Spin generalization of the elliptic Macdonald-Ruijsenaars operators and elliptic version of q-deformed Haldane-Shastry model

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Based on two recent joint papers with Maria Matushko:

M. Matushko, A. Zotov, *Anisotropic spin generalization of elliptic Macdonald-Ruijsenaars operators and R-matrix identities*, arXiv:2201.05944 [math.QA].
M. Matushko, A. Zotov, *Elliptic generalization of integrable q-deformed Haldane-Shastry long-range spin chain*, arXiv:2202.01177 [math-ph].

Plan of the talk:

- Macdonald-Ruijsenaars operators and many-body systems
- Commuting spin XYZ Macdonald-Ruijsenaars operators and R-matrix identities
- Classical analogues – relativistic interacting tops
- Polychronakos freezing trick and elliptic integrable long-range spin chains

Elliptic Macdonald-Ruijsenaars operators

For $i=1,\dots,N$ denote by p_i the shift operator acting on function $f(z_1,\dots,z_N)$ as follows:

$$(p_i f)(z_1,z_2,\dots z_N)=\exp\left(-\eta\frac{\partial}{\partial z_i}\right)f(z_1,\dots,z_N)=f(z_1,\dots,z_i-\eta,\dots,z_N). \qquad (1)$$

The Kronecker elliptic function on elliptic curve $\mathbb{C}/\Gamma$, $\Gamma=\mathbb{Z}\oplus\mathbb{Z}\tau$ with moduli τ ($\mathrm{Im}(\tau)>0$):

$$\phi(x,y)=\frac{\vartheta'(0)\vartheta(x+y)}{\vartheta(x)\vartheta(y)}, \quad \text{and denote} \quad \phi(z)=\phi(\hbar,z) \qquad (2)$$

S.N.M. Ruijsenaars proved (1987 Comm. Math. Phys.) commutativity for the following set of operators:

$$D_k=\sum_{|I|=k}\prod_{\substack{i\in I\\ j\notin I}}\phi(z_j-z_i)\prod_{i\in I}p_i, \qquad k=1,\dots,N, \qquad (3)$$

where the sum is taken over all subsets I of $\{1,\dots,N\}$ of size k. In the trigonometric limit the operators D_k turn into the Macdonald operators:

$$D_k^{Macd}=t^{\frac{k(k-N)}{2}}\sum_{|I|=k}\prod_{\substack{i\in I\\ j\notin I}}\frac{tx_i-x_j}{x_i-x_j}\prod_{i\in I}q^{x_i\partial_{x_i}}, \qquad k=1,\dots,N \qquad (4)$$

with $t=\exp(-2\pi\imath\hbar)$, $x_k=\exp(2\pi\imath z_k)$ and $q=\exp(-\eta)$.

Many-body systems:

The first Macdonald operator

$$D_1 = \sum_{j=1}^{N} \left(\prod_{k:k\neq j}^{N} \frac{tz_j - z_k}{z_j - z_k} \right) q^{z_j \partial_{z_j}} , \quad z_j = e^{2\pi\imath q_j} , \quad t = e^{2\pi\imath\eta} , \quad q = e^{\hbar/c} .$$

is the (first) Hamiltonian of the trigonometric Ruijsenaars-Schneider model. In the classical elliptic case it is the elliptic Ruijsenaars-Schneider model

$$H^{\mathrm{RS}} = \sum_{j=1}^{N} \left(\prod_{k:k\neq j}^{N} \frac{\vartheta(q_j - q_k - \eta)}{\vartheta(q_j - q_k)} \right) e^{v_j/c} .$$

In the non-relativistic limit we come to the Calogero-Moser-Sutherland model

$$H^{\mathrm{CM}} = \sum_{i=1}^{N} \frac{v_i^2}{2} - \nu^2 \sum_{i>j}^{N} \wp(q_i - q_j) .$$

At classical level our results are related to spin Ruijsenaars-Schneider model introduced by Krichever and Zabrodin. We will come to its anisotropic version.

Commutativity of elliptic Macdonald-Ruijsenaars operators

Following Ruijsenaars introduce notations. Let I, J be disjoint subsets of $\{1, \ldots, N\}$. Denote

$$(I, J) = \prod_{\substack{i \in I \\ j \in J}} \phi(z_i - z_j) \tag{5}$$

and

$$\mathbf{p}_I = \prod_{i \in I} p_i \,, \quad p_i = e^{-\eta \partial_{q_i}} \tag{6}$$

where p_i are shift operators. Then the operators D_k take the following form:

$$D_k = \sum_{I : |I| = k} (I^c, I)\, \mathbf{p}_I \,, \qquad k = 1, \ldots, N \,, \tag{7}$$

where I^c means the complement of a set I in $\{1, \ldots, N\}$, and $|I|$ is the number of elements in I. We also use the notations I_+ and I_- to highlight the shifts of all z_i, $i \in I$ by $\pm\eta$ respectively:

$$(I_\pm, J) = (\mathbf{p}_I^{\mp 1} I \mathbf{p}_I^{\pm 1}, J). \tag{8}$$

I_+ means that the arguments z_i with $i \in I$ are shifted as $z_i - \eta$.

The commutativity

$$[D_k, D_l] = 0 \qquad \forall k, l = 1, \dots, N \tag{9}$$

holds if and only if

$$\sum_{|I|=k} ((I^c, I)(I_-, I^c) - (I, I^c)(I_-^c, I)) = 0 \qquad \forall k \in \{1, \dots, N\} \quad \forall N. \tag{10}$$

It was argued that the identities considered as a functional equations for the function ϕ are reduced to a single equation, which provides (among meromorphic functions on elliptic curve) solution given by the elliptic Kronecker function only (up to some normalization factors). Trigonometric and rational solutions are obtained by degeneration

$$\phi^{\mathrm{trig}}(z, u) = \pi \cot(\pi z) + \pi \cot(\pi u)\,, \qquad \phi^{\mathrm{rat}}(z, u) = 1/z + 1/u\,. \tag{11}$$

Our plan is to construct spin generalization of D_k and prove their commutativity in a similar way - through some set of identities.

Spin XYZ elliptic Macdonald-Ruijsenaars operators. Main idea goes back to results by Uglov and Cherednik. It was formulated clearly in recant paper by Lamers, Pasquier and Serban. Our goal is to use representation of symmetric group through R-matrices.

By definition, any quantum R-matrix satisfies the quantum Yang-Baxter equation (QYB):

$$R^{\hbar}_{12}(u)R^{\hbar}_{13}(u+v)R^{\hbar}_{23}(v) = R^{\hbar}_{23}(v)R^{\hbar}_{13}(u+v)R^{\hbar}_{12}(u) \tag{12}$$

or

$$R^{\hbar}_{ij}(u)R^{\hbar}_{ik}(u+v)R^{\hbar}_{jk}(v) = R^{\hbar}_{jk}(v)R^{\hbar}_{ik}(u+v)R^{\hbar}_{ij}(u) \tag{13}$$

for any distinct integers $1 \leq i, j, k \leq N$. Here $R^{\hbar}_{ij}(u) \in \mathrm{End}(\mathcal{H})$, $\mathcal{H} = (\mathbb{C}^M)^{\otimes N}$. Also,

$$[R^{\hbar}_{ij}(u), R^{\hbar'}_{kl}(v)] = 0 \quad \text{for any distinct integers} \quad 1 \leq i, j, k, l \leq N \tag{14}$$

