Characteristic determinant and Manakov triple for double elliptic integrable system

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Workshop Elliptic Integrable Systems

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The talk is based on joint recent paper with A. Grekov

A. Grekov, A. Zotov, *Characteristic determinant and Manakov triple for the double elliptic integrable system*, SciPost Physics, 10:3 (2021), 055; `arXiv:2010.08077`.

In the end we also discuss a part of
A. Grekov, A. Zotov, *On Cherednik and Nazarov-Sklyanin large N limit construction for double elliptic integrable system*, `arXiv:2102.06853`.

Plan of the talk:

- Integrable many-body systems and Ruijsenaars (action-angle) duality
- Determinant representation
- IRF-Vertex relation and factorized Lax operators
- Manakov triple
- Classification of L-matrices
- Some open problems
- Dell version of Dunkl-Cherednik operators

Calogero-Moser-Sutherland integrable many-body systems

The Hamiltonian

$$H^{\mathrm{CM}} = \sum_{i=1}^{N} \frac{p_i^2}{2} - \nu^2 \sum_{i>j}^{N} U(q_i - q_j)\,,$$

where $\nu \in \mathbb{C}$ – constant of interaction and
$U(q_i - q_j)$ – potential of pairwise interaction (on a complex plane):

$$\vartheta(z) := \begin{cases} z\,, \\ \sinh(z)\,, \\ \vartheta(z) \end{cases} \qquad U(z) = -\partial_z^2 \log \vartheta(z) = \begin{cases} 1/z^2\,, \\ 1/\sinh^2(z)\,, \\ \wp(z) + \mathrm{const} \end{cases}$$

Theta-function: (quasi-periodic function on the lattice $z \to z+1$, $z \to z+\tau$)

$$\vartheta(z) = \vartheta(z|\tau) = -i \sum_{k\in\mathbb{Z}} (-1)^k e^{\pi i (k+\frac{1}{2})^2 \tau} e^{\pi i (2k+1) z}\,, \quad \mathrm{Im}(\tau) > 0\,.$$

$$p = e^{2\pi i \tau}\,, \quad p \to 0: \quad \vartheta(w) \to -i p^{\frac{1}{8}} \left(\sqrt{x} - 1/\sqrt{x}\right), \quad x = e^{2\pi i z}\,.$$

Briefly about elliptic functions (the Kronecker function)

$$\phi(\eta,z)=\begin{cases}1/\eta+1/z\,,\\ \coth(\eta)+\coth(z)\,,\\ \dfrac{\vartheta'(0)\vartheta(\eta+z)}{\vartheta(\eta)\vartheta(z)}\,.\end{cases}\quad E_1(z)=\begin{cases}1/z\,,\\ \coth(z)\,,\\ \dfrac{\vartheta'(z)}{\vartheta(z)}\end{cases}\quad \wp(z)=\begin{cases}1/z^2\,,\\ 1/\sinh^2(z)+1/3\,,\\ -E_1'(z)+\dfrac{1}{3}\dfrac{\vartheta'''(0)}{\vartheta'(0)}\end{cases}$$

Properties:

1. it has a simple pole at $z=0$: $\operatorname*{Res}_{z=0}\phi(\eta,z)=1\,.$
2. The quasi-periodic behavior on the lattice of periods $\mathbb{Z}\oplus\tau\mathbb{Z}$ ($\Sigma_\tau=\mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}$)

$$\phi(\eta,z+1)=\phi(\eta,z)\,,\qquad \phi(\eta,z+\tau)=e^{-2\pi\imath\eta}\phi(\eta,z)\,.$$

3. Fay identity – Riemann identities for theta functions (addition formula)

$$\phi(\eta_1,z_{12})\phi(\eta_2,z_{23})=\phi(\eta_2,z_{13})\phi(\eta_1-\eta_2,z_{12})+\phi(\eta_2-\eta_1,z_{23})\phi(\eta_1,z_{13})\,.$$

Addition formulae for $1/x$ function:

$$(z_1 - z_2) + (z_2 - z_3) + (z_3 - z_1) = 0$$

or, dividing by $(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)$

$$\frac{1}{(z_1 - z_2)} \frac{1}{(z_2 - z_3)} + \frac{1}{(z_2 - z_3)} \frac{1}{(z_3 - z_1)} + \frac{1}{(z_3 - z_1)} \frac{1}{(z_1 - z_2)} = 0$$

For the function of two variables

$$\phi(\eta, z) = \frac{1}{\eta} + \frac{1}{z}$$

we have the (genus 1) Fay identity (can be viewed as functional equation)

$$\phi(\eta_1, z_{12})\phi(\eta_2, z_{23}) = \phi(\eta_2, z_{13})\phi(\eta_1 - \eta_2, z_{12}) + \phi(\eta_2 - \eta_1, z_{23})\phi(\eta_1, z_{13}),$$

where $z_{ij} = z_i - z_j$. The upper addition formula is reproduced when $\eta \to \infty$.
Numerous applications in classical and quantum integrable systems: Lax equations, r-matrix structures, quadratic algebras of Sklyanin type, Yang-Baxter equations, Dunkl operators, Knizhnik-Zamolodchikov equations...

