FROM ELIPTIC HYPERGEOMETRIC INTEGRALS TO COMPLEX HYPERGEOMETRIC FUNCTIONS

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Abstract. Elliptic hypergeometric integrals are top transcendental special functions of hypergeometric type. They have found applications in $4d$ supersymmetric quantum fields theories (superconformal indices), in integrable systems (wave functions in quantum N-body problems) and $2d$ statistical mechanics (partition functions). I will describe how these integrals can be degenerated in a chain of limits to complex hypergeometric functions related to the representation theory of $SL(2,\mathbb{C})$.

An elliptic analogue of the Euler-Gauss ${}_2F_1$-function   V.S., 2003

$$V(t_1,\ldots,t_8;p,q)=\frac{(p;p)_\infty(q;q)_\infty}{4\pi\mathrm{i}}\int_{\mathbb{T}}\frac{\prod_{j=1}^8\Gamma(t_jx^{\pm1};p,q)}{\Gamma(x^{\pm2};p,q)}\frac{dx}{x},$$

where $\prod_{j=1}^8 t_j=(pq)^2$, $|t_j|<1$ and $(z;p)_\infty=\prod_{j=0}^\infty(1-zp^j)$,

$$\Gamma(z;p,q)=\prod_{j,k=0}^\infty\frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k},\quad |q|,|p|<1,$$

$$\Gamma(qz;p,q)=\theta(z;p)\Gamma(z;p,q),\qquad \theta(z;p):=(z;p)_\infty(pz^{-1};p)_\infty.$$

Inversion relation: $\Gamma\left(\frac{pq}{z};p,q\right)\Gamma(z;p,q)=1.$

The elliptic beta integral

$$V(t_1,\ldots,t_7,\frac{pq}{t_7};p,q)=\prod_{1\le j<k\le 6}\Gamma(t_jt_k;p,q).$$

V.S., 2000

Applications

- V = a special eigenfunction of $N = 1$ van Diejen model Hamiltonian (a BC_N version of the Ruijsenaars model). Similar picture for BC_N extension of the V-function. V.S., 2004

- Superconformal indices (counting BPS states) of $4d$ SUSY theories with $G = SU(2)$, $F = SU(8)$ or $SU(6)$ and chiral superfields in the fundamental representation. The elliptic beta integral evaluation proves the Seiberg duality conjecture in the BPS-sector. Dolan, Osborn, 2008

- In statistical mechanics: the star-triangle, star-star relations, top solutions of the Yang-Baxter equation guaranteeing integrability of the corresponding $2d$ Ising type models and $1d$ Heisenberg type spin chains.

  NB. Seiberg duality = integrability.

  V.S., 2003; Bazhanov, Sergeev, 2010; V.S., 2010; Derkachov, V.S., 2012.

The trigonometric limit: $p \to 0$ for fixed q

For fixed z and q:

$$\Gamma(z;0,q) = \frac{1}{(z;q)_\infty}, \quad \lim_{p\to 0} \Gamma(p^\alpha z;p,q) = 1,\ 0<\alpha<1,$$
$$\lim_{p\to 0} \Gamma(pz;p,q) = (qz^{-1};p,q)_\infty.$$

Cleaning the V-balancing condition:

$$t_j = p^{\alpha_j} g_j, \quad \sum_{j=1}^{8} \alpha_j = 2, \quad \prod_{j=1}^{8} g_j = q^2,$$

for fixed q and g_j the limit $p \to 0$ becomes well defined. E.g.,

$$\alpha_j = 0,\ j=1,\ldots,6, \quad \alpha_7 = \alpha_8 = 1, \quad \Rightarrow$$
$$\lim_{p\to 0} V(\underline{t};p,q) = \frac{(q;q)_\infty}{4\pi \mathrm{i}} \int_{\mathbb{T}} \frac{(z^{\pm 2};q)_\infty \prod_{k=7,8}(qg_k^{-1}z^{\pm 1};q)_\infty}{\prod_{k=1}^{6}(g_k z^{\pm 1};q)_\infty} \frac{dz}{z}.$$

Special choice of g_j: $6j$-symbols for $sl_q(2,\mathbb{R})$, or Askey-Wilson functions.
Goes down to Euler integral representation for ${}_2F_1$-function.

