: $\quad 2, 3, 4, 5$

$n=4$: $\quad 3, 7, 6, 4, 5$

Let $a_1, \ldots, a_{n+1}$ - integers $\le 2n$ $\quad a_j = 2^{k_j} \cdot q_j$, $\quad k_j \geq 0$, integers, $q_j \in \mathbb{Z}$

$a_j = 5 \quad k_j = 0$

$q_j = 5$

$n=3$, $2n = 6$

$1, 3, 5$

q_j - odd integers, $\le 2n$

$\{q_1, \ldots, q_{n+1}\}$

But there are only n odd integers $\le 2n$, so by the pigeon hole principle at least two q's are the same: $q_i = q_j = q$

then $a_i = 2^{k_i} \cdot q$

$a_j = 2^{k_j} \cdot q$

so if $k_i \le k_j \Rightarrow a_i \mid a_j$

if $k_i \geq k_j \Rightarrow a_j \mid a_i$

Combinatorics (6.3 - 6.6)

Pascal triangle

$$(x+y)^n = \underbrace{(x+y)(x+y) \ldots (x+y)}_{n \text{ times}} = \sum_{k=0}^{n} \underbrace{c_k}_{\binom{n}{k}} x^{n-k} \cdot y^k$$

$$(x+y)^2 = (x+y)(x+y) = 1 \cdot x^2 + 2xy + 1 \cdot y^2$$

$$(x+y)^3 = (x+y)(x+y)(x+y) = 1 \cdot x^3 + 3x^2y + 3xy^2 + 1 \cdot y^3$$

$\boxed{O_1 \quad O_2 \quad O_3}$

In how many ways we can draw 2 balls out of 3?

$O_1 \ O_2$

$O_1 \ O_3$

$\Big\}$ 3 choices

O_2, O_3)

$$(x+y)^n = \underbrace{(x+y)(x+y)(x+y)\cdots(x+y)}_{n} \leftarrow x^{n-k}\cdot y^k$$

draw k y's and $(n-k)$ x's

draw the remaining $(n-k)$ x's

n choose k

$$\binom{n}{k} = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}{k!}$$

↑ permutation factor

$$\prod_{i=1}^{n}(x_i+y_i) = (x_1+y_1)(x_2+y_2)(x_3+y_3)\cdots(x_n+y_n)$$

Lemma: $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

Proof: substitution

- $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
- $\binom{n}{1} = n$
- $\binom{n}{2} = \dfrac{n(n-1)}{2}$

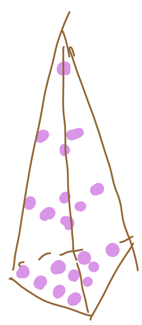
$\binom{7}{4} + \binom{7}{5} = \binom{8}{5}$

integers

triangular #

$\binom{n}{0}$ $\binom{n}{n}$

T_1
T_2
T_3
T_4
$\|$
t_4



tetrahedra numbers

$t_n = \binom{n}{3}$

$n \rightarrow$

0 1

0 1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

n

$\binom{7}{3}$

Distinguishable objects: Permutations

21 students Math 55

4 seating on the front row. Q: In how many ways we can do that?

4-permutation

$21 \cdot 20 \cdot 19 \cdot 18$ – number of permutations of 4 people out of 21.

Def: A permutation is a set of distinct objects in ordered pattern

r-permutation is a permutation of r objects.

$21 \cdot 20 \cdot \ldots \ldots 2 \cdot 1 = 21!$

$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \sim 8 \cdot 8^{21} \sim 8^{22}$ ← Stirling approximation

$\sim e^{n(\log n - \frac{1}{2})}$

Ex: $\{a\, b\, c\}$

$$\left.\begin{matrix} a\ b\ c \\ c\ a\ b \\ b\ c\ a \end{matrix}\right\} \text{cyclic}$$

$b\ a\ c$

$a\ c\ b$

$c\ b\ a$

$a - b$, c

$P(n,r)$ – r-permutation of n objects

$$P(n,r) = n \cdot (n-1) \cdot \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

$$0! = 1 \qquad = \frac{n \cdot (n-1)(n-2) \ldots (n-r+1)\cancel{(n-r)(n-r-1) \ldots 1}}{\cancel{(n-r)(n-r-1) \ldots 1}}$$

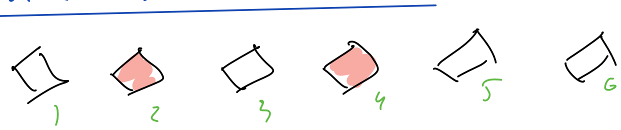
Ex: $\overset{X}{(ABC)}DEFGH$

Q1 How many permutations contain string (ABC)?

XDEFGH $\qquad$ 6! – # permutations of six elements

$6! = 720$

Indistinguishable objects: Combinations



6 beads

Color 2 out of 6

$$\frac{6 \times 5}{2!} = 15 \qquad \text{2 out of 6} \qquad \binom{6}{2} = \binom{6}{4}$$

$$\frac{6 \times 5 \times 4}{3!} \qquad \text{3 out of 6}$$

Def: r-combination is a set of unordered r elements

$\binom{n}{r}$ $\left(C_n^r,\ C(n,r)\right)$ – # of r-combinations out of n elements

'n choose r'

Theorem: $\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$

$P(n,r) = r!\binom{n}{r}$

- $\binom{n}{r} = \binom{n}{n-r}$ $r \mapsto n-r$ $\binom{n}{r} \mapsto \frac{n!}{(n-r)!\underbrace{(n-n+r)!}_{r!}}$

Ex: Bit string of length n. How many strings contain exactly r 1's?