Lax matrices $L(z)$ - $N \times N$ matrix (function of z):

$$\dot{L}(z) = [L(z), M(z)] \quad - \quad \text{equations of motion } \forall z$$

$\mathrm{tr}\, L^k(z)$ – conservation laws.
for the Calogero-Moser model (the Krichever's Lax pair)

$$L^{\mathrm{CM}}_{ij} = p_i \delta_{ij} + \nu(1 - \delta_{ij})\phi(z, q_{ij}) \,.$$

$$M^{\mathrm{CM}}_{ij} = \nu d_i \, \delta_{ij} + \nu(1 - \delta_{ij})\phi'(z, q_{ij}) \,, \quad d_i = \sum_{k \neq i} \wp(q_{ik}) \,,$$

The Ruijsenaars-Schneider model:

$$\ddot{q}_i = \sum_{k \neq i}^{N} \dot{q}_i \dot{q}_k (2E_1(q_{ik}) - E_1(q_{ik} + \eta) - E_1(q_{ik} - \eta)) \,, \quad i = 1 \ldots N \,,$$

where $q_{ij} = q_i - q_j$, η – the coupling constant.
The Lax matrix has trigonometric (exponential) dependence on momenta:

$$L^{\mathrm{RS}}_{ij} = \phi(z, q_{ij} + \eta) \prod_{k \neq j} \frac{\vartheta(q_{jk} - \eta)}{\vartheta(q_{jk})} \, e^{p_j / c} \,.$$

A short summary:

We have families of many-body integrable systems (Calogero-Ruijsenaars) with rational, trigonometric and elliptic potentials in coordinates.

Their existence (integrability) is based on solutions of the functional equation

$$\phi(\eta_1, z_{12})\phi(\eta_2, z_{23}) = \phi(\eta_2, z_{13})\phi(\eta_1 - \eta_2, z_{12}) + \phi(\eta_2 - \eta_1, z_{23})\phi(\eta_1, z_{13}) \,,$$

Main set of solutions:

$$\vartheta(z) = \begin{cases} z \,, \\ \sinh(z) \,, \\ \vartheta(z) \end{cases} \qquad \phi(\eta, z) = \begin{cases} 1/\eta + 1/z \,, \\ \coth(\eta) + \coth(z) \,, \\ \dfrac{\vartheta'(0)\vartheta(\eta + z)}{\vartheta(\eta)\vartheta(z)} \,. \end{cases}$$

The Lax matrices and other structures are constructed through these functions. In momenta the dependence is either rational (Calogero) or trigonometric (Ruijsenaars). What about elliptic case (dependence in momenta)?

P-Q duality (or Ruijsenaars duality or action-angle duality)

It interchanges action variables and coordinates in two models.

From the group-theoretical approach we have the moment map equations of type

$$L_{ij}^{\mathrm{CM}}(q_i - q_j) = \nu(1-\delta_{ij}) \qquad L_{ij}^{\mathrm{CM}} = p_i\delta_{ij} + \nu(1-\delta_{ij})/(q_i - q_j)$$

or using notation $e = (1, ..., 1)$

$$[Q, L^{\mathrm{CM}}] = \nu(e^T \otimes e - 1_N)\,, \quad Q = \mathrm{diag}(q_1, ..., q_N)\,.$$

Using the eigenvector problem $L^{\mathrm{CM}}\Psi = \Psi\Lambda$ (or $L^{\mathrm{CM}} = \Psi\Lambda\Psi^{-1}$)

$$Ad_{\Psi^{-1}} : [Q, L^{\mathrm{CM}}] = \nu(e \otimes e - 1_N) \to [\tilde{L}, \Lambda] = \nu(\xi \otimes \eta - 1_N)$$

$$\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_N) - \text{action variables for } L^{\mathrm{CM}}$$

The matrix $\tilde{L} = \Psi^{-1}Q\Psi$ is again Lax matrix of (another) Calogero model with coordinates λ_i and action variables q_i: $\tilde{L}_{ij}^{\mathrm{CM}} = \tilde{p}_i\delta_{ij} - \nu(1-\delta_{ij})/(\lambda_i - \lambda_j)$.

Similarly (but already not so easy):

$$A - gAg^{-1} = \nu(e^T \otimes e - 1_N)$$

can be solved

1. with respect to A with diagonal g – trigonometric Calogero model
2. or with respect to g with diagonal A – rational Ruijsenaars model.

The duality map interchanges the types of dependence. However, in the elliptic case the group-theoretical approach does not work in this way.

But one can find the Hamiltonians for the systems dual to models elliptic in coordinates. The answer is given in terms of higher genus theta-functions.

H.W. Braden, A. Marshakov, A. Mironov, A. Morozov, Nuclear Physics B 573 (2000) 553–572; hep-th/9906240.
V. Fock, A. Gorsky, N. Nekrasov, V. Rubtsov, JHEP 0007 (2000) 028; arXiv:hep-th/9906235.
A. Mironov, A. Morozov, Physics Letters B 475 (2000) 71–76; arXiv:hep-th/9912088.
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A. Mironov, Theoret. and Math. Phys., 129:2 (2001) 1581-–1585; arXiv:hep-th/0104253.
A. Mironov, A. Morozov, Physics Letters B 524 (2002) 217–226; arXiv:hep-th/0107114.
A. Mironov, Theoret. and Math. Phys., 135:3 (2003) 814-–827; arXiv:hep-th/0205202.
H.W. Braden, T.J. Hollowood, JHEP 0312 (2003) 023; arXiv:hep-th/0311024.

Lax formulation is unknown.

Table of many-body systems

red arrows connect p-q dual models

moment. \ coor	rat	trig	ell
rat	rat Calogero	trig Calogero	ell Calogero
trig	rat Ruijsenaars	trig Ruijsenaars	ell Ruijsenaars
ell	dual to ell. CM	dual to ell. RS	double elliptic model

We are going to deal with an alternative formulation.

P. Koroteev, Sh. Shakirov, *The Quantum DELL System*, Lett. Math. Phys. 110 (2020) 969–999; arXiv:1906.10354 [hep-th].

In what follows we also use the following (slightly modified) theta-function

$$\theta_p(x) = \sum_{n\in\mathbb{Z}} p^{\frac{n^2-n}{2}}(-x)^n\,, \quad p = e^{2\pi i\tau}\,.$$

It is easily related to the previous:

$$\theta_p(x) = ip^{-\frac{1}{8}}x^{\frac{1}{2}}\vartheta(w|\,\tau)\,, \quad x = e^{2\pi i w}\,.$$

In the trigonometric limit $p \to 0$

$$\theta_p(x) \to (1-x)\,, \qquad \vartheta(w) \to -ip^{\frac{1}{8}}\left(\sqrt{x} - 1/\sqrt{x}\right).$$

The Koroteev-Shakirov version of generating function of commuting Hamiltonians.