Rational degeneration: $q \to 1$ (the limit $q = 0$ is trivial).

Jackson's q-gamma function

$$\Gamma_q(u) := \frac{(q;q)_\infty}{(q^u;q)_\infty}(1-q)^{1-u}, \qquad |q| < 1,$$

$$\Gamma_q(u+1) = \frac{1-q^u}{1-q}\Gamma_q(u), \quad \lim_{q\to 1}\Gamma_q(u) = \Gamma(u).$$

Koornwinder (1990): the limit $q \to 1^-$ is uniform on compacta

Rains (2006): OK for $|q| \to 1$ from inside $\mathbb{T}$ under fixed angle

Substitute $z = q^u$, $g_j = q^{a_j}$, $\sum_{j=1}^8 a_j = 2$, infinite products $\to 1/\Gamma_q(u)$
$\Rightarrow$ Mellin-Barnes type integral

$$\lim_{q\to 1}\lim_{p\to 0} V(\underline{t};p,q) = \frac{\kappa(q)}{4\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\prod_{j=1}^6 \Gamma(a_j \pm u)}{\Gamma(\pm 2u)\prod_{j=7,8}\Gamma(1-a_j \pm u)} du,$$

where $\kappa(q)$ is a diverging factor

$$\kappa(q) = (q;q)^5 (1-q)^{-6} \log q^{-1}.$$

Goes down to the Barnes representation for ${}_2F_1$-function.

Hyperbolic degeneration

Parametrize

$$t_j = e^{-2\pi v g_j}, \quad z = e^{-2\pi v u}, \quad p = e^{-2\pi v \omega_1}, \quad q = e^{-2\pi v \omega_2}.$$

In the limit $v \to 0^+$, Ruijsenaars, 1997

$$\Gamma(e^{-2\pi v u}; e^{-2\pi v \omega_1}, e^{-2\pi v \omega_2}) \underset{v\to 0^+}{=} e^{-\pi \frac{2u-\omega_1-\omega_2}{12 v \omega_1 \omega_2}} \gamma^{(2)}(u;\omega_1,\omega_2).$$

Faddeev's (1994) modular dilogarithm, or hyperbolic gamma function

$$\gamma^{(2)}(u;\omega) = \gamma^{(2)}(u;\omega_1,\omega_2) := e^{-\frac{\pi \mathrm{i}}{2} B_{2,2}(u;\omega)} \gamma(u;\omega),$$

second order multiple Bernoulli polynomial

$$B_{2,2}(u;\omega) = \frac{1}{\omega_1\omega_2}\left((u - \frac{\omega_1+\omega_2}{2})^2 - \frac{\omega_1^2+\omega_2^2}{12} \right),$$

$$\gamma(u;\omega) := \frac{(\tilde{\mathbf{q}} e^{2\pi \mathrm{i} \frac{u}{\omega_1}}; \tilde{\mathbf{q}})_\infty}{(e^{2\pi \mathrm{i} \frac{u}{\omega_2}}; \mathbf{q})_\infty} = \exp\left(- \int_{\mathbb{R}+\mathrm{i}0} \frac{e^{ux}}{(1-e^{\omega_1 x})(1-e^{\omega_2 x})} \frac{dx}{x} \right),$$

$$\mathbf{q} = e^{2\pi \mathrm{i} \frac{\omega_1}{\omega_2}}, \qquad \tilde{\mathbf{q}} = e^{-2\pi \mathrm{i} \frac{\omega_2}{\omega_1}}.$$

It is well defined for $\omega_1, \omega_2 > 0$ (i.e., $|\mathbf{q}| = 1$) and $0 < \mathrm{Re}(u) < \omega_1 + \omega_2$.

Rains (2006): this limit is uniform on compacta!