We deal with the elliptic GL_M Baxter-Belavin R-matrix normalized in a way that the unitarity property is as follows:

$$R^{\hbar}_{ij}(z)R^{\hbar}_{ji}(-z) = \mathrm{Id}\,\phi(\hbar, z)\phi(\hbar, -z) = \mathrm{Id}\,(\wp(\hbar) - \wp(z))\,, \tag{15}$$

where $\mathrm{Id} = 1_{M^N}$ is the identity matrix in $\mathrm{End}(\mathcal{H})$. In what follows we also use R-matrices $\bar{R}^{\hbar}_{ij}(z)$, which are related to $R^{\hbar}_{ij}(z)$ through

$$R^{\hbar}_{ij}(z) = \phi(\hbar, z)\bar{R}^{\hbar}_{ij}(z)\,. \tag{16}$$

Then

$$\bar{R}^{\hbar}_{ij}(z)\bar{R}^{\hbar}_{ji}(-z) = \mathrm{Id}\,. \tag{17}$$

In $M=2$ it is the Baxter's 8-vertex R-matrix. It has the form

$$R^{\hbar}_{12}(z)=\frac{1}{2}\Big(\varphi_{00}\,\sigma_0\otimes\sigma_0+\varphi_{01}\,\sigma_1\otimes\sigma_1+\varphi_{11}\,\sigma_2\otimes\sigma_2+\varphi_{10}\,\sigma_3\otimes\sigma_3\Big)\,, \tag{18}$$

$$\varphi_{00}=\phi(z,\frac{\hbar}{2})\,,\quad \varphi_{10}=\phi(z,\frac{1}{2}+\frac{\hbar}{2})\,,\quad \varphi_{01}=e^{\pi\imath z}\phi(z,\frac{\tau}{2}+\frac{\hbar}{2})\,,\quad \varphi_{11}=e^{\pi\imath z}\phi(z,\frac{1+\tau}{2}+\frac{\hbar}{2})\,.$$

In 4×4 form it is as follows:

$$R^{\hbar}_{12}(z)=\tfrac{1}{2}\left(\begin{array}{cccc} \varphi_{00}+\varphi_{10} & 0 & 0 & \varphi_{01}-\varphi_{11} \\ 0 & \varphi_{00}-\varphi_{10} & \varphi_{01}+\varphi_{11} & 0 \\ 0 & \varphi_{01}+\varphi_{11} & \varphi_{00}-\varphi_{10} & 0 \\ \varphi_{01}-\varphi_{11} & 0 & 0 & \varphi_{00}+\varphi_{10} \end{array}\right) \tag{19}$$

Our construction is valid for GL_M Baxter-Belavin R-matrix and its trigonometric and rational degenerations. The simplest one is the Yang's rational R-matrix

$$R^{\hbar}_{12}(z)=\frac{\mathrm{Id}}{\hbar}+\frac{P_{12}}{z}\,,\qquad \bar{R}^{\hbar}_{12}(z)=\frac{z\mathrm{Id}+\hbar P_{12}}{\hbar+z}\,. \tag{20}$$

In this case we will obtain Hamiltonians for the rational (isotropic) spin Ruijsenaars model. Possible degenerations of elliptic case include 11-vertex rational R-matrix, 7-vertex trigonometric R-matrix and their higher rank versions.

Spin operators. Consider symmetric group S_N generated by relations:

$$\sigma_{i-1}\sigma_i\sigma_{i-1} = \sigma_i\sigma_{i-1}\sigma_i \,, \tag{21}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{for } j \neq i \pm 1 \tag{22}$$

and

$$(\sigma_i)^2 = 1\,. \tag{23}$$

Obviously, it has representation $\sigma_i = s_{i,i+1}$, where $s_{i,i+1}$, $i = 1, ..., N-1$ are permutations of variables $z_1, ..., z_N$:

$$s_{i,i+1} f(z_1, ..., z_i, z_{i+1}, ..., z_N) = f(z_1, ..., z_{i+1}, z_i, ..., z_N)\,. \tag{24}$$

Here f is any function (for which the action of D_k operators is well-defined). Denote by s_ω the permutation operator representing $w \in S_N$. For example, for the cycle $(12\ldots j)$ we have

$$s_{(12\ldots j)} = s_{12}\, s_{23}\, \ldots\, s_{j-1,j}\,. \tag{25}$$

For $i_1 < i_2 < \cdots < i_k$ denote by $\{i_1, i_2, \ldots, i_k\} \in S_N$ the (shortest) permutation $i_m \to m$ for all $1 \leq m \leq k$. It can be presented as a product of cycles:

$$\{i_1, i_2, \ldots, i_k\} = (k, \ldots, i_k)(k-1, \ldots, i_{k-1}) \ldots (2, \ldots, i_2)(1, \ldots, i_1)\,. \tag{26}$$

The Macdonald-Ruijsenaars operators are symmetric with respect to the action of permutation group. Therefore, each of these operators can be represented as a sum over certain permutations acting on some "first" term:

$$D_k = \sum_{i_1<i_2<\cdots<i_k} s^{-1}_{\{i_1,i_2,\ldots,i_k\}}(I_0^c, I_0)\,\mathbf{p}_{I_0}\, s_{\{i_1,i_2,\ldots,i_k\}}\,, \tag{27}$$

where we denote by $I_0 = \{1, 2, \ldots, k\}$ the subset in k elements and its complement $I_0^c = \{k+1, \ldots N\}$. The sum is over all (ordered) k-element subsets of $\{1, \ldots, N\}$.
Another well known representation of the (braid) relations is given by
$\sigma_i = R^{\hbar}_{i,i+1}(z_i - z_{i+1})P_{i,i+1} \in \mathrm{End}(\mathcal{H})$, $\mathcal{H} = (\mathbb{C}^M)^{\otimes N}$, where P_{ij} are permutation matrix-valued operators. In this case (21)-(22) are equivalent to the Yang-Baxter equations. If the R-matrix entering representation is unitary, then the involution property holds as well.
Consider representation given by the composition of the previously discussed:

$$\sigma_i = \bar{R}_{i,i+1}(z_i - z_{i+1})P_{i,i+1}s_{i,i+1}\,. \tag{28}$$

Then

$$\begin{aligned}\sigma_{(12\ldots j)} = \sigma_{12}\sigma_{23}\ldots\sigma_{j-1,j} = \bar{R}_{12}(z_1 - z_2)\bar{R}_{13}(z_1 - z_3)\ldots\bar{R}_{1j}(z_1 - z_j)P_{(12\ldots j)}s_{(12\ldots j)} = \\ = P_{(12\ldots j)}s_{(12\ldots j)}\bar{R}_{j1}(z_j - z_1)\bar{R}_{j2}(z_j - z_2)\ldots\bar{R}_{j,j-1}(z_j - z_{j-1})\,.\end{aligned} \tag{29}$$

Introduce the operators $\mathcal{D}_k$ (matrix generalization of the scalar operators acting in $\mathrm{End}(\mathcal{H})$):

$$\mathcal{D}_k = \sum_{i_1<i_2<\dots<i_k} \sigma^{-1}_{\{i_1,i_2,\dots,i_k\}}(I_0^c, I_0)\,\mathbf{p}_{I_0}\,\sigma_{\{i_1,i_2,\dots,i_k\}}\,. \tag{30}$$

After some transformations the dependence on permutations s_{ij} and P_{ij} are cancelled out:

$$\mathcal{D}_k = \sum_{1\le i_1<\dots<i_k\le N} \left(\prod_{\substack{j=1\\ j\neq i_1\dots i_{k-1}}}^{N} \phi(z_j - z_{i_1})\,\phi(z_j - z_{i_2})\,\cdots\,\phi(z_j - z_{i_k}) \right)\times$$

$$\times \left(\overleftarrow{\prod_{j_1=1}^{i_1-1}} \bar{R}_{j_1 i_1} \overleftarrow{\prod_{\substack{j_2=1\\ j_2\neq i_1}}^{i_2-1}} \bar{R}_{j_2 i_2} \dots \overleftarrow{\prod_{\substack{j_k=1\\ j_k\neq i_1\dots i_{k-1}}}^{i_k-1}} \bar{R}_{j_k i_k} \right)\times \tag{31}$$

$$\times p_{i_1}\cdot p_{i_2}\cdots p_{i_k}\times \left(\overrightarrow{\prod_{\substack{j_k=1\\ j_k\neq i_1\dots i_{k-1}}}^{i_k-1}} \bar{R}_{i_k j_k} \overrightarrow{\prod_{\substack{j_{k-1}=1\\ j_{k-1}\neq i_1\dots i_{k-2}}}^{i_{k-1}-1}} \bar{R}_{i_{k-1} j_{k-1}} \dots \overrightarrow{\prod_{j_1=1}^{i_1-1}} \bar{R}_{i_1 j_1} \right),$$

where $k = 1, \dots, N$ and $\bar{R}_{ij} = \bar{R}^{\hbar}_{ij}(z_i - z_j)$. In the scalar case $(M = 1)$ $\bar{R}_{ij} = 1$ and (31) coincides with the Macdonald-Ruijsenaars operators.