Consider

$$\hat{\mathcal{O}}(\lambda) = \sum_{n_1,\dots,n_N\in\mathbb{Z}} \omega^{\sum_i \frac{n_i^2-n_i}{2}} (-\lambda)^{\sum_i n_i} \prod_{i<j}^{N} \frac{\theta_p(t^{n_i-n_j}\frac{x_i}{x_j})}{\theta_p(\frac{x_i}{x_j})} \prod_{i}^{N} q^{n_i x_i \partial_i} = \sum_{n\in\mathbb{Z}} \lambda^n \hat{\mathcal{O}}_n \,.$$

and

$$\hat{H}_n = \hat{\mathcal{O}}_0^{-1}\hat{\mathcal{O}}_n$$

The positions of particles q_i enter through $x_i = e^{q_i}$, $t = e^{\eta}$ – is exponent of the coupling constant η, $q = e^{\hbar}$ – is exponent of the Planck constant $\hbar$; $\partial_i = \partial_{x_i}$, so that $\partial_{q_i} = x_i\partial_i$. $\omega = e^{2\pi i\tilde{\tau}}$ is the second modular parameter, λ is the (spectral) parameter of generating function.

When $N = 1$ we have $\theta_\omega(\lambda q^{x_1\partial_{x_1}})$.

Commutativity

$$[\hat{H}_n, \hat{H}_m] = 0$$

is an experimental result.

We define a modified version of the generating function $\hat{\mathcal{O}}$ depending also on the spectral parameter z:

$$\hat{\mathcal{O}}'(z,\lambda) = \sum_{k\in\mathbb{Z}} \frac{\vartheta(z-k\eta)}{\vartheta(z)} \lambda^k \hat{\mathcal{O}}'_k =$$

$$\sum_{n_1,\dots,n_N\in\mathbb{Z}} \frac{\vartheta(z-\eta\sum_{i=1}^N n_i)}{\vartheta(z)} \omega^{\sum_i \frac{n_i^2-n_i}{2}} (-\lambda)^{\sum_i^N n_i} \prod_{i<j}^N \frac{\vartheta(q_i-q_j+\eta(n_i-n_j))}{\vartheta(q_i-q_j)} \prod_i^N e^{n_i\hbar\partial_{q_i}} .$$

Using the elliptic Ruijsenaars-Schneider Lax operator

$$\hat{L}^{RS}_{ij}(z,q_i-q_j,\eta,\hbar) = \frac{\vartheta(-\eta)\vartheta(z+q_{ij}-\eta)}{\vartheta(z)\vartheta(q_{ij}-\eta)} \prod_{k\neq j} \frac{\vartheta(q_{jk}+\eta)}{\vartheta(q_{jk})} e^{\hbar\partial_{q_j}} . \quad q_{ij}=q_i-q_j .$$

we are going to prove

$$\hat{\mathcal{O}}'(z,\lambda) =: \det_{1\le i,j\le N} \left\{ \hat{\mathcal{L}}^{\rm Dell}_{ij}(z,\lambda\,|\,\hbar,\eta\,|\,\tau,\omega) \right\} : ,$$

where

$$\hat{\mathcal{L}}^{\rm Dell}_{ij}(z,\lambda\,|\,\hbar,\eta\,|\,\tau,\omega) = \sum_{k\in\mathbb{Z}} \omega^{\frac{k^2-k}{2}} (-\lambda)^k \hat{L}^{RS}_{ij}(z,q_i-q_j,k\eta,k\hbar)$$

Sketch of the proof: Consider the determinant

$$:\det \hat{\mathcal{L}} := \sum_{\sigma} (-1)^{|\sigma|} : \hat{\mathcal{L}}_{\sigma(1)1} \hat{\mathcal{L}}_{\sigma(2)2} \dots \hat{\mathcal{L}}_{\sigma(N)N} : \, .$$

Let us represent it as a sum of determinants. For this purpose collect all the terms with $\prod_i^N q^{n_i x_i \partial_i}$:

$$:\det \hat{\mathcal{L}} := \sum_{n_1,\dots,n_N \in \mathbb{Z}} \omega^{\sum_i \frac{n_i^2 - n_i}{2}} (-\lambda)^{\sum_i n_i} \times$$

$$\times \sum_{\sigma} (-1)^{|\sigma|} : \hat{L}^{\mathrm{RS}}_{\sigma(1)1}(z, n_1\eta, n_1\hbar) \hat{L}^{\mathrm{RS}}_{\sigma(2)2}(z, n_2\eta, n_2\hbar) \dots \hat{L}^{\mathrm{RS}}_{\sigma(N)N}(z, n_N\eta, n_N\hbar) :$$

$$= \sum_{n_1,\dots,n_N \in \mathbb{Z}} \omega^{\sum_i \frac{n_i^2 - n_i}{2}} (-\lambda)^{\sum_i n_i} : \det_{1 \le i,j \le N} \hat{L}^{\mathrm{RS}}_{ij}(z, q_i - q_j, n_j\eta, n_j\hbar) : ,$$

where the matrix $\hat{L}^{\mathrm{RS}}_{ij}(z, q_i - q_j, n_j\eta, n_j\hbar)$ is constructed by combining rows from different terms of the sum.