Dedekind function: $\eta(\tau)=q^{1/24}(q;q)_\infty,\ q=e^{2\pi \mathrm{i}\tau}$,

$$\eta(-1/\tau)=\sqrt{-\mathrm{i}\tau}\,\eta(\tau).$$

It yields the asymptotics of $(q;q)_\infty$ for $\tau\to 0$ (or $q\to 1$).

Then, Rains, 2006

$$V(e^{-2\pi v g_k};e^{-2\pi v\omega_1},e^{-2\pi v\omega_2}) \underset{v\to 0^+}{=} e^{\frac{\pi}{4v}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)} I_h(\underline{g}),$$

$$I_h(\underline{g})=\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\prod_{j=1}^{8}\gamma^{(2)}(g_j\pm z;\omega_1,\omega_2)}{\gamma^{(2)}(\pm 2z;\omega_1,\omega_2)}\frac{dz}{2\mathrm{i}\sqrt{\omega_1\omega_2}},$$

$$\mathrm{Re}(g_j)>0,\qquad \sum_{j=1}^{8} g_j=2(\omega_1+\omega_2).$$

Special choice of parameters $\Rightarrow$ eigenfunction of $N=1$ Hamiltonian of hyperbolic Ruijsenaars model (Ruijsenaars, 1994) $\propto$ $6j$ symbols for the Faddeev modular double $sl_q(2,\mathbb{R})\times sl_{\tilde{q}}(2,\mathbb{R})$ (Ponsot, Teschner, 2001)

Remind

$$q = e^{2\pi \mathrm{i}\frac{\omega_1}{\omega_2}}, \qquad \tilde{q} = e^{-2\pi \mathrm{i}\frac{\omega_2}{\omega_1}}.$$

Then,

$$\gamma^{(2)}(\omega_1 x;\omega) = \Gamma_q(x)\frac{e^{-\frac{\pi \mathrm{i}}{2}B_{2,2}(\omega_1 x;\omega)}}{(q;q)_\infty (1-q)^{1-x}}(\tilde{q}e^{2\pi \mathrm{i}x};\tilde{q})_\infty$$

$$\underset{\omega_1\to 0}{\approx} \frac{\Gamma(x)}{\sqrt{2\pi}}\left(\frac{\omega_2}{2\pi\omega_1}\right)^{\frac{1}{2}-x}, \qquad q\to 1^-, \quad \tilde{q}\to 0. \qquad \text{Ruijsenaars, 1997}$$

Replace $g_{7,8} \to g_{7,8} + \omega_1$ and apply the inversion formula

$$\gamma^{(2)}(x;\omega)\gamma^{(2)}(\omega_1+\omega_2-x;\omega) = 1.$$

Then the limit $\omega_1 \to 0 \;\Rightarrow\;$ the previous plain hypergeometric integral.

Complex hypergeometric functions

Euler's beta integral:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0.$$

Take $\alpha, \alpha' \in \mathbb{C}$, $\alpha - \alpha' = n_\alpha \in \mathbb{Z}$ and

$$[z]^\alpha := z^\alpha \bar{z}^{\alpha'} = |z|^{2\alpha'} z^{n_\alpha}, \quad \int_{\mathbb{C}} d^2z := \int_{\mathbb{R}^2} d(\mathrm{Re}\, z)\, d(\mathrm{Im}\, z),$$

$\bar{z}$ is a complex conjugate of z.

Then, the complex beta integral is Gelfand, Graev, Vilenkin, 1962

$$\int_{\mathbb{C}} [w-z_1]^{\alpha-1}[z_2-w]^{\beta-1}\frac{d^2w}{\pi} = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\alpha'-\beta')}{\Gamma(1-\alpha')\Gamma(1-\beta')\Gamma(\alpha+\beta)}[z_2-z_1]^{\alpha+\beta-1}$$