Introduce new short notations (useful for the proof of commutativity). For any pair I, J of disjoint subsets in $\{1, \dots, N\}$ define the product

$$\mathcal{R}_{I,J} = \prod_{i \in I, j \in J, i<j} R_{ij}\left(z_i - z_j\right) , \tag{32}$$

where the ordering of R-matrices is as follows:

$$\mathcal{R}_{I,J} = \overleftarrow{\prod_{\substack{i_1 \in I \\ i_1 < j_1}}} R_{i_1,j_1}\left(z_{i_1} - z_{j_1}\right) \overleftarrow{\prod_{\substack{i_2 \in I \\ i_2 < j_2}}} R_{i_2,j_2}\left(z_{i_2} - z_{j_2}\right) \dots \overleftarrow{\prod_{\substack{i_k \in I \\ i_k < j_k}}} R_{i_k,j_k}\left(z_{i_k} - z_{j_k}\right) . \tag{33}$$

and arrows mean the ordering $\overrightarrow{\prod_{j=1}^{N}} R_{ij} = R_{i1} R_{i2} \dots R_{iN}$ and $\overleftarrow{\prod_{j=1}^{N}} R_{ij} = R_{iN} R_{i,N-1} \dots R_{i1}$

Here $J = \{j_1, j_2, \dots, j_k\}$ and the elements j_m are in increasing order $j_1 < j_2 < \dots < j_k$. Let also $I = \{i_1, i_2, \dots, i_l\}$ and $i_1 < i_2 < \dots < i_l$. In what follows we assume that the above ordering of indices in I and J is fixed. By moving R-matrices the definition (33) is equivalently rewritten as

$$\mathcal{R}_{I,J} = \overrightarrow{\prod_{\substack{j_l \in J \\ j_l > i_l}}} R_{i_l,j_l}\left(z_{i_l} - z_{j_l}\right) \overrightarrow{\prod_{\substack{j_{l-1} \in J \\ j_{l-1} > i_{l-1}}}} R_{i_{l-1},j_{l-1}}\left(z_{i_{l-1}} - z_{j_{l-1}}\right) \dots \overrightarrow{\prod_{\substack{j_1 \in J \\ j_1 > i_1}}} R_{i_1,j_1}\left(z_{i_1} - z_{j_1}\right) . \tag{34}$$

Similarly, define the product

$$\mathcal{R}'_{I,J} = \prod_{i \in I, j \in J, i > j} R_{ij}\left(z_i - z_j\right) \,, \tag{35}$$

with the following ordering:

$$\mathcal{R}'_{I,J} = \overrightarrow{\prod_{\substack{j_l \in J \\ j_l < i_l}}} R_{i_l,j_l}(z_{i_l} - z_{j_l}) \overrightarrow{\prod_{\substack{j_{l-1} \in J \\ j_{l-1} < i_{l-1}}}} R_{i_{l-1},j_{l-1}}(z_{i_{l-1}} - z_{j_{l-1}}) \ldots \overrightarrow{\prod_{\substack{j_1 \in J \\ j_1 < i_1}}} R_{i_1,j_1}(z_{i_1} - z_{j_1})\,. \tag{36}$$

Again, we may rewrite it equivalently as

$$\mathcal{R}'_{I,J} = \overleftarrow{\prod_{\substack{i_1 \in I \\ i_1 > j_1}}} R_{i_1,j_1}\left(z_{i_1} - z_{j_1}\right) \overleftarrow{\prod_{\substack{i_2 \in I \\ i_2 > j_2}}} R_{i_2,j_2}\left(z_{i_2} - z_{j_2}\right) \ldots \overleftarrow{\prod_{\substack{i_k \in I \\ i_k > j_k}}} R_{i_k,j_k}\left(z_{i_k} - z_{j_k}\right) \,. \tag{37}$$

The product of $\mathcal{R}'_{I,J}$ and $\mathcal{R}_{I,J}$ provides R-matrix analogue for the notation (I, J) used in the scalar case.

It can be verified that

$$\mathcal{R}'_{I,J}\mathcal{R}_{I,J} = \overleftarrow{\prod_{i_1\in I}} R_{i_1,j_1}(z_{i_1}-z_{j_1}) \overleftarrow{\prod_{i_2\in I}} R_{i_2,j_2}(z_{i_2}-z_{j_2}) \dots \overleftarrow{\prod_{i_k\in I}} R_{i_k,j_k}(z_{i_k}-z_{j_k}) \tag{38}$$

and

$$\mathcal{R}'_{I,J}\mathcal{R}_{I,J} = \overrightarrow{\prod_{j_l\in J}} R_{i_l,j_l}(z_{i_l}-z_{j_l}) \overrightarrow{\prod_{j_{l-1}\in J}} R_{i_{l-1},j_{l-1}}(z_{i_{l-1}}-z_{j_{l-1}}) \dots \overrightarrow{\prod_{j_1\in J}} R_{i_1,j_1}(z_{i_1}-z_{j_1})\,. \tag{39}$$

Up till now we did not use the Yang-Baxter equation. Let us formulate it for the above products since it plays a key role in deriving and proving R-matrix identities.
Let $R(u)$ be an R-matrix satisfying the quantum Yang-Baxter equation and $\mathcal{R}_{A,B}$ and $\mathcal{R}'_{A,B}$ are the corresponding products of R-matrices defined by (33) and (36) respectively. For any disjoint subsets A, B, C of $\{1, 2, \dots, N\}$ the following identities hold true:

$$\mathcal{R}_{C,A\cup B}\mathcal{R}_{B,A} = \mathcal{R}_{B\cup C,A}\mathcal{R}_{C,B}\,, \tag{40}$$

$$\mathcal{R}'_{A,B}\mathcal{R}'_{A\cup B,C} = \mathcal{R}'_{B,C}\mathcal{R}'_{A,B\cup C}\,. \tag{41}$$

Notice that we did not use the unitarity property in the above definitions and properties. For a unitary R-matrix satisfying (15) we have (in addition to all the above mentioned statements):

$$\mathcal{R}_{I,J}\mathcal{R}'_{J,I} = \mathrm{Id} \prod_{\substack{i\in I, j\in J \\ i<j}} \phi(h, z_i - z_j)\phi(h, z_j - z_i) \tag{42}$$

and

$$\mathcal{R}'_{I,J}\mathcal{R}_{J,I} = \mathrm{Id} \prod_{\substack{i\in I, j\in J \\ i>j}} \phi(h, z_i - z_j)\phi(h, z_j - z_i)\,. \tag{43}$$

All the same notations are used for the normalized unitary R-matrix $\bar{R}^h(u)$. In this case we have

$$\bar{\mathcal{R}}_{I,J}\bar{\mathcal{R}}'_{J,I} = \bar{\mathcal{R}}'_{J,I}\bar{\mathcal{R}}_{I,J} = \mathrm{Id}\,. \tag{44}$$

In the above given notations our difference operators take the form:

$$\mathcal{D}_k = \sum_{|I|=k} (I^c, I) \cdot \bar{\mathcal{R}}_{I^c,I} \cdot \mathbf{p}_I \cdot \bar{\mathcal{R}}'_{I,I^c} \,, \tag{45}$$

where R-matrices are those with bars ($\bar{R}_{ij}\bar{R}_{ji} = \mathrm{Id}$).