Using its explicit form let us rewrite it through the elliptic Cauchy matrix:

$$\hat{L}^{\rm RS}_{ij}(z, q_i - q_j, n_j\eta, n_j\hbar) = \vartheta(-n_j\eta) L^{\rm Cauchy}_{ij}(z, q_i - \tilde{q}_j) \prod_{k:k\neq j} \frac{\vartheta(\tilde{q}_j - q_k)}{\vartheta(q_j - q_k)}\, e^{n_j\hbar\partial_j}\,,$$

where

$$L^{\rm Cauchy}_{ij}(z, q_i - \tilde{q}_j) = \frac{\vartheta(z + q_i - \tilde{q}_j)}{\vartheta(z)\vartheta(q_i - \tilde{q}_j)}\,, \qquad \tilde{q}_j = q_j + n_j\eta\,.$$

Therefore,

$$: \det_{1\leq i,j\leq N} \hat{L}^{\rm RS}_{ij}(z, q_i - q_j, n_j\eta, n_j\hbar) :=$$

$$= \det_{1\leq i,j\leq N} L^{\rm Cauchy}_{ij}(z, q_i - \tilde{q}_j) \prod_{k=1}^{N} \vartheta(-n_k\eta) \prod_{k,j:k\neq j} \frac{\vartheta(\tilde{q}_j - q_k)}{\vartheta(q_j - q_k)} \prod_{k=1}^{N} e^{n_k\hbar\partial_k}\,.$$

Plugging the Cauchy determinant identity

$$\det_{1\le i,j\le N} \frac{\vartheta(z+u_i-w_j)}{\vartheta(z)\vartheta(u_i-w_j)} = \frac{\vartheta\Big(z+\sum_{i=1}^{N}(u_i-w_j)\Big)}{\vartheta(z)} \frac{\prod_{p<q}^{N} \vartheta(u_p-u_q)\vartheta(w_q-w_p)}{\prod_{r,s=1}^{N} \vartheta(u_r-w_s)}.$$

we finish the proof. ■

Recall the definition of the Hamiltonians: $\hat{H}_n = \hat{\mathcal{O}}_0^{-1}\hat{\mathcal{O}}_n$.
Generating function for these Hamiltonians is

$$\hat{H}(\lambda) = \sum_{n\in\mathbb{Z}} \lambda^n \hat{H}_n = \hat{\mathcal{O}}_0^{-1}\hat{\mathcal{O}}(\lambda),$$

where

$$\hat{\mathcal{O}}(\lambda) = \sum_{n\in\mathbb{Z}} \lambda^n \hat{\mathcal{O}}_n.$$

Define the operator $\hat{\mathcal{O}}(1) = \hat{\mathcal{O}}(\lambda)|_{\lambda=1}$.
Let us prove that

$$\hat{\mathcal{H}}(\lambda) = \hat{\mathcal{O}}(1)^{-1}\hat{\mathcal{O}}(\lambda) = \sum_{n\in\mathbb{Z}} \lambda^n \hat{\mathcal{H}}_n\,, \quad \hat{\mathcal{H}}_n = \hat{\mathcal{O}}(1)^{-1}\hat{\mathcal{O}}_n$$

is also a generating function of the commuting Hamiltonians.
So that commutativity of $\hat{\mathcal{H}}_n$ follows from commutativity of $\hat{H}_n$.

Proof: First, let us notice that the operators $\hat{H}_{k\,n} = \hat{\mathcal{O}}_k^{-1}\hat{\mathcal{O}}_n = \hat{H}_k^{-1}\hat{H}_n$ also commute with each other due to commutativity of $\hat{H}_k$. Therefore, $\hat{H}_{mk}\hat{H}_{nk} = \hat{H}_{nk}\hat{H}_{mk}$, or acting on this equality by $\hat{\mathcal{O}}_k^{-1}$ from the right

$$\hat{\mathcal{O}}_m^{-1}\hat{\mathcal{O}}_k\hat{\mathcal{O}}_n^{-1} = \hat{\mathcal{O}}_n^{-1}\hat{\mathcal{O}}_k\hat{\mathcal{O}}_m^{-1}\,.$$

Next, summing up over $k \in \mathbb{Z}$ gives

$$\hat{\mathcal{O}}_m^{-1}\hat{\mathcal{O}}(1)\hat{\mathcal{O}}_n^{-1} = \hat{\mathcal{O}}_n^{-1}\hat{\mathcal{O}}(1)\hat{\mathcal{O}}_m^{-1}\,.$$

By taking its inverse we get $\hat{\mathcal{O}}_n\hat{\mathcal{O}}(1)^{-1}\hat{\mathcal{O}}_m = \hat{\mathcal{O}}_m O(1)^{-1}\hat{\mathcal{O}}_n$. Finally, multiplying both sides by $\hat{\mathcal{O}}(1)^{-1}$ from the left yields

$$\hat{\mathcal{O}}(1)^{-1}\hat{\mathcal{O}}_n\hat{\mathcal{O}}(1)^{-1}\hat{\mathcal{O}}_m = \hat{\mathcal{O}}(1)^{-1}\hat{\mathcal{O}}_m\hat{\mathcal{O}}(1)^{-1}\hat{\mathcal{O}}_n$$

for any n and m, which is equivalent to $[\hat{\mathcal{H}}_n, \hat{\mathcal{H}}_m] = 0$. ■

Importance of

$$\hat{\mathcal{H}}(\lambda) = \hat{\mathcal{O}}(1)^{-1}\hat{\mathcal{O}}(\lambda) = \sum_{n\in\mathbb{Z}} \lambda^n \hat{\mathcal{H}}_n\,, \quad \hat{\mathcal{H}}_n = \hat{\mathcal{O}}(1)^{-1}\hat{\mathcal{O}}_n\,.$$

is as follows.
On the one hand $\hat{\mathcal{O}}(1)$, compared to $\hat{\mathcal{O}}_0$ is hard to invert since its Taylor series expansion in ω starts not with 1.
On the other hand the advantage of $\hat{\mathcal{O}}(1)$ is its determinant representation, while there is no natural way to find a determinant representation for $\hat{\mathcal{O}}_0$.
The generating function of the quantum Hamiltonians takes the form:

$$\hat{\mathcal{H}}(z,\lambda) =: \big(\det_{1\le i,j\le N} \mathcal{L}_{ij}(z,1)\big) :^{-1} : \det_{1\le i,j\le N} \mathcal{L}_{ij}(z,\lambda) : \;.$$