Complex gamma function

$$\mathbf{\Gamma}(x,n) = \mathbf{\Gamma}(\alpha|\alpha') := \frac{\Gamma(\alpha)}{\Gamma(1-\alpha')} = \frac{\Gamma(\frac{n+\mathrm{i}x}{2})}{\Gamma(1+\frac{n-\mathrm{i}x}{2})},$$

$$\alpha = \frac{n+\mathrm{i}x}{2}, \quad \alpha' = \frac{-n+\mathrm{i}x}{2}, \quad x \in \mathbb{C},\ n \in \mathbb{Z}.$$

$$\text{From the reflection relation}\quad \Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}\quad\Rightarrow$$

$$\mathbf{\Gamma}(\alpha|\alpha')=(-1)^{\alpha-\alpha'}\mathbf{\Gamma}(\alpha'|\alpha),\qquad \mathbf{\Gamma}(x,-n)=(-1)^n\mathbf{\Gamma}(x,n),$$

$$\mathbf{\Gamma}(\alpha|\alpha')\mathbf{\Gamma}(1-\alpha|1-\alpha')=(-1)^{\alpha-\alpha'},\qquad \mathbf{\Gamma}(x,n)\mathbf{\Gamma}(-x-2\mathrm{i},n)=1.$$

$$\mathbf{\Gamma}(\alpha{+}1|\alpha')=\mathbf{\Gamma}(x{-}\mathrm{i},n{+}1)=\alpha\mathbf{\Gamma}(\alpha|\alpha'),\quad \mathbf{\Gamma}(\alpha|\alpha'{+}1)=\mathbf{\Gamma}(x{-}\mathrm{i},n{-}1)=-\alpha'\mathbf{\Gamma}(\alpha|\alpha').$$

$$\Rightarrow\qquad \int_{\mathbb{C}}[w-z_1]^{\alpha-1}[z_2-w]^{\beta-1}\frac{d^2w}{\pi}$$
$$=\frac{\mathbf{\Gamma}(\alpha|\alpha')\mathbf{\Gamma}(\beta|\beta')}{\mathbf{\Gamma}(\alpha+\beta|\alpha'+\beta')}[z_2-z_1]^{\alpha+\beta-1}=\frac{\mathbf{\Gamma}(\alpha,\beta,\gamma)}{[z_1-z_2]^{\gamma}},$$

where $\alpha+\beta+\gamma=\alpha'+\beta'+\gamma'=1$ and

$$\mathbf{\Gamma}(\alpha_1,\ldots,\alpha_k):=\mathbf{\Gamma}(\alpha_1|\alpha_1')\cdots\mathbf{\Gamma}(\alpha_k|\alpha_k').$$

Inversion $w \to w^{-1}$, $z_1 \to z_1^{-1}$, $z_2 \to z_2^{-1}$ and shifts $w \to w - z_3$, $z_1 \to z_1 - z_3$, $z_2 \to z_2 - z_3$ $\Rightarrow$ the star-triangle relation:

$$\int_{\mathbb{C}} [z_1 - w]^{\alpha-1} [z_2 - w]^{\beta-1} [z_3 - w]^{\gamma-1} \frac{d^2 w}{\pi} = \frac{\mathbf{\Gamma}(\alpha, \beta, \gamma)}{[z_3 - z_2]^{\alpha} [z_1 - z_3]^{\beta} [z_2 - z_1]^{\gamma}}, \quad \alpha + \beta + \gamma = 1.$$

Nice applications to Feynman diagrams and non-compact spin chains:
Vasil'ev, Pismak, Khonkonen, Derkachov, Manashov, Valinevich, 1981 - ...

Complex hypergeometric functions:
Emergence in 2d conformal field theory: Dotsenko, Fateev, 1985
Complex Selberg integral: Aomoto, 1987
Mellin-Barnes representations of complex hypergeometric functions:
$3j$-symbols of $SL(2, \mathbb{C})$ for unitary principal series (real x): Naimark, 1959
$6j$-symbols of $SL(2, \mathbb{C})$: Ismagilov, 2006; Derkachov, V.S., 2017
Rigorous spectral analysis: Molchanov, Neretin, 2018