Example. $N = 2$:

$$\mathcal{D}_1 = \phi(z_2 - z_1)p_1 + \phi(z_1 - z_2)\bar{R}^{\hbar}_{12}(z_1 - z_2)\, p_2 \bar{R}^{\hbar}_{21}(z_2 - z_1) \,, \tag{46}$$

and $\mathcal{D}_2 = p_1 p_2$.

Example. $N = 3$:

$$\begin{aligned}
\mathcal{D}_1 \;&=\; \phi(z_2 - z_1)\phi(z_3 - z_1)p_1 + \\
&+\phi(z_1 - z_2)\phi(z_3 - z_2)\bar{R}^{\hbar}_{12}(z_1 - z_2)\, p_2 \bar{R}^{\hbar}_{21}(z_2 - z_1) + \\
&+\phi(z_1 - z_3)\phi(z_2 - z_3)\bar{R}^{\hbar}_{23}(z_2 - z_3)\, \bar{R}^{\hbar}_{13}(z_1 - z_3)\, p_3 \bar{R}^{\hbar}_{31}(z_3 - z_1)\, \bar{R}^{\hbar}_{32}(z_3 - z_2) \,, \\
\mathcal{D}_2 \;&=\; \phi(z_3 - z_1)\phi(z_3 - z_2)p_1 p_2 + \\
&+\phi(z_2 - z_1)\phi(z_2 - z_3)\bar{R}^{\hbar}_{23}(z_2 - z_3)\, p_1 p_3 \bar{R}^{\hbar}_{32}(z_3 - z_2) + \\
&+\phi(z_1 - z_2)\phi(z_1 - z_3)\bar{R}^{\hbar}_{12}(z_1 - z_2)\, \bar{R}^{\hbar}_{13}(z_1 - z_3)\, p_2 p_3 \bar{R}^{\hbar}_{31}(z_3 - z_1)\, \bar{R}^{\hbar}_{21}(z_2 - z_1) \,,
\end{aligned}$$

and $\mathcal{D}_3 = p_1 p_2 p_3$.

Example. $N = 4$:

$$\begin{aligned}
\mathcal{D}_1 &= \phi(z_{21})\phi(z_{31})\phi(z_{41})p_1+\phi(z_{12})\phi(z_{32})\phi(z_{42})\bar{R}_{12}p_2\bar{R}_{21}+ \\
&+\phi(z_{13})\phi(z_{23})\phi(z_{43})\bar{R}_{23}\bar{R}_{13}p_3\bar{R}_{31}\bar{R}_{32}+\phi(z_{14})\phi(z_{24})\phi(z_{34})\bar{R}_{34}\bar{R}_{24}\bar{R}_{14}p_4\bar{R}_{41}\bar{R}_{42}\bar{R}_{43}\,, \\
\mathcal{D}_2 &= \phi(z_{31})\phi(z_{32})\phi(z_{41})\phi(z_{42})p_1p_2 + \phi(z_{21})\phi(z_{23})\phi(z_{41})\phi(z_{43})\bar{R}_{23}p_1p_3\bar{R}_{32}+ \\
&+\phi(z_{21})\phi(z_{24})\phi(z_{31})\phi(z_{34})\bar{R}_{34}\bar{R}_{24}p_1p_4\bar{R}_{42}\bar{R}_{43}+ \\
&+\phi(z_{12})\phi(z_{13})\phi(z_{42})\phi(z_{43})\bar{R}_{12}\bar{R}_{13}p_2p_3\bar{R}_{31}\bar{R}_{21}+ \\
&+\phi(z_{12})\phi(z_{14})\phi(z_{32})\phi(z_{34})\bar{R}_{12}\bar{R}_{34}\bar{R}_{14}p_2p_4\bar{R}_{41}\bar{R}_{43}\bar{R}_{21}+ \\
&+\phi(z_{13})\phi(z_{14})\phi(z_{23})\phi(z_{24})\bar{R}_{23}\bar{R}_{13}\bar{R}_{24}\bar{R}_{14}p_3p_4\bar{R}_{41}\bar{R}_{42}\bar{R}_{31}\bar{R}_{32}\,, \\
\mathcal{D}_3 &= \phi(z_{41})\phi(z_{42})\phi(z_{43})p_1p_2p_3 + \phi(z_{31})\phi(z_{32})\phi(z_{34})\bar{R}_{34}p_1p_2p_4\bar{R}_{43}+ \\
&+\phi(z_{21})\phi(z_{23})\phi(z_{24})\bar{R}_{23}\bar{R}_{24}p_1p_3p_4\bar{R}_{42}\bar{R}_{32}+ \\
&+\phi(z_{12})\phi(z_{13})\phi(z_{14})\bar{R}_{12}\bar{R}_{13}\bar{R}_{14}p_2p_3p_4\bar{R}_{41}\bar{R}_{31}\bar{R}_{21}
\end{aligned}$$

and $\mathcal{D}_4 = p_1p_2p_3p_4$.

Identities

The operators $\mathcal{D}_k$ commute with each other iff the following set of identities holds for any $k = 1, 2, \ldots m$ and any $m \leq N$:

$$\sum_{|I|=k} \left(\mathcal{R}_{I^c,I} \cdot \mathcal{R}'_{I_-,I^c} \cdot \mathcal{R}_{I_-,I^c} \cdot \mathcal{R}'_{I^c,I} - \mathcal{R}_{I,I^c} \cdot \mathcal{R}'_{I^c_-,I} \cdot \mathcal{R}_{I^c_-,I} \cdot \mathcal{R}'_{I,I^c} \right) = 0. \tag{47}$$

It is important that R-matrices here are without bars (not $\bar{R}$. That is in $M = 1$ case these are the Ruijsenaars identities.

Example. $k = 1$ $N = 3$:

$$\begin{aligned}
R^h_{12}(z_1 - z_2)R^h_{13}(z_1 - z_3)R^h_{31}(z_3 - z_1 - \eta)R^h_{21}(z_2 - z_1 - \eta) \\
-R^h_{12}(z_1 - z_2 - \eta)R^h_{13}(z_1 - z_3 - \eta)R^h_{31}(z_3 - z_1)R^h_{21}(z_2 - z_1) \\
+R^h_{23}(z_2 - z_3)R^h_{32}(z_3 - z_2 - \eta)R^h_{12}(z_1 - z_2 - \eta)R^h_{21}(z_2 - z_1) \\
-R^h_{12}(z_1 - z_2)R^h_{21}(z_2 - z_1 - \eta)R^h_{23}(z_2 - z_3 - \eta)R^h_{32}(z_3 - z_2) \\
+R^h_{23}(z_2 - z_3 - \eta)R^h_{13}(z_1 - z_3 - \eta)R^h_{31}(z_3 - z_1)R^h_{32}(z_3 - z_2) \\
-R^h_{23}(z_2 - z_3)R^h_{13}(z_1 - z_3)R^h_{31}(z_3 - z_1 - \eta)R^h_{32}(z_3 - z_2 - \eta) = 0
\end{aligned} \tag{48}$$

