# Dell systems and the Seiberg-Witten prepotentials

Workshop on Elliptic Integrable Systems

March 8, 2021

# Plan

1. Classical p–q duality
2. Two-particle ($N = 2$) Dell system
3. $N > 2$ construction of Mironov, Morozov
4. Necessary conditions for Poisson commutativity
5. Theta-constant representation for SW curves ($N > 4$)
6. Non-linear equations for the elliptic SW prepotentials
7. Modular properties of Dell systems

# References


- S.N.M. Ruijsenaars, "Complete integrability of relativistic Calogero-Moser systems and elliptic function identities", Comm. Math. Phys. 1987.
- Braden, Marshakov, Mironov, and Morozov, "On double-elliptic integrable systems: 1. A duality argument for the case of SU(2)", Nucl. Phys. B 573, 2000.
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- Mironov and Morozov, "Commuting Hamiltonians from Seiberg-Witten theta functions", Phys. Lett. B 475 71, 2000.
- Aminov, Braden, Mironov, Morozov, and Zotov, 10.1007/JHEP01 033, 2015.
- Aminov, Mironov, Morozov, "Modular properties of 6d (DELL) systems", J. High Energ. Phys. 2017, 23 (2017).

# Classical p–q duality.

- First introduced by Ruijsenaars as an action–angle transformation for the Calogero-Moser systems and for their relativistic generalizations:

| | | |
|---|---|---|
| rational Calogero $\frac{1}{2}\sum_i p_i^2 + \sum_{i,j} \frac{m^2}{(q_i - q_j)^2}$ | $\longleftrightarrow$ | rational Calogero $\sum_i Q_i^2 + \sum_{i,j} \frac{m^2}{(P_i - P_j)^2}$ |
| trig. Calogero $\frac{1}{2}\sum_i p_i^2 + \sum_{i,j} \frac{m^2}{\sinh^2(q_i - q_j)}$ | $\longleftrightarrow$ | rational Ruijsenaars $\sum_i \exp(P_i) \prod_{j\neq i} \sqrt{1 - \frac{g^2}{(Q_i - Q_j)^2}}$ |
| trig. Ruijsenaars $\sum_i \exp(p_i) \prod_{j\neq i} \sqrt{1 - \frac{g^2}{\sinh^2(q_i - q_j)}}$ | $\longleftrightarrow$ | trig. Ruijsenaars $\sum_i \exp(Q_i) \prod_{j\neq i} \sqrt{1 - \frac{g^2}{\sinh^2(P_i - P_j)}}$ |

# $N = 2$ Dell system

- The elliptic p–q dual systems were constructed explicitely for the two-particle case by Braden, Marshakov, Mironov, Morozov

| elliptic Calogero | p–q transform. | dual system |
|---|---|---|
| $\frac{p^2}{2}+\frac{g^2}{\mathrm{sn}^2(q)}$ | $\longleftrightarrow$ | $\alpha_1(Q)\,\mathrm{cn}\left(\beta_1 P \middle\| k\frac{\alpha_1}{\beta_1}\right)$ |
| elliptic Ruijsenaars | | dual system |
| $\cosh(p)\sqrt{1-\frac{g^2}{\mathrm{sn}^2(q)}}$ | $\longleftrightarrow$ | $\alpha_2(Q)\,\mathrm{cn}\left(\beta_2 P \middle\| k\frac{\alpha_2}{\beta_2}\right)$ |
| Double-elliptic system | | Double–elliptic system |
| $\alpha_3(Q)\,\mathrm{cn}\left(\beta_3 P \middle\| k\frac{\alpha_3}{\beta_3}\right)$ | $\longleftrightarrow$ | $\alpha_3(P)\,\mathrm{cn}\left(\beta_3 Q \middle\| k\frac{\alpha_3}{\beta_3}\right)$ |

Where

$$\alpha_1(Q)=\sqrt{1-\frac{2g^2}{Q^2}},\quad \alpha_2(Q)=\sqrt{1-\frac{2g^2}{\sinh^2(Q)}},\quad \alpha_3\left(Q|\tilde{k}\right)=\sqrt{1-\frac{2g^2}{\mathrm{sn}^2\left(Q|\tilde{k}\right)}}.$$