From hyperbolic integrals to complex hypergeometric functions

Formally: Bazhanov, Mangazeev, Sergeev, 2008; Kels, 2014;
Complete rigorous consideration: Sarkissian, V.S, 2019

Take

$$\gamma(u;\omega_1,\omega_2)=\frac{(e^{2\pi \mathrm{i}\frac{u}{\omega_1}}e^{-2\pi \mathrm{i}\frac{\omega_2}{\omega_1}};e^{-2\pi \mathrm{i}\frac{\omega_2}{\omega_1}})_\infty}{(e^{2\pi \mathrm{i}\frac{u}{\omega_2}};e^{2\pi \mathrm{i}\frac{\omega_1}{\omega_2}})_\infty}$$

and set

$$\text{“}b\text{”}=\sqrt{\frac{\omega_1}{\omega_2}}=\mathrm{i}+\delta,\quad \delta\to 0^+\qquad\qquad (c_{CFT}\to 1).$$

Then $\omega_1+\omega_2=2\delta\sqrt{\omega_1\omega_2}+O(\delta^2)\to 0$ and

$$\sqrt{\frac{\omega_2}{\omega_1}}=-\mathrm{i}+\delta+O(\delta^2),\quad \frac{\omega_1}{\omega_2}=-1+2\mathrm{i}\delta+\delta^2,\quad \frac{\omega_2}{\omega_1}=-1-2\mathrm{i}\delta+O(\delta^2).$$

Special choice of the argument u:

$$u=\mathrm{i}\sqrt{\omega_1\omega_2}(n+x\delta),\quad n\in\mathbb{Z},\ x\in\mathbb{C}.$$

Then, for $q=e^{2\pi \mathrm{i}\frac{\omega_1}{\omega_2}}\to 1$

$$(e^{2\pi \mathrm{i}\frac{u}{\omega_2}};e^{2\pi \mathrm{i}\frac{\omega_1}{\omega_2}})_\infty=(e^{-2\pi\delta(n+\mathrm{i}x+\delta x)};q)_\infty \underset{\delta\to 0^+}{=}\frac{(q;q)_\infty(1-q)^{1-\frac{n+\mathrm{i}x}{2}+O(\delta)}}{\Gamma_q\left(\frac{n+\mathrm{i}x}{2}+O(\delta)\right)}.$$

Similarly for $\tilde{q} = e^{-2\pi\mathrm{i}\frac{\omega_2}{\omega_1}} \to 1$

$$(e^{2\pi\mathrm{i}\frac{u}{\omega_1}} e^{-2\pi\mathrm{i}\frac{\omega_2}{\omega_1}}; e^{-2\pi\mathrm{i}\frac{\omega_2}{\omega_1}})_\infty \underset{\delta\to 0^+}{=} \frac{(\tilde{q};\tilde{q})_\infty (1-\tilde{q})^{\frac{-n+\mathrm{i}x}{2}+O(\delta)}}{\Gamma_{\tilde{q}}\left(1+\frac{n-\mathrm{i}x}{2}+O(\delta)\right)}.$$

As a result,

$$\gamma(u;\omega_1,\omega_2) \underset{\delta\to 0^+}{=} \frac{\Gamma_q\left(\frac{n+\mathrm{i}x}{2}+O(\log q)\right)}{\Gamma_{\tilde{q}}\left(1+\frac{n-\mathrm{i}x}{2}+O(\log\tilde{q})\right)} \frac{(\tilde{q};\tilde{q})_\infty}{(q;q)_\infty} \frac{(1-\tilde{q})^{\frac{-n+\mathrm{i}x}{2}+O(\log\tilde{q})}}{(1-q)^{1-\frac{n+\mathrm{i}x}{2}+O(\log q)}}.$$

$$\frac{(\tilde{q};\tilde{q})_\infty}{(q;q)_\infty} = e^{\frac{\pi\mathrm{i}}{12}\left(\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}\right)} \left(-\mathrm{i}\frac{\omega_1}{\omega_2}\right)^{\frac{1}{2}} \underset{\delta\to 0^+}{=} e^{\frac{\pi\mathrm{i}}{12}}.$$