The operator $\hat{\mathcal{O}}(1)^{-1}$ in really acts on $\hat{\mathcal{O}}(\lambda)$ as a quantum operator, so that we can not unify the normal orderings.
But in the classical case

$$\mathcal{H}(z,\lambda) = \det_{N\times N}[\mathcal{L}^{-1}(z,1)\mathcal{L}(z,\lambda)]\,,$$

That is, the matrix

$$L(z,\lambda) = \mathcal{L}^{-1}(z,1)\mathcal{L}(z,\lambda) \in \mathrm{Mat}(N,\mathbb{C})$$

with

$$\mathcal{L}(z,\lambda) = \sum_{n\in\mathbb{Z}} (-\lambda)^n \omega^{\frac{n^2-n}{2}} L^{RS}(z,q^n,t^n)$$

arises, which determinant $\mathcal{H}(z,\lambda)$ is the generating function of the classical Hamiltonians.

We call it spectral L-matrix. It is not Lax matrix since its traces (eigenvalues) are not the Hamiltonians. Only determinant.

Expression $\mathcal{H}(z,\lambda)$ can be considered as an analogue of the expression

$$\det(\lambda - l(z))$$

for the spectral curve of an integrable system with the Lax matrix $l(z)$. It can be seen from the limit $\omega = 0$ we have

$$\mathcal{L}(z,\lambda)|_{\omega=0} = 1_N - \lambda L^{\mathrm{RS}}(z,q,t)\,,$$

where 1_N is the identity $N \times N$ matrix. Then

$$L(z,\lambda) = \mathcal{L}^{-1}(z,1)\mathcal{L}(z,\lambda) = \lambda 1_N + (1-\lambda)\Big(1_N - L^{\mathrm{RS}}(z,q,t)\Big)^{-1}.$$

Therefore, equation $\mathcal{H}(z,\lambda)|_{\omega=0} = 0$ is indeed the spectral curve of the elliptic Ruijsenaars-Schneider model (written in some complicated way).

If we had a true Lax matrix for the Dell model then $\det L(z,\lambda)$ should represent its spectral curve. So, if the Lax representation exists we need to find a matrix $\breve{L}$ of a size $M \times M$ (possibly $M = \infty$) and a change of variables $u = u(z,\lambda)$, $\zeta = \zeta(z,\lambda)$ satisfying

$$\det_{N\times N} \mathcal{L}(z,\lambda) = \det_{M\times M}\Big(u - \breve{L}(\zeta)\Big)\,.$$

L-A-B Manakov triple

$$\frac{d}{dt_k}L(z,\lambda) = [L(z,\lambda), M_k(z)] + B_k(z,\lambda)L(z,\lambda)\,,$$

where

$$\operatorname{tr} B_k(z,\lambda) = 0\,.$$

Indeed, by differentiating $L(z,\lambda)$ we get

$$M_k(z) = \mathcal{L}^{-1}(z,\lambda)\Big(\frac{d}{dt_k}\mathcal{L}(z,\lambda)\Big)$$

and

$$B_k(z,\lambda) = \mathcal{L}^{-1}(z,\lambda)\Big(\frac{d}{dt_k}\mathcal{L}(z,\lambda)\Big) - \mathcal{L}^{-1}(z,1)\Big(\frac{d}{dt_k}\mathcal{L}(z,1)\Big)\,.$$

The property of $\operatorname{tr} B = 0$ follows from

$$\frac{d}{dt_k}\det L(z,\lambda) = \frac{d}{dt_k}\frac{\det\mathcal{L}(z,\lambda)}{\det\mathcal{L}(z,1)}\,.$$

The l.h.s. of equals zero since $\det L(z,\lambda) = \mathcal{H}(z,\lambda)$, while the r.h.s. is proportional to the trace of $B_k(z,\lambda)$.

IRF-Vertex relation

Consider the intertwining matrix (Baxter, Jimbo, Miwa..)

$$g(z,q) = \Xi(z,q)\left(D^0\right)^{-1}$$

with

$$\Xi_{ij}(z,q) = \vartheta\left[\begin{array}{c} \frac{1}{2} - \frac{i}{N} \\ \frac{N}{2} \end{array}\right]\left(z - Nq_j + \sum_{m=1}^{N} q_m \,\middle|\, N\tau\right),$$

and

$$D^0_{ij}(z,q) = \delta_{ij} D^0_j = \delta_{ij} \prod_{k\neq j} \vartheta(q_j - q_k)\,.$$

Relates dynamical and non-dynamical R-matrices:

$$g_2(z_2,q)\, g_1(z_1, q+\hbar^{(2)})\, R^{\mathrm{F}}_{12}(\hbar, z_1 - z_2 |\, q) = R^{\mathrm{B}}_{12}(\hbar, z_1 - z_2)\, g_1(z_1,q)\, g_2(z_2, q+\hbar^{(1)})\,.$$

where

$$R^{\mathrm{F}}_{12}(\hbar, z_1, z_2 |\, q) = R^{\mathrm{F}}_{12}(\hbar, z_1 - z_2 |\, q) =$$

$$= \sum_{i\neq j} E_{ii}\otimes E_{jj}\,\phi(\hbar, -q_{ij}) + \sum_{i\neq j} E_{ij}\otimes E_{ji}\,\phi(z_1 - z_2, q_{ij}) + \phi(\hbar, z_1 - z_2)\sum_i E_{ii}\otimes E_{ii}\,.$$

and $R^{\mathrm{B},\hbar}_{12}(z) = \sum T_\alpha \otimes T_{-\alpha}\,\varphi_\alpha(z, \omega_\alpha + \hbar)\,.$