# Basic facts about genus $g$ theta functions.

- The Riemann theta function with characteristic $\left[\begin{array}{c}\vec{\alpha}\\ \vec{\beta}\end{array}\right]$ is

$$\theta^{(g)}\left[\begin{array}{c}\vec{\alpha}\\ \vec{\beta}\end{array}\right](\mathbf{z}|\Omega)=\sum_{\mathbf{n}\in\mathbb{Z}^g}\exp\left(\pi\imath\,(\mathbf{n}+\vec{\alpha})^t\,\Omega\,(\mathbf{n}+\vec{\alpha})+2\pi\imath\,(\mathbf{n}+\vec{\alpha})\cdot(\mathbf{z}+\vec{\beta})\right),$$

where $z\in\mathbb{C}^g$ and $\Omega$ is a symmetric $g\times g$ matrix with positive definite imaginary part.

- Alternatively, an entire function $f(\mathbf{z})$ on $\mathbb{C}^g$ such that

$$f(\mathbf{z}+\mathbf{m})=\exp\left(2\pi\imath\,\vec{\alpha}\cdot\mathbf{m}\right)f(\mathbf{z}),$$

$$f(\mathbf{z}+\Omega\mathbf{m})=\exp\left(-2\pi\imath\,\vec{\beta}\cdot\mathbf{m}\right)\exp\left(-\pi\imath\,\mathbf{m}^t\,\Omega\,\mathbf{m}-2\pi\imath\,\mathbf{m}\cdot\mathbf{z}\right)f(\mathbf{z})$$

is the Riemann theta function with characteristic $\left[\begin{array}{c}\vec{\alpha}\\ \vec{\beta}\end{array}\right]$.

# $N > 2$ Construction of Mironov, Morozov

- In the center-of-mass frame one has $N-1$ Hamiltonians:

$$H_a(\mathbf{z}\,|\,\Omega) = \frac{\theta^{(g)}\begin{bmatrix} 0 & \dots & 0 \\ \frac{a}{N} & \dots & \frac{a}{N} \end{bmatrix}(\mathbf{z}\,|\,T)}{\theta^{(g)}(\mathbf{z}\,|\,T)}, \quad g = N, \quad a = 1, \dots, N-1.$$

- The hypothesis is that the Hamiltonians are Poisson commuting with respect to the Seiberg-Witten symplectic structure:

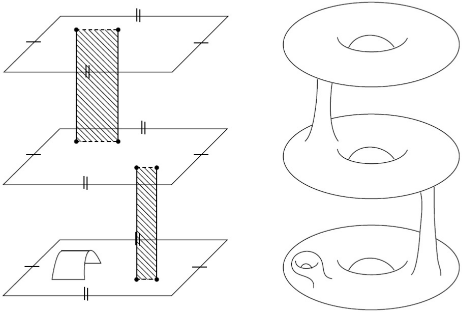
$$\omega^{SW} = \sum_{i=1}^{N} \mathrm{d}z_i \wedge \mathrm{d}a_i$$

and $T$ is given by the second derivative of the corresponding SW prepotential:

$$T_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}.$$

# Alternative construction by Braden and Hollowood

- Braden and Hollowood proposed a geometric setting for the Dell Hamiltonians:



$$\sum_{a=0}^{N-1} \Theta \begin{bmatrix} 0 & \frac{a}{N} \\ 0 & 0 \end{bmatrix} (z_1, z_2 \,|\, \Gamma)\, \theta_a = 0.$$

# Poisson commutativity for $N = 3, 4$.

- Following the initial idea one can introduce new Poisson brackets:

$$\{z_i, T_{jk}\} = P_{ijk}(T), \quad \{z_i, z_j\} = 0, \quad \{T_{ij}, T_{kl}\} = 0.$$

Now we can forget about the Seiberg–Witten prepotential and consider $T$ as an arbitrary period matrix of genus $N$ Riemann surface.

- Then the Poisson commutativity means:

$$\exists P_{ijk} \neq 0 : \quad \sum_{i=1}^{N-1} \sum_{j \leqslant k} P_{ijk} \left( \frac{\partial H_a}{\partial z_i} \frac{\partial H_b}{\partial T_{jk}} - \frac{\partial H_b}{\partial z_i} \frac{\partial H_a}{\partial T_{jk}} \right) = 0 \qquad (1)$$

- For $N = 3$ and $N = 4$ there exist a nontrivial set of such quantities $P_{ijk}$.
- Thus, the Poisson commutativity is related to some higher genus theta-function identities.