A: $\binom{n}{r}$

Binomial theorem:

$n \in \mathbb{Z}$, $x, y \in \mathbb{R}$ $\quad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

$\underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}} = \underbrace{\binom{n}{0}}_{1} x^n + \underbrace{\binom{n}{1}}_{n} x^{n-1}y + \underbrace{\binom{n}{2}}_{n(n-1)} x^{n-2}y^2 + \cdots + \underbrace{\binom{n}{n-1}}_{n} x y^{n-1} + \underbrace{\binom{n}{n}}_{1} y^n$

- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Proof: plug in $x=y=1$

- $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ $(1-1)^n = \sum_{k=0}^{n} \binom{n}{k}(-1)^k$

- $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$ $x=1, y=2$

Pascal's triangle identity:

$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

Proof: RHS: # ways to draw k out of $n+1$

S - set of $n+1$ elements $S' = S - \{a\}$

$|S| = n+1$ $|S'| = n$

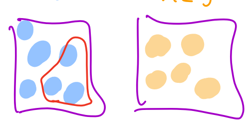
$\binom{n}{k-1}$ – # ways to draw $k-1$ elements from S' (a is picked)

$\binom{n}{k}$ – # ways to draw k elements from S' (a is not picked)

Vandermonde identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\cdot\binom{n}{k}$$

$m=6$ $n=5$



A B

$r=3$

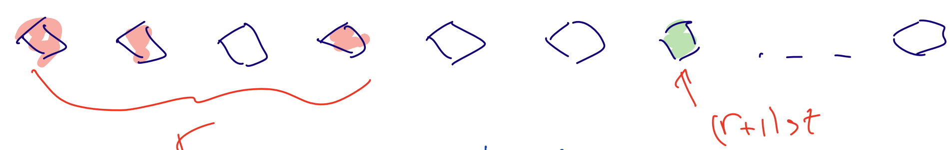
$$= \binom{m}{r}\binom{n}{0} + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r-2}\binom{n}{2} + \dots + \binom{m}{0}\binom{n}{r}$$

• $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$ plug $m=n$, $r=n$

RHS: $\sum_{k=0}^{r} \underset{\binom{n}{k}}{\underset{\parallel}{\binom{n}{n-k}}}\binom{n}{k}$

• $\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r} = \binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \dots + \binom{n}{r}$

$n+1$



r $(r+1)$st

think of how to interprete the RHS via arranging $(r+1)$st element.

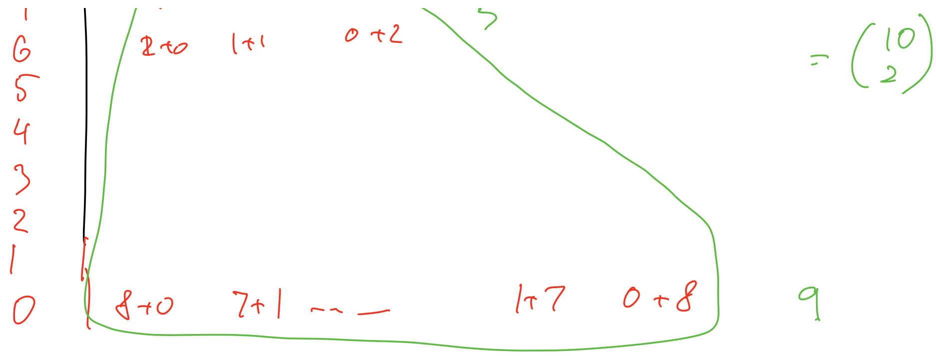
Problem: 8 apples for 3 students

In how many ways can you distribute 8 apples among 3 students?

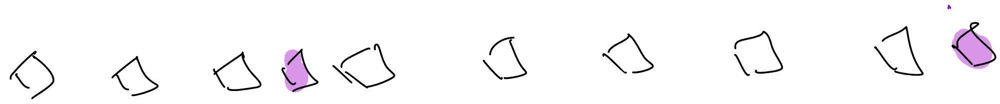
(okay if someone does not get an apple)

Ann	Bob + Claire	
8	0+0	1
7	1+0 0+1	2

$$\sum_{i=1}^{9} i = 45 = \frac{10\cdot(10-1)}{2}$$



$= \binom{10}{2}$



10 kids
2 out of 10

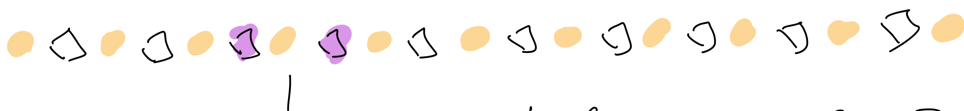
A B C

$\binom{8+2}{2} = \binom{10}{2}$

- <u>11 oranges among 3 kids</u>

How many ways to give 11 oranges to 3 kids such that each kid should have at least one?

$A = 45$ (see above), give each kid an orange, there are 8 left.



10 beads, color 2 $\Rightarrow \binom{10}{2}$

$\overset{2}{\text{M}_1\text{M}_2}$, $\overset{2}{\text{T}_1\text{T}_2}$, $\overset{2}{\text{E}_1\text{E}_2}$

$\text{C O M}_1\text{M}_2\text{ I T}_1\text{T}_2\text{ E}_1\text{E}_2$

In how many ways can you rearrange the letters in this word?

$$\frac{9!}{\underset{M}{2!}\cdot\underset{T}{2!}\cdot\underset{E}{2!}}$$

<u>Theorem:</u> If a set of n objects contains subsets of $n_1, n_2 \dots n_k$ indistinguishable objects then the total number of orderings is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

$n_1!\, n_2! \cdots n_k!$

$$\binom{n}{k} = \frac{n!}{\underbrace{k!}_{n_1}\,\underbrace{(n-k)!}_{n_2}}$$