For non-dynamical R-matrices we have ordinary exchange relations:

$$\hat{L}_1(z)\hat{L}_2(w)R^{\mathrm{B},\hbar}_{12}(z-w) = R^{\mathrm{B},\hbar}_{12}(z-w)\hat{L}_2(w)\hat{L}_1(z)\,.$$

Factorization of Lax operator (K. Hasegawa)

The Ruijsenaars Lax operator is represented in the form:

$$\hat{L}^{\mathrm{RS}}(z) = g^{-1}(z)g(z-N\eta)\, q^{\mathrm{diag}(\partial_{q_1},\ldots,\partial_{q_N})/c}$$

Classical version

$$L^{\mathrm{RS}}(z) = g^{-1}(z)g(z-N\eta)e^{P/c} \in \mathrm{Mat}(N,\mathbb{C})\,, \quad P = \mathrm{diag}(p_1,\ldots,p_N)$$

and $\eta = \nu/c \to 0$ provides

$$L^{\mathrm{CM}}(z) = P + \nu' g^{-1}\partial_z g\,.$$

The gauged transformed is the Sklyanin's Lax operator

$$\hat{L}^{\mathrm{Skl}}(z) =: g(z-N\eta)\, q^{\mathrm{diag}(\partial_{q_1},\ldots,\partial_{q_N})/c}\, g^{-1}(z) := \sum_{k=1}^{N} g_{ik}(z-N\eta)\, g^{-1}_{kj}(z)\, e^{(\hbar/c)\partial_{q_k}}\,.$$

For the systems of Calogero-Ruijsenaars type the Lax matrix can be specified by a choice of two ingredients:

- the function f,
- intertwining matrix $\Xi(z)$: Then the Lax matrix is of the form:

$$L^{\rm CR}(z) = G^{-1}(z)\, f(-{\rm ad}_{N\eta\partial_z})\, G(z)\,, \qquad {\rm ad}_{\partial_z} * = [\partial_z, *\,]\,,$$

where the matrix $G(z)$ is defined in terms of $\Xi(z)$:

$$G(z) = g(z,\tau)e^{\frac{z}{Nc\eta}P} = \Xi(z)D^{-1}e^{\frac{z}{Nc\eta}P}\,, \qquad P = {\rm diag}(p_1, ..., p_N)$$

The function $f(w)$ is either: 1) linear: $f(w) = w$; or 2) exponent: $f(w) = e^w$. The first choice of the function f provides the Lax matrix of the Calogero-Moser-Sutherland systems. The second choice of f gives rise to the Lax matrices of the Ruijsenaars-Schneider models.

What is elliptic version for function f?

The choices of $g(z) = \Xi(z)(D^0)^{-1}$

1. Elliptic

$$\Xi_{ij}(z,q) = \vartheta\left[\begin{array}{c} \frac{1}{2} - \frac{i}{N} \\ \frac{N}{2} \end{array}\right]\left(z - Nq_j + \sum_{m=1}^{N} q_m \,\middle|\, N\tau\right),$$

$$D^0_{ij}(z,q) = \delta_{ij} D^0_j = \delta_{ij} \prod_{k \neq j} \vartheta(q_j - q_k)\,.$$

2. Trigonometric with spectral parameter

$$D^0_{ij} = \delta_{ij} \prod_{k \neq i} (e^{-2q_i} - e^{-2q_k})\,,$$

$$\tilde{\Xi}_{ij}(z) = \begin{cases} x_j^{i-1},\ i \leq N, \\ x_j^{N-1} + \dfrac{(-1)^N}{x_j},\ i = N \end{cases}$$

with $x_j = e^{-2q_j + 2z + 2\bar{q}}$. Here $\bar{q} = \frac{1}{N}\sum_{k=1}^{N} q_k$ is the center of mass coordinate.

2*. Trigonometric without spectral parameter

$$\Xi_{ij}(z) = \exp\left((2i-1-N)(z-q_j)\right),$$

$$(D^0)_{ij} = \delta_{ij} \prod_{k\neq i} \sinh(q_i - q_k)$$

3. Rational with spectral parameter

$$(D^0)_{ij}(q) = \delta_{ij} \prod_{k\neq i}^{n} (q_i - q_k),$$

$$\Xi_{ij}(q,z) = (z - q_j + \bar{q})^{\varrho(i)}, \quad \bar{q} = \frac{1}{N}\sum_{k=1}^{N} q_k$$

with

$$\varrho(i) = \begin{cases} i-1 \text{ for } 1 \leq i \leq N-1, \\ N \text{ for } i = N. \end{cases}$$

3*. Rational without spectral parameter

$$\Xi_{ij}(z) = (z - q_j + \bar{q})^{i-1}$$

is the Vandermonde matrix

Based on the Koroteev-Shakirov Hamiltonians and/or the Manakov L-matrix structure we come to elliptic version for the function f:

$$3)\ \text{ratio of theta-functions:}\ f_\lambda(w) = \frac{\theta_\omega(\lambda e^w)}{\theta_\omega(e^w)}\,.$$

Let us slightly change the definition of $f_\lambda(w)$ as in transition from $\theta_p(e^w)$ to $\vartheta(w)$ together with additional normalization factor $\vartheta'(0)/\vartheta(\log(\lambda))$.
Then the function $f_\lambda(w)$ turns into the Kronecker elliptic function depending on the moduli $\tilde{\tau}$ (defined through $\omega = e^{2\pi\imath\tilde{\tau}}$):

$$f_u(w) \to \lambda^{1/2} f_u(w) \frac{\vartheta'(0|\tilde{\tau})}{\vartheta(u|\tilde{\tau})} = \Phi(u,w|\tilde{\tau}) = \frac{\vartheta'(0|\tilde{\tau})\vartheta(w+u|\tilde{\tau})}{\vartheta(u|\tilde{\tau})\vartheta(w|\tilde{\tau})}\,.$$

where $u = \log(\lambda)$.
In this way we come to the classification for the function $f_u(w)$ (responsible for the momenta type dependence), which is parallel to the well known classification of the coordinates dependence.