# Schottky problem.

- The dimension of a moduli space $M_g$ of compact Riemann surfaces with genus $g > 1$ is $3g - 3$.
- The dimension of a moduli space $A_g$ of the corresponding abelian varieties is $g(g+1)/2$.
- When $g > 3$ one needs to place restrictions on $g \times g$ period matrix $T$.

Schottky applied theta constants to define the Schottky locus in $A_g$:

$$16 \sum_{\vec{\alpha},\vec{\beta}\in\mathbb{Z}^g/2\mathbb{Z}^g} \left(\theta\left[\begin{array}{c}\vec{\alpha}/2\\ \vec{\beta}/2\end{array}\right](0|T)\right)^{16} - \sum_{\vec{\alpha},\vec{\beta}\in\mathbb{Z}^g/2\mathbb{Z}^g} \left(\theta\left[\begin{array}{c}\vec{\alpha}/2\\ \vec{\beta}/2\end{array}\right](0|T)\right)^{8} = 0.$$

# Theta-constant representation for the Seiberg-Witten curves.

- For $N > 4$ the involutivity condition places restrictions on the period matrix.

The restrictions (1) can be formulated as

$$\exists P_{ijk} \neq 0 : \quad \sum_{i,j,k=1}^{g} P_{ijk} \, C^{\vec{\alpha}}_{\{ijk\}}(T) = 0, \qquad (2)$$

where we used the fact, that $\{H_a, H_b\}$ is connected to the linear space of weight 3 theta functions.

- This provides us with an independent method for calculating the Seiberg-Witten prepotentials.
- The hypothesis is that the constraints (2) define the Seiberg-Witten locus in $A_g$.

# The linear space of weight $\lambda$ theta functions.

- An entire function $f(\mathbf{z})$ on $\mathbb{C}_g$ is called theta function of weight $\lambda$, if

$$f(\mathbf{z}+\mathbf{m}) = f(\mathbf{z}),$$
$$f(\mathbf{z}+T\mathbf{m}) = \exp\left(-\pi\imath\lambda\mathbf{m}T\mathbf{m} - 2\pi\imath\lambda\mathbf{z}\cdot\mathbf{m}\right) f(\mathbf{z})$$

for all $\mathbf{m} \in \mathbb{Z}^g$.

- Such functions form a linear space $R_\lambda^T$ of dimension $\lambda^g$.

It turns out, that

$$\{H_a, H_b\} = \frac{1}{\theta^3} \sum_{i\leqslant j\leqslant k} P_{ijk} H^{ab}_{\{ijk\}},$$

$$H^{ab}_{\{ijk\}} = H^{ab}_{ijk} + H^{ab}_{jki} + H^{ab}_{kij} \in R_3^T.$$

- We decompose $H^{ab}_{\{ijk\}}$ in the basis of $R_3^T$ and obtain vectors $C^{\vec{\alpha}}_{\{ijk\}}$.

# Explicit form of the vectors $C_{ijk}^{\vec{\alpha}}$

- Using the notation $\theta'_i = \partial_{z_i}\theta$ we get

$$C_{ijk}^{\vec{\alpha}} = \sum_{\vec{\beta}\in\mathbb{Z}^g/2\mathbb{Z}^g} \left( 9\theta'_i \begin{bmatrix} \frac{\vec{\beta}-\vec{\alpha}}{2} \\ a/N \dots a/N \end{bmatrix} (0\,|\,2T) \right.$$

$$\times \theta''_{jk} \begin{bmatrix} \frac{\vec{\beta}}{2} - \frac{\vec{\alpha}}{6} \\ (2b-a)/N \dots (2b-a)/N \end{bmatrix} (0\,|\,6T)$$

$$-\theta'''_{ijk} \begin{bmatrix} \frac{\vec{\beta}-\vec{\alpha}}{2} \\ a/N \dots a/N \end{bmatrix} (0\,|\,2T)$$

$$\left. \times\, \theta \begin{bmatrix} \frac{\vec{\beta}}{2} - \frac{\vec{\alpha}}{6} \\ (2b-a)/N \dots (2b-a)/N \end{bmatrix} (0\,|\,6T) \right).$$