Finally, for $\sqrt{\frac{\omega_1}{\omega_2}} = \mathrm{i}+\delta$, uniformly on the compacta

$$\gamma^{(2)}(\mathrm{i}\sqrt{\omega_1\omega_2}(n+x\delta);\omega_1,\omega_2) \underset{\delta\to 0^+}{\approx} (4\pi\delta)^{\mathrm{i}x-1} e^{\frac{\pi\mathrm{i}}{2}n^2} \mathbf{\Gamma}(x,n).$$

Degeneration of integrals

Consider

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Delta(z)\frac{dz}{\mathrm{i}\sqrt{\omega_1\omega_2}} = \int_{-\infty}^{\infty} \Delta(\mathrm{i}\sqrt{\omega_1\omega_2}x)dx, \quad x = \frac{z}{\mathrm{i}\sqrt{\omega_1\omega_2}},$$

$$= \sum_{N\in\mathbb{Z}} \int_{N-1/2}^{N+1/2} \Delta(\mathrm{i}\sqrt{\omega_1\omega_2}\, x)dx = \sum_{N\in\mathbb{Z}} \int_{-1/2}^{1/2} \Delta(\mathrm{i}\sqrt{\omega_1\omega_2}(N+x))dx$$

$$= \sum_{N\in\mathbb{Z}} \int_{-1/2\delta}^{1/2\delta} \delta\Delta(\mathrm{i}\sqrt{\omega_1\omega_2}(N+y\delta))dy \underset{\delta\to 0^+}{=} \sum_{N\in\mathbb{Z}} \int_{-\infty}^{\infty} \left[\delta\Delta(\mathrm{i}\sqrt{\omega_1\omega_2}(N+y\delta))\right] dy, \quad x = y\delta.$$

Apply this limit to the general univariate hyperbolic beta integral.

$$I_h(g_1,\ldots,g_7,\omega_1+\omega_2-g_7) = \prod_{1\le j<k\le 6} \gamma^{(2)}(g_j+g_k;\omega).$$

Parametrization of the integral kernel for $I_h(\underline{g})$:

$$z = \mathrm{i}\sqrt{\omega_1\omega_2}(N+\delta y), \quad y \in \mathbb{C}, \quad N \in \mathbb{Z}+\nu,\ \nu = 0, \frac{1}{2},$$

$$g_k = \mathrm{i}\sqrt{\omega_1\omega_2}(N_k+\delta \alpha_k), \quad \alpha_k \in \mathbb{C}, \quad N_k \in \mathbb{Z}+\nu,\ \nu = 0, \frac{1}{2},$$

The limit $\delta \to 0^+ \ \Rightarrow$ the balancing condition

$$\sum_{k=1}^{6} \alpha_k = -2\mathrm{i}, \quad \sum_{k=1}^{6} N_k = 0.$$

The parameter $\nu = 0, 1/2$ emerges because only the sums $N_k \pm N$ should be integers. Now, $\delta \to 0^+$ limiting relations:

$$\prod_{k=1}^{6} \gamma^{(2)}(g_k \pm z;\omega) \to \frac{(-1)^{2\nu}}{(4\pi\delta)^8} \prod_{k=1}^{6} \mathbf{\Gamma}(\alpha_k + y, N_k + N)\mathbf{\Gamma}(\alpha_k - y, N_k - N),$$

$$\prod_{1\le j<k\le 6}\gamma^{(2)}(g_j+g_k;\omega)\to\frac{(-1)^{2\nu}}{(4\pi\delta)^5}\prod_{1\le j<k\le 6}\mathbf{\Gamma}(\alpha_j+\alpha_k,N_j+N_k),$$

$$\gamma^{(2)}(\pm 2z;\omega)\to\frac{(-1)^{2\nu}}{(4\pi\delta)^2}\frac{\Gamma(N+\mathrm{i}y)}{\Gamma(1+N-\mathrm{i}y)}\frac{\Gamma(-N-\mathrm{i}y)}{\Gamma(1-N+\mathrm{i}y)}=\frac{(4\pi\delta)^{-2}}{y^2+N^2}.$$