Example. $k = 1$ $N = 4$:

$$
\begin{aligned}
&R^h_{12}(z_1 - z_2 - \eta)R^h_{13}(z_1 - z_3 - \eta)R^h_{14}(z_1 - z_4 - \eta)R^h_{41}(z_4 - z_1)R^h_{31}(z_3 - z_1)R^h_{21}(z_2 - z_1) \\
&-R^h_{12}(z_1 - z_2)R^h_{13}(z_1 - z_3)R^h_{14}(z_1 - z_4)R^h_{41}(z_4 - z_1 - \eta)R^h_{31}(z_3 - z_1 - \eta)R^h_{21}(z_2 - z_1 - \eta) \\
&+R^h_{12}(z_1 - z_2)R^h_{21}(z_2 - z_1 - \eta)R^h_{23}(z_2 - z_3 - \eta)R^h_{24}(z_2 - z_4 - \eta)R^h_{42}(z_4 - z_2)R^h_{32}(z_3 - z_2) \\
&-R^h_{23}(z_2 - z_3)R^h_{24}(z_2 - z_4)R^h_{42}(z_4 - z_2 - \eta)R^h_{32}(z_3 - z_2 - \eta)R^h_{12}(z_1 - z_2 - \eta)R^h_{21}(z_2 - z_1) \\
&+R^h_{23}(z_2 - z_3)R^h_{13}(z_1 - z_3)R^h_{31}(z_3 - z_1 - \eta)R^h_{32}(z_3 - z_2 - \eta)R^h_{34}(z_3 - z_4 - \eta)R^h_{43}(z_4 - z_3) \\
&-R^h_{34}(z_3 - z_4)R^h_{43}(z_4 - z_3 - \eta)R^h_{23}(z_2 - z_3 - \eta)R^h_{13}(z_1 - z_3 - \eta)R^h_{31}(z_3 - z_1)R^h_{32}(z_3 - z_2) \\
&+R^h_{34}(z_3 - z_4)R^h_{24}(z_2 - z_4)R^h_{14}(z_1 - z_4)R^h_{41}(z_4 - z_1 - \eta)R^h_{42}(z_4 - z_2 - \eta)R^h_{43}(z_4 - z_3 - \eta) \\
&-R^h_{34}(z_3 - z_4 - \eta)R^h_{24}(z_2 - z_4 - \eta)R^h_{14}(z_1 - z_4 - \eta)R^h_{41}(z_4 - z_1)R^h_{42}(z_4 - z_2)R^h_{43}(z_4 - z_3) \\
&= 0 \quad .
\end{aligned}
\tag{49}
$$

Example. $k=2$ $N=5$:

$$
\begin{gathered}
R^-_{23}R^-_{24}R^-_{25}R^-_{13}R^-_{14}R^-_{15}R_{51}R_{41}R_{31}R_{52}R_{42}R_{32} - R_{23}R_{24}R_{25}R_{13}R_{14}R_{15}R^-_{51}R^-_{41}R^-_{31}R^-_{52}R^-_{42}R^-_{32} \\
+R_{23}R^-_{32}R^-_{34}R^-_{35}R^-_{12}R^-_{14}R^-_{15}R_{51}R_{41}R_{21}R_{53}R_{43} - R_{34}R_{35}R_{12}R_{14}R_{15}R^-_{51}R^-_{41}R^-_{21}R^-_{53}R^-_{43}R^-_{23}R_{32} \\
+R_{34}R_{24}R^-_{42}R^-_{43}R^-_{45}R^-_{12}R^-_{13}R^-_{15}R_{51}R_{31}R_{21}R_{54} - R_{45}R_{12}R_{13}R_{15}R^-_{51}R^-_{31}R^-_{21}R^-_{54}R^-_{34}R^-_{24}R_{42}R_{43} \\
+R_{45}R_{35}R_{25}R^-_{52}R^-_{53}R^-_{54}R^-_{12}R^-_{13}R^-_{14}R_{41}R_{31}R_{21} - R_{12}R_{13}R_{14}R^-_{41}R^-_{31}R^-_{21}R^-_{45}R^-_{35}R^-_{25}R_{52}R_{53}R_{54} \\
+R_{12}R_{13}R^-_{31}R^-_{21}R^-_{34}R^-_{35}R^-_{24}R^-_{25}R_{52}R_{42}R_{53}R_{43} - R_{34}R_{35}R_{24}R_{25}R^-_{52}R^-_{42}R^-_{53}R^-_{43}R^-_{12}R^-_{13}R_{31}R_{21} \\
+R_{12}R_{34}R_{14}R^-_{41}R^-_{43}R^-_{21}R^-_{45}R^-_{23}R^-_{25}R_{52}R_{32}R_{54} - R_{45}R_{23}R_{25}R^-_{52}R^-_{32}R^-_{54}R^-_{12}R^-_{34}R^-_{14}R_{41}R_{43}R_{21} \\
+R_{12}R_{45}R_{35}R_{15}R^-_{51}R^-_{53}R^-_{54}R^-_{21}R^-_{23}R^-_{24}R_{42}R_{32} - R_{23}R_{24}R^-_{42}R^-_{32}R^-_{12}R^-_{45}R^-_{35}R^-_{15}R_{51}R_{53}R_{54}R_{21} \\
+R_{23}R_{13}R_{24}R_{14}R^-_{41}R^-_{42}R^-_{31}R^-_{32}R^-_{45}R^-_{35}R_{53}R_{54} - R_{45}R_{35}R^-_{53}R^-_{54}R^-_{23}R^-_{13}R^-_{24}R^-_{14}R_{41}R_{42}R_{31}R_{32} \\
+R_{23}R_{13}R_{45}R_{25}R_{15}R^-_{51}R^-_{52}R^-_{54}R^-_{31}R^-_{32}R^-_{34}R_{43} - R_{34}R^-_{43}R^-_{23}R^-_{13}R^-_{45}R^-_{25}R^-_{15}R_{51}R_{52}R_{54}R_{31}R_{32} \\
+R_{34}R_{24}R_{14}R_{35}R_{25}R_{15}R^-_{51}R^-_{52}R^-_{53}R^-_{41}R^-_{42}R^-_{43} - R^-_{34}R^-_{24}R^-_{14}R^-_{35}R^-_{25}R^-_{15}R_{51}R_{52}R_{53}R_{41}R_{42}R_{43} =
\end{gathered}
$$

$$= 0$$

Here $R^-_{ij}(z) = R^\hbar_{ij}(z-\eta)$.
The proof of R-matrix identities is given in our first paper.

What kind of spin many-body systems appeared?

Classical integrability: Lax matrices $L(z)$ - $N \times N$ matrix (function of z):

$$\dot{L}(z) = [L(z), M(z)] \quad - \quad \text{equations of motion } \forall z$$

Then $\operatorname{tr} L^k(z)$ – (generating functions in z of) conservation laws.
For the Calogero-Moser model we have the Krichever's Lax pair with spectral parameter:

$$L^{\rm CM}_{ij} = p_i \delta_{ij} + \nu(1-\delta_{ij})\phi(z, q_{ij})\,.$$

$$M^{\rm CM}_{ij} = \nu d_i\, \delta_{ij} + \nu(1-\delta_{ij})\phi'(z, q_{ij})\,, \quad d_i = \sum_{k\neq i} \wp(q_{ik})\,,$$

$$L^{\rm CM}(z) = \begin{pmatrix} p_1 & \nu\phi(z, q_1-q_2) & \nu\phi(z, q_1-q_3) & \dots & \nu\phi(z, q_1-q_N) \\ \nu\phi(z, q_2-q_1) & p_2 & \nu\phi(z, q_2-q_3) & \dots & \nu\phi(z, q_2-q_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu\phi(z, q_N-q_1) & \nu\phi(z, q_N-q_2) & \nu\phi(z, q_N-q_3) & \dots & p_N \end{pmatrix}$$

The Kronecker function is the Green function for the operator $\bar{\partial}$ with the above given boundary conditions:

$$\bar{\partial}\phi(z,u) = \delta^2(z,\bar{z})\,.$$

Similarly, the Lax matrix on elliptic curve is a section $L(z) \in \Gamma(\mathrm{End}V)$ of some holomorphic vector bundle V. It has simple pole at $z=0$, and it is fixed by its residue and boundary conditions:

$$\bar{\partial}L(z) = S\delta^2(z,\bar{z})\,, \quad \operatorname*{Res}_{z=0} L(z) = S\,,$$

$$L(z+1) = g_1 L(z) g_1^{-1}\,, \quad L(z+\tau) = g_\tau L(z) g_\tau^{-1}\,.$$

This viewpoint provides Hitchin type approach to many-body systems and their generalizations. It was developed by Gorsky-Nekrasov and Levin-Olshanetsky.