# Constraints for the $N = 5$ case

- We assume the instanton expansion

$$T_{ij} = \tau\delta_{ij} - \frac{1}{\pi\imath}\ln F_{ij}^{(pert)} + \sum_{k\in\mathbb{N}} q^k \frac{\partial^2 F^{(k)}}{\partial\hat{a}_i\partial\hat{a}_j}, \quad q \equiv \mathrm{e}^{2\pi\imath\tau}.$$

- Using the perturbative expansions in the trigonometric limit:

$$Q_{ij} \equiv \exp\left(2\pi\imath\, T_{ij}\right) = q_{ij} + \sum_{k\in\mathbb{N}} q^k\, q_{ij}^{(k)},$$

$$P_{ijk} = p_{ijk} + \sum_{l\in\mathbb{N}} q^l\, p_{ijk}^{(l)},$$

- we arrive at the following linear system in the leading order:

$$L\vec{p} = 0 \quad \Rightarrow \quad \det L = 0$$

- Matrix $L$ can be written explicitely:

$$
L=\left|\begin{array}{cccccccccc}
q_{13}-q_{23} & 0 & q_{13}-q_{23} & q_{13}-q_{23} & q_{12}-q_{23} & q_{12}-q_{23} & q_{12}-q_{23} & 0 & 0 & 0\\
q_{24}-q_{14} & q_{24}-q_{14} & 0 & q_{24}-q_{14} & 0 & q_{12}-q_{24} & 0 & q_{24}-q_{12} & q_{24}-q_{12} & 0\\
q_{25}-q_{15} & q_{25}-q_{15} & q_{25}-q_{15} & 0 & 0 & 0 & q_{12}-q_{25} & 0 & q_{12}-q_{25} & q_{25}-q_{12}\\
0 & 0 & 0 & 0 & 0 & q_{35}-q_{45} & q_{34}-q_{45} & 0 & q_{34}-q_{35} & 0\\
0 & q_{14}-q_{34} & q_{13}-q_{34} & 0 & q_{34}-q_{14} & 0 & q_{34}-q_{14} & q_{34}-q_{13} & q_{34}-q_{13} & 0\\
0 & q_{15}-q_{35} & 0 & q_{13}-q_{35} & q_{35}-q_{15} & q_{35}-q_{15} & 0 & 0 & q_{13}-q_{35} & q_{35}-q_{13}\\
0 & 0 & q_{25}-q_{45} & q_{24}-q_{45} & 0 & 0 & 0 & 0 & q_{24}-q_{25} & 0\\
0 & q_{25}-q_{35} & 0 & q_{23}-q_{35} & 0 & 0 & q_{23}-q_{25} & 0 & 0 & 0\\
0 & q_{24}-q_{34} & q_{23}-q_{34} & 0 & 0 & q_{23}-q_{24} & 0 & 0 & 0 & 0\\
0 & 0 & q_{15}-q_{45} & q_{14}-q_{45} & 0 & q_{15}-q_{45} & q_{14}-q_{45} & q_{45}-q_{15} & 0 & q_{45}-q_{14}
\end{array}\right|.
$$

- The most general solution to the above constraint is

$$
q_{ij}=\left(1-\frac{m^2}{\mathrm{sn}\left(a_i-a_j\,|\,\hat{\tau}\right)^2}\right)^{-1}.
$$

# Non-linear equations for elliptic prepotentials

- The same constraints provide WDVV-like equations

$$\forall \vec{\alpha} \in \mathbb{Z}^g / 3\mathbb{Z}^g : \quad \sum_{i,j,k=1}^{N} \frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k} C_{ijk}^{\vec{\alpha}} = 0.$$

- With the help of the $q$-expansions

$$T_{ij} = \tau\, \delta_{ij} + \frac{\partial^2 \mathcal{F}^{\mathrm{pert}}}{\partial a_i \partial a_j} + \sum_{k \in \mathbb{N}} q^k \frac{\partial^2 \mathcal{F}^{(k)}}{\partial a_i \partial a_j},$$

$$\frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k} = \frac{\partial^3 \mathcal{F}^{\mathrm{pert}}}{\partial a_i \partial a_j \partial a_k} + \sum_{k \in \mathbb{N}} q^k \frac{\partial^3 \mathcal{F}^{(k)}}{\partial a_i \partial a_j \partial a_k},$$

one can compute both perturbative and instanton part of the prepotential for $N > 2$.