The Ruijsenaars-Schneider Lax matrix takes the form

$$\tilde{L}^{\mathrm{RS}}(z) = G^{-1}(z)\,\mathrm{Ad}_{e^{-N\eta\partial_z}}\,G(z)$$

with

$$G(z) = \Xi(z) D^{-1} e^{\frac{z}{Nc\eta}P} = g(z) e^{\frac{z}{Nc\eta}P}\,, \qquad P = \mathrm{diag}(p_1, ..., p_N)$$

Up to gauge transformation with the diagonal matrix $\exp\left(\frac{z}{Nc\eta}P\right)$:

$$\tilde{L}^{\mathrm{RS}}(z) = G^{-1}(z) G(z - N\eta) = e^{-\frac{z}{Nc\eta}P} L^{\mathrm{RS}}(z)\, e^{\frac{z}{Nc\eta}P}\,.$$

For the double elliptic case we get

$$\mathcal{L}(z,\lambda) = g^{-1}(z) \sum_{k\in\mathbb{Z}} (-\lambda)^k \omega^{\frac{k^2-k}{2}}\, g(z - kN\eta)\, e^{kP/c}\,,$$

or

$$\mathcal{L}'(z,\lambda) = e^{-\frac{z}{c\eta}P} \mathcal{L}(z,\lambda) e^{\frac{z}{c\eta}P} = G^{-1}(z)\theta_\omega\Big(\lambda \mathrm{Ad}_{e^{-N\eta\partial_z}}\Big) G(z)\,.$$

By introducing also

$$\Theta(z,\lambda) = \theta_\omega\Big(\lambda \mathrm{Ad}_{e^{-N\eta\partial_z}}\Big) G(z) =$$

$$= \sum_{k\in\mathbb{Z}} (-\lambda)^k \omega^{\frac{k^2-k}{2}}\, g(z-kN\eta)\, e^{kP/c} e^{\frac{z}{c\eta}P}$$

we come to the following expression for the Manakov's L-matrix:

$$L'(z,\lambda) = \Theta^{-1}(z,1)\Theta(z,\lambda)\,.$$

In terms of the Kronecker function we may write the Manakov L-matrix as

$$\check{L}(z,\lambda) = \Phi[G(z,\tau), u|\tilde{\tau}] :=$$

$$= \frac{\vartheta'(0|\tilde{\tau})}{\vartheta(u|\tilde{\tau})}\Big[\vartheta(-\,\mathrm{ad}_{N\eta\partial_z}|\tilde{\tau})\,G(z)\Big]^{-1}\vartheta(u-\,\mathrm{ad}_{N\eta\partial_z}|\tilde{\tau})\,G(z)\,,$$

where $u = \log(\lambda)$.
This gives a universal receipt for construction of the L-matrices in the table of many-body systems.

Table of integrable many-body systems

A rule for constructing L-matrix:

moment. $\bar{\tau}$ \ coord. τ	rat $G(z,\tau)$	trig $G(z,\tau)$	ell $G(z,\tau)$
rat. $\Phi(*,u\mid\bar{\tau})$	rat CM	trig CM	ell CM
trig $\Phi(*,u\mid\bar{\tau})$	rat RS	trig RS	ell RS
ell $\Phi(*,u\mid\bar{\tau})$	dual to ell. CM	dual to ell. RS	Dell

intertwining matrix from IRF-Vertex relation

Sklyanin type L-operator

$$L^{\mathrm{Skl}}(z) = G(z)\tilde{L}^{\mathrm{RS}}(z)G^{-1}(z) =$$

$$= G(z-N\eta)\,G^{-1}(z) = \Xi(z-N\eta)\,e^{P/c}\,\Xi^{-1}(z)\,.$$

In quantum case

$$\hat{L}^{\mathrm{Skl}}(z) =: \Xi(z-N\eta)\,q^{\mathrm{diag}(\partial_{q_1},\dots,\partial_{q_N})/c}\,\Xi^{-1}(z) := \sum_{k=1}^{N}\Xi_{ik}(z-N\eta)\,\Xi^{-1}_{kj}(z)\,e^{(\hbar/c)\partial_{q_k}}\,.$$

Then for the Dell model we have

$$\mathcal{L}^{\mathrm{Dell}}(z,\lambda) = G(z)\mathcal{L}(z,\lambda)G^{-1}(z) = \Big(f(-\mathrm{ad}_{N\eta\partial_z})\,G(z)\Big)G^{-1}(z) =$$

$$= \sum_{m\in\mathbb{Z}}(-\lambda)^m\omega^{\frac{m^2-m}{2}}\,\Xi(z-mN\eta)\,e^{mP/c}\,\Xi^{-1}(z)\,.$$

Hence, we have the answer similar to the one obtained through Ruijsenaars model:

$$\mathcal{L}^{\mathrm{Dell}}(z,\lambda) = \sum_{m\in\mathbb{Z}}(-\lambda)^m\omega^{\frac{m^2-m}{2}}\,L^{\mathrm{Skl}}(z,\{p_i\},\{q_i\},m\eta,mc^{-1})\,.$$