Geometrical interpretation also shows how these models could be extended and classified.

Classification of holomorphic bundles over elliptic curve is due to Atiyah.

$\underline{deg(V) = 0}$: $g_1 = 1_N$, $g_\tau = \mathrm{diag}(e^{-2\pi\imath q_1}, ..., e^{-2\pi\imath q_N})$

In this case the integrable system is the spin Calogero-Moser model
The residue matrix S is an element of Lie coalgebra, and when the Casimir functions are fixed it reduces to a coadjoint orbit. The Poisson structure is given by the Poisson-Lie brackets: $\{S_{ij}, S_{kl}\} = -S_{il}\delta_{kj} + S_{kj}\delta_{il}$ and canonical $\{p_i, q_j\} = \delta_{ij}$.

$$H^{\mathrm{spin}} = \sum_{i=1}^{N} \frac{p_i^2}{2} - \sum_{i>j}^{N} S_{ij}S_{ji}\wp(q_i - q_j)\,.$$

In the case of orbit of minimal dimension $S_{ij} = a_i b_j$ the model turns into the spinless CM.
In quantum case the potential contains spin exchange operator

$$\hat{H}^{\mathrm{spin}} = \sum_{i=1}^{N} \frac{\hbar^2\partial_{q_i}^2}{2} - \sum_{i>j}^{N} P_{ij}\wp(q_i - q_j)\,, \quad P_{12} = \sum_{k,l=1}^{M} E_{kl} \otimes E_{lk}$$

Permutation operator $P_{12}(u \otimes v) = v \otimes u$.

$\underline{deg = 1}$: $g_1 = Q$, $g_\tau = \Lambda$ (Heisenberg group: $\exp(\frac{2\pi\imath}{N})Q\Lambda = \Lambda Q$)

$$Q_{jk} = \delta_{jk} \exp\left(\frac{2\pi i}{N} k\right), \quad \Lambda_{jk} = \delta_{j-k+1=0 \mod N}, \quad Q^N = \Lambda^N = 1_N .$$

Integrable systems are tops like models. Dynamical variables: $S = \sum S_{ij} E_{ij} \in \mathrm{Mat}(M)$

Euler-Arnold equations:

$$\dot{S} = [S, J(S)] , \qquad J(S) = \sum J_{ij,kl} S_{kl} E_{ij}$$

J – inverse inertia tensor. The equations comes from the Hamiltonian $H = \frac{1}{2}\,\mathrm{tr}(SJ(S))$ and the linear Poisson brackets $\{S_{ij}, S_{kl}\} = \delta_{il} S_{kj} - \delta_{kj} S_{il}$.

Using special matrix basis

$$T_\alpha = \exp\left(\alpha_1 \alpha_2 \frac{\pi i}{N}\right) Q^{\alpha_1} \Lambda^{\alpha_2}, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_N \times \mathbb{Z}_N, \quad T_0 = T_{(0,0)} = 1_N,$$

we come to the Sklyanin's type Lax matrix:

$$L(z) = \sum_\alpha S_\alpha T_\alpha \varphi_\alpha(z) , \quad \varphi_\alpha = \exp(2\pi\imath z \frac{\alpha_2}{N}) \phi(z, \omega_\alpha) , \quad \omega_\alpha = \frac{\alpha_1 + \alpha_2 \tau}{N}$$

$deg = M$, $rk = NM$: g.c.d.(rk,deg)=M: Intermediate case: interacting tops

It can be viewed as anisotropic version of the spin Calogero model: N interacting GL_M tops, i.e. $S_{ij} \to \mathcal{S}^{ij} \in \mathrm{Mat}_M$ and

$$\mathcal{H}^{\text{tops}} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i=1}^{N} H^{\text{top}}(\mathcal{S}^{ii}) + \frac{1}{2} \sum_{i,j:\, i\neq j}^{N} \mathcal{V}(\mathcal{S}^{ii}, \mathcal{S}^{jj}, q_i - q_j)\,.$$

$$H^{tops} = \sum_{i=1}^{M} \frac{p_i^2}{2} - \frac{1}{2} \sum_{i=1}^{M} \sum_{\alpha\neq 0} S_\alpha^{ii} S_{-\alpha}^{ii} \wp(\omega_\alpha) - \frac{1}{2N} \sum_{i\neq j}^{M} \sum_{\alpha,\beta} \kappa_{\alpha,\beta}^2 S_\beta^{jj} S_{-\beta}^{ii} \wp(\omega_\alpha + \frac{q_{ij}}{N})\,.$$

$N = 1$ case is the single Euler-Arnold top

$M = 1$ case is the (spin) Calogero-Moser model

$$\mathcal{L}(z) = \begin{pmatrix} \mathcal{L}^{11}(z) & \mathcal{L}^{12}(z) & \dots & \mathcal{L}^{1M}(z) \\ \mathcal{L}^{21}(z) & \mathcal{L}^{22}(z) & \dots & \mathcal{L}^{2M}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}^{M1}(z) & \mathcal{L}^{M2}(z) & \dots & \mathcal{L}^{MM}(z) \end{pmatrix} \Bigg\} \begin{array}{c}\text{in one column} \\ M \text{ blocks} \\ \text{of size } N \times N\end{array}$$

Relativistic interacting tops.

$$\mathcal{L}(z)=\sum_{i,j=1}^{M}E_{ij}\otimes\mathcal{L}^{ij}(z)\in\mathrm{Mat}(NM,\mathbb{C})\,,\quad \mathcal{L}^{ij}(z)\in\mathrm{Mat}(N,\mathbb{C})\,.$$

$$\mathcal{L}^{ij}(z)=\sum_{\alpha}T_\alpha\mathcal{S}^{ij}_\alpha\varphi_\alpha(z,\omega_\alpha+q_{ij}+\eta)\,,\quad q_{ij}=q_i-q_j\,,\quad \omega_\alpha=\frac{\alpha_1+\alpha_2\tau}{N}\,,$$

By introducing

$$J^{\eta,\,q_{ij}}(\mathcal{S}^{ij})=\sum_{\alpha}T_\alpha\mathcal{S}^{ij}_\alpha\Big(E_1(\omega_\alpha+q_{ij}+\eta)-E_1(\omega_\alpha+q_{ij})\Big)\,,\quad E_1(x)=\vartheta'(x)/\vartheta(x)$$

equations of motion take the form

$$\dot{\mathcal{S}}^{ij}=\mathcal{S}^{ij}J^{\eta}(\mathcal{S}^{jj})-J^{\eta}(\mathcal{S}^{ii})\mathcal{S}^{ij}+\sum_{k:k\neq j}^{M}\mathcal{S}^{ik}J^{\eta,\,q_{kj}}(\mathcal{S}^{kj})-\sum_{k:k\neq i}^{M}J^{\eta,\,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{kj}\,.$$

$$\ddot{q}_i=\frac{1}{N}\,\mathrm{tr}\Big(\dot{\mathcal{S}}^{ii}\Big)=\frac{1}{N}\sum_{k:k\neq i}^{M}\mathrm{tr}\Big(\mathcal{S}^{ik}J^{\eta,\,q_{ki}}(\mathcal{S}^{ki})-J^{\eta,\,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{ki}\Big)\,,$$

For $M=1$ case these are equations of motion introduced by Krichever and Zabrodin.
For $N=1$ one obtains relativistic top described by the classical Sklyanin algebra.