The diverging factors $(4\pi\delta)^{-5}$ on both sides cancel.
The final complex beta integral evaluation formula:

$$\frac{1}{8\pi}\sum_{N\in\mathbb{Z}+\nu}\int_{-\infty}^{\infty}(y^2+N^2)\prod_{k=1}^{6}\mathbf{\Gamma}(\alpha_k\pm y,N_k\pm N)dy$$
$$=\prod_{1\le j<k\le 6}\mathbf{\Gamma}(\alpha_j+\alpha_k,N_j+N_k),$$

where $\sum_{k=1}^6\alpha_k=-2\mathrm{i}$, $\sum_{k=1}^6 N_k=0$, $N_k\in\mathbb{Z}+\nu$, $\nu=0,\frac{1}{2}$, and

$$\mathbf{\Gamma}(x_1\pm x_2,n_1\pm n_2):=\mathbf{\Gamma}(x_1+x_2,n_1+n_2)\mathbf{\Gamma}(x_1-x_2,n_1-n_2)\,.$$

Special cases (non-rigorously): Bazhanov, Mangazeev, Sergeev, 2008; Kels, 2014
General case (rigorously): Sarkissian, V.S., 2019; Derkachov, Manashov, 2019.

Degeneration of the $W(E_7)$-group transformation law for V-function: Seiberg duality for $4d$ SCI on $S^3\times S^1 \to$ mirror symmetry of $3d$ PF on $S_b^3 \to 2d$ duality of what on $\mathbb{C}$? vortices ?)

$$\begin{aligned} &\sum_{N\in\mathbb{Z}+\nu}\int_{-\infty}^{\infty}(y^2+N^2)\prod_{k=1}^{8}\boldsymbol{\Gamma}(\alpha_k\pm y, N_k\pm N)dy \\ &= (-1)^L\prod_{1\le j<k\le 4}\boldsymbol{\Gamma}(\alpha_j+\alpha_k, N_j+N_k)\prod_{5\le j<k\le 8}\boldsymbol{\Gamma}(\alpha_j+\alpha_k, N_j+N_k) \\ &\quad\times\sum_{N\in\mathbb{Z}+\mu}\int_{-\infty}^{\infty}(y^2+N^2)\prod_{k=1}^{4}\boldsymbol{\Gamma}(\alpha_k\pm y-\tfrac{1}{2}X-\mathrm{i}, N_k\pm N-\tfrac{1}{2}L) \\ &\quad\times\prod_{k=5}^{8}\boldsymbol{\Gamma}(\alpha_k\pm y+\tfrac{1}{2}X+\mathrm{i}, N_k\pm N+\tfrac{1}{2}L)dy, \end{aligned}$$

with $X:=\sum_{j=1}^4\alpha_j$, $L:=\sum_{j=1}^4 N_j$ and the balancing

$$\sum_{k=1}^{8}\alpha_k=-4\mathrm{i},\quad \alpha_k\in\mathbb{C},\quad \sum_{k=1}^{8}N_k=0,\quad N_k\in\mathbb{Z}+\nu,\ \nu=0,\frac{1}{2}.$$

Limits of parameters $g_k \to \infty$.

Degenerate form of the hyperbolic beta integral

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_{j=1}^{3} \gamma^{(2)}(f_j + z; \omega)\gamma^{(2)}(h_j - z; \omega) \frac{dz}{\mathrm{i}\sqrt{\omega_1\omega_2}} = \prod_{j,k=1}^{3} \gamma^{(2)}(f_j + h_k; \omega),$$