Let us quantize the Sklyanin L-operators in the fundamental representation. These quantizations are described by Belavin's R-matrices.
Then for

$$\hat{L}^{\mathrm{Dell}}(z,\lambda) = \Big(:\hat{\mathcal{L}}^{\mathrm{Skl}}(z,1): \Big)^{-1} :\hat{\mathcal{L}}^{\mathrm{Skl}}(z,\lambda):$$

we get the following answer for $\hat{L}^{\mathrm{Dell}}(z,\lambda)$

$$\mathbf{R}_{12}(z,\lambda) = \mathcal{R}_{12}(z,1)^{-1}\,\mathcal{R}_{12}(z,\lambda) \in \mathrm{Mat}(N,\mathbb{C})\,.$$

with

$$\mathcal{R}_{12}(z,\lambda) = \sum_{a,b,c,d=1}^{N} E_{ab}\otimes E_{cd}\,\mathcal{R}^{\eta}_{ab,cd}(z,\lambda)$$

and

$$\mathcal{R}^{\eta}_{ab,cd}(z,\lambda) = \sum_{m\in\mathbb{Z}} (-\lambda)^m \omega^{\frac{m^2-m}{2}} R^{\mathrm{B}}_{ab,cd}(m\eta,z)\,.$$

What is equation for $\mathbf{R}_{12}(z,\lambda)$?

Cherednik operators

Consider the coordinate trigonometric case in

$$\hat{H}_n = \hat{\mathcal{O}}_0^{-1}\hat{\mathcal{O}}_n \,,$$

$$\hat{\mathcal{O}}(u) = \sum_{n_1,\ldots,n_N\in\mathbb{Z}} \omega^{\sum_i \frac{n_i^2-n_i}{2}}(-u)^{\sum_i n_i}\prod_{i<j}^N \frac{\theta_p(t^{n_i-n_j}\frac{x_i}{x_j})}{\theta_p(\frac{x_i}{x_j})}\prod_i^N q^{n_i x_i\partial_i} = \sum_{n\in\mathbb{Z}} u^n \hat{\mathcal{O}}_n \,.$$

When $p=0$: together with the change t to t^{-1}, $q\leftrightarrow q^{-1}$ and conjugation by the function $\prod_{i<j} x_i x_j$, the limit $p\to 0$ yields

$$D_N(u) = D_N(u|x_1,\ldots,x_N) = \sum_{n_1,\ldots,n_N\in\mathbb{Z}} \omega^{\sum_i \frac{n_i^2-n_i}{2}}(-u)^{\sum_i n_i}\prod_{i<j}^N \frac{t^{n_j}x_i - t^{n_i}x_j}{x_i-x_j}\prod_i^N \gamma^{n_i} \,.$$

where we have introduced the notation $\gamma_i = q^{-x_i\partial_i}$.

One more trigonometric limit $\omega \to 0$ being applied to provides (the trigonometric) Macdonald-Ruijsenaars operators. Then the generating function is represented in the following form:

$$D_N(u)\Big|_{\omega=0} = \left(\det\left[x_i^{N-j}\right]_{i,j=1}^{N}\right)^{-1} \det\left[x_i^{N-j}(1-ut^{j-1}\gamma_i)\right]_{i,j=1}^{N} . \tag{1}$$

For the latter model there exists a set of N commuting operators (the Cherednik operators)

$$C_i(t,q) = t^{i-1} R_{i,i+1}(t) \dots R_{iN}(t)\gamma_i R_{1,i}(t)^{-1} \dots R_{i-1,i}(t)^{-1} ,$$

acting on $\mathbb{C}[x_1, \dots, x_N]$, where the R-operators are of the form:

$$R_{ij}(t) = \frac{x_i - tx_j}{x_i - x_j} + \frac{(t-1)x_j}{x_i - x_j}\sigma_{ij} ,$$

and σ_{ij} permutes the variables x_i and x_j.

The commutativity of the Macdonald-Ruijsenaars operators for different values of spectral parameter u

$$\left[D_N(u) \Big|_{\omega=0}, D_N(u') \Big|_{\omega=0} \right] = 0$$

follows from the commutativity of $C_i(t,q)$ and the following relation between $D_N(u)\Big|_{\omega=0}$ and the Cherednik operators $C_i(t,q)$:

$$D_N(u)\Big|_{\omega=0} = \prod_{i=1}^{N} (1 - uC_i)\,\Big|_{\Lambda_N}\,, \qquad (2)$$

where Λ_N is the space of symmetric functions in variables $x_1, ..., x_N$.

Introduce the double-elliptic ($p=0$) version of the Cherednik operators:

$$\mathrm{P}\theta_\omega(uC_i) = \sum_{n\in\mathbb{Z}} \omega^{\frac{n^2-n}{2}} (-u)^n t^{n(i-1)} R_{i,i+1}(t^n)...R_{iN}(t^n)\gamma_i^n R_{1,i}(t^n)^{-1} \ldots R_{i-1,i}(t^n)^{-1}\,,$$

where $R_{ij}(t)$ is the previously defined trigonometric R-operator, and u is a spectral parameter.

These operators do not commute with each other. However, we prove the following relation:

$$D_N(u) = \prod_{i=1}^{N} \mathrm{P}\theta_\omega(uC_i)\Big|_{\Lambda_N} = \mathrm{P}\theta_\omega(uC_1)...\mathrm{P}\theta_\omega(uC_N)\Big|_{\Lambda_N}\,.$$

This allows to compute large N limit for the Dell model. See A. Grekov, A. Zotov, *On Cherednik and Nazarov-Sklyanin large N limit construction for double elliptic integrable system* arXiv:2102.06853.

Table of spectral dualities

dual spec param λ \ spectral param z	rat. XXX	trig XXZ	elliptic XYZ
Gaudin models (classical r-matrices)	XXX Gaudin	XXZ Gaudin	XYZ Gaudin
Spin chains Quantum R-matrices RTT relations	XXX chain	XXZ chain	XYZ chain
Dell spin chains	Algebraic structures are to be found		

Thank you!