Long-range spin chains

The Hamiltonian of the Haldane-Shastry spin chain

$$H^{\rm HS} = \frac{1}{2} \sum_{i \neq j}^{N} \frac{1 - P_{ij}}{\sin^2(\pi(x_i - x_j))} \tag{50}$$

describes pairwise interaction of N spins being attached to equidistant points on a circle: $x_k = k/N$, $k = 1, ..., N$. Here P_{ij} are the permutation operators (or spin exchange operators), which act on the Hilbert space $\mathcal{H} = (\mathbb{C}^M)^{\otimes N}$ by permuting i-th and j-th tensor components. For su_2 case $M = 2$ and

$$P_{ij} = \frac{1}{2} \sum_{a=0}^{3} \sigma_a^{(i)} \sigma_a^{(j)} , \qquad \sigma_a^{(i)} = \underbrace{1_2 \otimes ... 1_2 \otimes \sigma_a \otimes 1_2 ... \otimes 1_2}_{\sigma_a \text{ is on the i−th place}} \in \mathrm{Mat}_{2^N} , \tag{51}$$

Integrable isotropic long-range gl_M spin chains on N sites of the Haldane-Shastry type are defined by Hamiltonians of the form:

$$H^{\rm XXX} = \frac{g}{2} \sum_{i \neq j}^{N} P_{ij} \, U(x_i - x_j) \in \mathrm{End}(\mathcal{H}) , \tag{52}$$

where $U(x)$ is a certain function, $g \in \mathbb{C}$ is a constant parameter and $x_1, ..., x_N$ is a special set of points.

Quantum spin Calogero-Moser-Sutherland models are defined by the Hamiltonian of the form:

$$H^{\mathrm{spin\,CM}} = \frac{\mathrm{Id}}{2}\sum_{k=1}^{N}\eta^2\partial_{z_k}^2 + \frac{1}{2}\sum_{i\neq j}^{N}(\hbar^2\mathrm{Id} - \eta\hbar P_{ij})\,U(z_i - z_j)\,, \qquad \mathrm{Id} = 1_{M^N}\,, \tag{53}$$

where η is the Planck constant and $\hbar$ is a coupling constant

The procedure relating Calogero-Moser-Sutherland models and Haldane-Shastry chains is called the Polychronakos freezing trick. Loosely speaking, it states that one should remove the terms with differential operators from the spin Calogero-Moser Hamiltonians and fix the positions of particles as equilibrium positions of the underlying spinless classical model.

$$h^{\mathrm{CM}} = \frac{1}{2}\sum_{k=1}^{N}v_k^2 - \frac{\nu^2}{2}\sum_{i\neq j}^{N}U(z_i - z_j)\,, \tag{54}$$

where v_k are momenta (with the canonical Poisson brackets $\{v_i, z_j\} = \delta_{ij}$) and ν is the classical coupling constant. The set $z_k = x_k = k/N$, $k = 1...N$ solves the system of equations

$$\dot{v}_k = \ddot{z}_k = \nu^2\sum_{j:j\neq i}U'(z_i - z_j) = 0\,, \quad i = 1,...,N \tag{55}$$

for $U(x) = 1/\sin^2(\pi x)$ and $U(x) = \wp(x)$.

Anisotropic models

A general form for anisotropic gl_M model is as follows:

$$H^{\mathrm{anis}} = \frac{g}{2} \sum_{i \neq j}^{N} \sum_{a,b,c,d=1}^{M} e_{ab}^{(i)} e_{cd}^{(j)} U_{ab,cd}(x_i - x_j) \in \mathrm{End}(\mathcal{H}) \,, \tag{56}$$

where $e_{ab}^{(i)}$ is the standard matrix basis matrix $e_{ab} \in \mathrm{Mat}_M$ in the i-th tensor component of $\mathcal{H}$.
This Hamiltonian becomes isotropic in the case $U_{ab,cd}(x_i - x_j) = \delta_{ad}\delta_{bc} U(x_i - x_j)$.
Anisotropic spin Calogero-Moser Hamiltonian is of the form:

$$H^{\mathrm{anis\,CM}} = \frac{\mathrm{Id}}{2} \sum_{k=1}^{N} \eta^2 \partial_{z_k}^2 + \frac{g}{2} \sum_{i \neq j}^{N} \sum_{a,b,c,d=1}^{M} e_{ab}^{(i)} e_{cd}^{(j)} U_{ab,cd}(z_i - z_j) \,. \tag{57}$$

Example: XXZ gl_2

$$H^{\mathrm{XXZ}} = \frac{g}{2} \sum_{i \neq j}^{N} \frac{\cos(\pi(x_i - x_j))(\sigma_1^{(i)}\sigma_1^{(j)} + \sigma_2^{(i)}\sigma_2^{(j)}) + \sigma_3^{(i)}\sigma_3^{(j)}}{\sin^2(\pi(x_i - x_j))} \,. \tag{58}$$

q-deformed models

The q-deformed Haldane-Shastry model was introduced by D. Uglov. His construction was revisited and clarified by Lamers, Pasquier and Serban.

The Hamiltonian is as follows:

$$\mathbf{H}_1^{\mathrm{trig}} = 2\pi\imath(1-t)\sum_{k<i} \frac{y_i y_k}{(t y_k - y_i)(t y_i - y_k)} \times$$

$$\times \bar{R}^{\mathrm{trig}}_{i-1,i}\left(\frac{y_{i-1}}{y_i}\right) \dots \bar{R}^{\mathrm{trig}}_{k+1,i}\left(\frac{y_{k+1}}{y_i}\right) C_{ik} \bar{R}^{\mathrm{trig}}_{i,k+1}\left(\frac{y_i}{y_{k+1}}\right) \dots \bar{R}^{\mathrm{trig}}_{i,i-1}\left(\frac{y_i}{y_{i-1}}\right),$$

where $y_j = \exp\left(\frac{2\pi\imath}{N} j\right)$ and

$$C_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -\sqrt{t} & 0 \\ 0 & -\sqrt{t} & t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the limit $t \to 1$ it reproduces the Haldane-Shastry model.

Recall anisotropic spin Macdonald-Ruijsenaars operators:

$$\mathcal{D}_k = \sum_{1\leq i_1<\ldots<i_k\leq N} \left(\prod_{\substack{j\,=\,1\\ j\neq i_1\ldots i_{k-1}}}^{N} \phi(z_j - z_{i_1})\, \phi(z_j - z_{i_2}) \,\cdots\, \phi(z_j - z_{i_k}) \right) \times$$

$$\times \left(\overleftarrow{\prod_{j_1=1}^{i_1-1}} \bar{R}_{j_1 i_1} \overleftarrow{\prod_{\substack{j_2\,=\,1\\ j_2\neq i_1}}^{i_2-1}} \bar{R}_{j_2 i_2} \;\ldots\; \overleftarrow{\prod_{\substack{j_k\,=\,1\\ j_k\neq i_1\ldots i_{k-1}}}^{i_k-1}} \bar{R}_{j_k i_k} \right) \times \tag{59}$$

$$\times p_{i_1}\cdot p_{i_2}\cdots p_{i_k} \times \left(\overrightarrow{\prod_{\substack{j_k=1\\ j_k\neq i_1\ldots i_{k-1}}}^{i_k-1}} \bar{R}_{i_k j_k} \overrightarrow{\prod_{\substack{j_{k-1}=1\\ j_{k-1}\neq i_1\ldots i_{k-2}}}^{i_{k-1}-1}} \bar{R}_{i_{k-1} j_{k-1}} \;\ldots\; \overrightarrow{\prod_{j_1=1}^{i_1-1}} \bar{R}_{i_1 j_1} \right),$$

where $k = 1, ..., N$, $\bar{R}_{ij} = \bar{R}^{\hbar}_{ij}(z_i - z_j)$ and p_i, $i = 1, ..., N$ are the shift operators.