# $N = 3$ example

- In the leading order there are 5 different equations:

$$\vec{\alpha} = (0,0,0): \quad \sum_{i=1}^{3} \mathrm{e}^{2\pi\imath\, \partial_i^2 \mathcal{F}^{\mathrm{pert}}} \partial_i^3 \mathcal{F}^{\mathrm{pert}} = 0, \qquad (3)$$

$$\vec{\alpha} = (0,1,2): \quad \frac{\partial^3 \mathcal{F}^{\mathrm{pert}}}{\partial a_2^2 \partial a_3} + \frac{\partial^3 \mathcal{F}^{\mathrm{pert}}}{\partial a_2 \partial a_3^2} = 0, \qquad (4)$$

$$\vec{\alpha} = (1,0,2): \quad \frac{\partial^3 \mathcal{F}^{\mathrm{pert}}}{\partial a_1^2 \partial a_3} + \frac{\partial^3 \mathcal{F}^{\mathrm{pert}}}{\partial a_1 \partial a_3^2} = 0, \qquad (5)$$

$$\vec{\alpha} = (1,2,0): \quad \frac{\partial^3 \mathcal{F}^{\mathrm{pert}}}{\partial a_1^2 \partial a_2} + \frac{\partial^3 \mathcal{F}^{\mathrm{pert}}}{\partial a_1 \partial a_2^2} = 0, \qquad (6)$$

$$\vec{\alpha} = (1,1,1): \quad \frac{\partial^3 \mathcal{F}^{\mathrm{pert}}}{\partial a_1 \partial a_2 \partial a_3} = 0. \qquad (7)$$

# Double-elliptic SW prepotential

- The expression for the $N \geq 2$ Dell prepotential can be written as

$$\mathcal{F}^{\mathrm{Dell}} = \frac{\tau}{2}\sum_{i=1}^{N} a_i^2 + \frac{\epsilon^2}{2\pi\imath\,\beta^2}\sum_{i<j}\log\,\theta_{11}\left(\pi^{-1}\beta\,a_{ij}\middle|\,\hat{\tau}\right)$$
$$-\frac{\epsilon^2}{\pi\imath\,\beta^2}\sum_{\substack{i_1,\,\ldots,\,i_n\,\in\,\mathbb{Z}_{\geq 0}\\ i_1+\cdots+i_n\neq 0}}\epsilon^{2\,i_1+\cdots+2\,i_n}\,\widehat{C}_{i_1,\ldots,i_n}(\epsilon,\tau,\hat{\tau})$$
$$\times\prod_{\substack{k\,=\,1\\ (\vec{\alpha}_k\,\in\,\Delta_+)}}^{n}\frac{1}{\sigma\left(\beta\,\vec{\alpha}_k\cdot\vec{a}\,\middle|\,\hat{\tau}\right)^{2\,i_k}},$$

where $n = N(N-1)/2$, and $\Delta_+$ is the set of all positive roots $\{\vec{e}_i - \vec{e}_j;\ i<j\}$ in the $A_{N-1}$ root system.

- The coefficients $\widehat{C}_{i_1,\dots,i_n}$ depend on both elliptic parameters only through the Eisenstein series:

$$E_2(\tau) = 1 - 24 \sum_{n\in\mathbb{N}} \frac{n\, q^n}{1-q^n}, \quad E_4(\tau) = 1 + 240 \sum_{n\in\mathbb{N}} \frac{n^3\, q^n}{1-q^n},$$

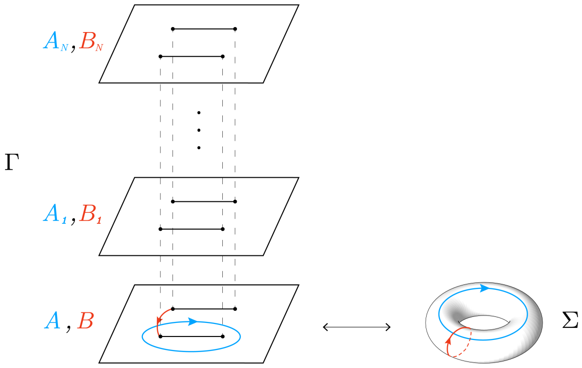
$$E_6(\tau) = 1 - 504 \sum_{n\in\mathbb{N}} \frac{n^5\, q^n}{1-q^n}.$$