Namely, consider expansion of $\mathcal{D}_k$ in variable η (near $\eta = 0$):

$$\mathcal{D}_k = \mathcal{D}_k^{[0]} + \eta \mathcal{D}_k^{[1]} + \eta^2 \mathcal{D}_k^{[2]} + O(\eta^3)\,, \quad k = 1, ..., N\,. \tag{60}$$

Since $\mathcal{D}_k^{[0]} = \mathcal{D}_k|_{\eta=0}$

$$\mathcal{D}_k^{[0]} = \mathrm{Id} \sum_{|I|=k} \prod_{\substack{i\in I\\ j\notin I}} \phi(z_j - z_i) = \mathrm{Id}\, D_k^{[0]}\,, \tag{61}$$

where Id is the identity matrix in $\mathrm{End}(\mathcal{H})$. For the set of $\mathcal{D}_k^{[1]}$ one gets

$$\begin{gathered} -\mathcal{D}_1^{[1]} = \mathrm{Id} \sum_{i=1}^{N} \prod_{\substack{j=1\\ j\neq i}}^{N} \phi(z_j - z_i) \frac{\partial}{\partial z_i} + \\ + \sum_{i=1}^{N} \prod_{\substack{j=1\\ j\neq i}}^{N} \phi(z_j - z_i) \sum_{k=1}^{i-1} \bar{R}_{i-1,i} \ldots \bar{R}_{k+1,i} \bar{R}_{k,i} \left(\frac{\partial}{\partial z_i} \bar{R}_{i,k} \right) \bar{R}_{i,k+1} \ldots \bar{R}_{i,i-1}\,. \end{gathered} \tag{62}$$

Therefore,

$$\mathcal{D}_k^{[1]} = \mathrm{Id}\, D_k^{[1]} - \tilde{H}_k\,, \quad k = 1, ..., N\,, \tag{63}$$

where $\tilde{H}_k \in \mathrm{End}(\mathcal{H})$ are some matrix-valued functions, which contain R-matrix derivatives but do not contain differential operators.

Let us now restrict the latter equality to the point $z_k = x_k = k/N$ and denote

$$H_i = \tilde{H}_i \Big|_{z_k = x_k}\,. \tag{64}$$

It can be proved that these are commutative Hamiltonians

$$[H_i, H_j] = 0\,, \quad i, j = 1, ..., N-1\,. \tag{65}$$

It is proved in our second paper.

Using short notation

$$\bar{F}^{\hbar}_{ij}(z) = \frac{\partial}{\partial z}\bar{R}^{\hbar}_{ij}(z)$$
$$\bar{R}_{ij} = \bar{R}^{\hbar}_{ij}(x_i - x_j)\,, \qquad \bar{F}_{ij} = \bar{F}^{\hbar}_{ij}(x_i - x_j)\,. \tag{66}$$

we have

$$\mathbf{H}_1 = \sum_{k<i}^{N} \bar{R}_{i-1,i}\ldots\bar{R}_{k+1,i}\bar{R}_{k,i}\bar{F}_{i,k}\bar{R}_{i,k+1}\ldots\bar{R}_{i,i-1}\,. \tag{67}$$

The second Hamiltonian is of the form:

$$\begin{gathered}\mathbf{H}_2 = \sum_{\substack{i,m,l=1\\ i<m<l}}^{N} \frac{1}{\wp(\hbar)-\wp(x_m-x_l)}\big(\bar{R}_{m-1,m}\ldots\bar{R}_{i+1,m}\bar{R}_{i,m}\bar{F}_{m,i}\bar{R}_{m,i+1}\ldots\bar{R}_{m,m-1}+\\ +\bar{R}_{m-1,m}\ldots\bar{R}_{1,m}\bar{R}_{l-1,l}\ldots\bar{R}_{m+1,l}\bar{R}_{m-1,l}\ldots\bar{R}_{i+1,l}\bar{R}_{i,l}\times\\ \times\bar{F}_{l,i}\bar{R}_{l,i+1}\ldots\bar{R}_{l,m-1}\bar{R}_{l,m+1}\ldots\bar{R}_{l,l-1}\bar{R}_{m,1}\ldots\bar{R}_{m,m-1}\big)+\\ +\sum_{\substack{i,m,l=1\\ i<m<l}}^{N} \frac{1}{\wp(\hbar)-\wp(x_i-x_l)}\bar{R}_{l-1,l}\ldots\bar{R}_{m+1,l}\bar{R}_{m,l}\bar{F}_{l,m}\bar{R}_{l,m+1}\ldots\bar{R}_{l,l-1}\,.\end{gathered} \tag{68}$$

Example: Hamiltonians for $N=3$

$$\begin{aligned}\mathbf{H}_1 &= \bar{R}^{\hbar}_{12}(x_1-x_2)\bar{F}^{\hbar}_{21}(x_2-x_1)+\bar{R}^{\hbar}_{23}(x_2-x_3)\bar{F}^{\hbar}_{32}(x_3-x_2)+ \\ &+\bar{R}^{\hbar}_{23}(x_2-x_3)\bar{R}^{\hbar}_{13}(x_1-x_3)\bar{F}^{\hbar}_{31}(x_3-x_1)\bar{R}^{\hbar}_{32}(x_3-x_2)\,,\end{aligned} \tag{69}$$

$$\begin{aligned}\mathbf{H}_2 &= \frac{1}{\wp(\hbar)-\wp(\frac{1}{3})}\Big(\bar{R}^{\hbar}_{23}(x_2-x_3)\bar{F}^{\hbar}_{32}(x_3-x_2)+\bar{R}^{\hbar}_{12}(x_1-x_2)\bar{F}^{\hbar}_{21}(x_2-x_1)+ \\ &+\bar{R}^{\hbar}_{12}(x_1-x_2)\bar{R}^{\hbar}_{13}(x_1-x_3)\bar{F}^{\hbar}_{31}(x_3-x_1)\bar{R}^{\hbar}_{21}(x_2-x_1)\Big)\,.\end{aligned} \tag{70}$$

Underlying identities. For $x_j = \frac{j}{N}$ the following relation holds:

$$\sum_{\substack{|I|=k \\ l\in I}} \prod_{\substack{i\in I \\ j\notin I}} \phi(x_j - x_i) = \sum_{\substack{|I'|=k \\ m\in I'}} \prod_{\substack{i\in I' \\ j\notin I'}} \phi(x_j - x_i) \qquad \text{for } l, m = 1 \dots N. \tag{71}$$

Denote by

$$g(x) = E_1(\hbar + x) - E_1(x)\,, \tag{72}$$

and

$$f(x) = g(x) - g(-x) = E_1(\hbar + x) + E_1(\hbar - x) - 2E_1(x)\,. \tag{73}$$

The following identities hold

$$\sum_{l\neq m} f_{lm} = 0\,, \tag{74}$$

$$\sum_{\substack{|I|=k \\ m\in I}} \left(\prod_{\substack{i\in I \\ j\notin I}} \phi(x_j - x_i) \sum_{\substack{l\in I \\ l\neq m}} f_{lm} \right) = 0\,. \tag{75}$$

The identities lead to equilibrium positions (equal velocities and vanishing accelerations) in all flows.

In the non-relativistic limit ($\bar{R}_{ij} = \mathrm{Id} + \hbar \bar{r}_{ij} + ...$)

$$\mathcal{H}_2 = \sum_{i>j}^{N} \partial \bar{r}_{ij}(x_i - x_j)\,. \tag{76}$$

$$\mathcal{H}_3 = \sum_{i<j<k}^{N} [\bar{r}_{ij}(x_i - x_j) + \bar{r}_{kj}(x_k - x_j), \partial \bar{r}_{ki}(x_k - x_i)]\,, \tag{77}$$

and

$$[\mathcal{H}_2, \mathcal{H}_3] = 0\,. \tag{78}$$

This model was previously introduced in our joint papers with I. Sechin through R-matrix valued Lax pairs. It is anisotropic version of the Inozemtsev (long-range) chain.
This model can be also viewed as a result of the freezing trick being applied to the model of interacting tops. Commutativity was verified by numerical calculations only. Now all higher Hamiltonians can be derived and their commutativity is proved.

Thank you!