- For example, $\widehat{C}_{i_1,\dots,i_n}$ can be decomposed in powers of $\epsilon$:

$$\widehat{C}_{i_1,\dots,i_n}(\epsilon,\tau,\hat{\tau}) = \sum_{k=0}^{+\infty} \epsilon^{2k} \sum_{\substack{m_1,\, m_2,\, m_3 \geq 0 \\ m_1 + 2m_2 + 3m_3 = k}} \hat{E}_2^{m_1}\, \hat{E}_4^{m_2}\, \hat{E}_6^{m_3}\, \widehat{C}_{i_1,\dots,i_n,k,(m)}(\tau),$$

- With $\widehat{C}_{i_1,\dots,i_n,k,(m)}(\tau)$ being quasimodular forms of weight $2\, i_1 + \cdots + 2\, i_n + 2k$.

# First modular anomaly equation




- We start with the modular transformations of $\tau$:

$$\tau \to \tau + 1 \quad \text{and} \quad \tau \to -1/\tau.$$

- The second transformation interchanges the cycles $A$ and $B$ on the base torus, which gives

$$a_i \xrightarrow{\tau\to-1/\tau} a_i^D, \quad a_i^D \xrightarrow{\tau\to-1/\tau} -a_i, \quad T \xrightarrow{\tau\to-1/\tau} -T^{-1}.$$

- Solving perturbatively in each order in $\epsilon$, we get

$$\tau \to -\frac{1}{\tau}, \quad \epsilon \to \frac{\epsilon}{\tau}, \quad \beta \to \frac{\beta}{\tau}, \quad \hat{\tau} \to \hat{\tau} - \frac{N\,\epsilon^2}{\pi^2\,\tau}, \quad a_i \to a_i^D, \quad T \to -T^{-1}.$$

- This leads to the first modular anomaly equation:

$$\frac{6}{\pi\imath}\frac{\partial \mathcal{F}^{\mathrm{Dell}}}{\partial E_2} - \frac{N\epsilon^2}{\pi^2}\frac{\partial \mathcal{F}^{\mathrm{Dell}}}{\partial \hat{\tau}} = -\frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial \mathcal{F}^{\mathrm{Dell}}}{\partial a_i} - \tau\, a_i\right)^2.$$

## Second modular anomaly equation

- The sum of the classical and perturbative parts of the period matrix is invariant under

$$\hat{\tau} \to \hat{\tau} + 1, \quad \epsilon \to \epsilon, \quad \beta \to \beta, \quad \tau \to \tau, \quad a_i \to a_i, \quad T_{ij} \to T_{ij}$$

and

$$\hat{\tau} \to -\frac{1}{\hat{\tau}}, \quad \epsilon \to \frac{\epsilon}{\hat{\tau}}, \quad \beta \to \frac{\beta}{\hat{\tau}}, \quad \tau \to \tau - \frac{N\,\epsilon^2}{\pi^2\,\hat{\tau}}, \quad a_i \to a_i, \quad T_{ij} \to T_{ij} - \frac{\epsilon^2}{\pi^2\,\hat{\tau}}.$$

- Thus, the second modular anomaly equation can be derived:

$$\frac{6}{\pi\imath}\frac{\partial \mathcal{F}^{\mathrm{Dell}}}{\partial \hat{E}_2} - \frac{N\,\epsilon^2}{\pi^2}\frac{\partial \mathcal{F}^{\mathrm{Dell}}}{\partial \tau} = -\frac{\epsilon^2}{2\,\pi^2}\left(\sum_{i=1}^{N} a_i\right)^2 .$$

# Modular properties

- The duality group is not just a product $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$.
- The modular transformation of one of the elliptic parameters is accompanied by the shift of the other.
- The four generators of this duality group are

$$1)\quad \tau \to \tau + 1, \quad \hat{\tau} \to \hat{\tau},$$

$$2)\quad \tau \to -1/\tau, \quad \hat{\tau} \to \hat{\tau} - \pi^{-2} N \epsilon^2 \tau^{-1},$$

$$3)\quad \hat{\tau} \to \hat{\tau} + 1, \quad \tau \to \tau,$$

$$4)\quad \hat{\tau} \to -1/\hat{\tau}, \quad \tau \to \tau - \pi^{-2} N \epsilon^2 \hat{\tau}^{-1}.$$

Thank you!