$\sum_{j=1}^{3}(f_j + h_j) = \omega_1 + \omega_2$. The limit $\omega_1 + \omega_2 \to 0$ ($b \to \mathrm{i}$) yields

$$\frac{\mathrm{i}^{2\nu}}{4\pi} \sum_{N \in \mathbb{Z}+\nu} \int_{-\infty}^{\infty} \prod_{j=1}^{3} \mathbf{\Gamma}(s_j + y, N + N_j)\mathbf{\Gamma}(t_j - y, N - M_j) dy$$

$$= \prod_{j,k=1}^{3} \mathbf{\Gamma}(s_j + t_k, N_j + M_k), \quad \sum_{j=1}^{3}(N_j + M_j) = 0, \ \sum_{j=1}^{3}(s_j + t_j) = -2\mathrm{i}.$$

For $\nu = 0$ this is the Mellin-Barnes representation of the complex star-triangle relation. Derkachov, Manashov, Valinevich, 2017.

$6j$**-symbols** for unitary principal series of $SL(2,\mathbb{C})$ = triple integral over complex plane (i.e. 6 real integrations) = infinite bilateral sum of Mellin-Barnes integrals.

Ismagilov, 2006; Derkachov, V.S., 2017

$$\mathrm{R}_\ell(c,c') = \int \mathrm{d}^2 z\, \Psi_2(a_1,a_2,a_3|\ell,c,z)\overline{\Psi_1(a_1,a_2,a_3|\ell,c',z)},$$

$$\overline{\Psi_1(a_1,a_2,a_3|\ell,c',z)} = \int \frac{\mathrm{d}^2 y}{[1-y]^{\frac{1-\ell-c'-a_3}{2}}[y]^{\frac{1+\ell-c'+a_3}{2}}[z-y]^{\frac{1-a_1+a_2+c'}{2}}},$$

$$\Psi_2(a_1,a_2,a_3|\ell,c,z) = \frac{1}{[z]^{\frac{1+a_1-\ell+c}{2}}}\int \frac{\mathrm{d}^2 z_0}{[1-z_0]^{\frac{1+a_1+\ell+c}{2}}[z_0]^{\frac{1+a_2-a_3-c}{2}}[z-z_0]^{\frac{1-a_2+a_3-c}{2}}}.$$

Mellin-Barnes representation:

$$a_1 = N_1/2 + \mathrm{i}\sigma_1 \qquad a_3 = N_3/2 + \mathrm{i}\sigma_3 \qquad c = M_1/2 + \mathrm{i}\rho_1$$

$$a_2 = N_2/2 + \mathrm{i}\sigma_2 \qquad l = N_4/2 + \mathrm{i}\sigma_4 \qquad c' = M_2/2 + \mathrm{i}\rho_2$$

$$s = N/2 + \mathrm{i}u/2$$

$$\left\{ \begin{smallmatrix} \sigma_1,N_1 & \sigma_2,N_2 \\ \sigma_3,N_3 & \sigma_4,N_4 \end{smallmatrix} \middle| \begin{smallmatrix} \rho_1,M_1 \\ \rho_2,M_2 \end{smallmatrix} \right\} = (-1)^{M_2-N_2+N_4} \frac{\pi^2}{4} \frac{\mathbf{\Gamma}\left(\sigma_1-\sigma_2+\rho_2-\mathrm{i}, A_1\right)\mathbf{\Gamma}\left(\sigma_2-\sigma_3+\rho_1-\mathrm{i}, A_2\right)}{\mathbf{\Gamma}\left(-\sigma_3-\sigma_4+\rho_2-\mathrm{i}, A_3\right)}$$

$$\times \mathbf{\Gamma}\left(\sigma_1+\sigma_4+\rho_1-\mathrm{i}, A_4\right) \sum_{N\in\mathbb{Z}} \int \prod_{k=1}^{4} \mathbf{\Gamma}(R_k-u, S_k-N)\mathbf{\Gamma}(U_k+u, T_k+N)du$$

where

$$\begin{aligned} R_1 &= -\sigma_1+\sigma_2-\rho_2-\mathrm{i} & U_1 &= -\rho_1-\sigma_2+\sigma_4+\rho_2 & S_1 &= (-N_1+N_2-M_2)/2 \\ R_2 &= \sigma_1+\sigma_2-\rho_2-\mathrm{i} & U_2 &= \rho_1-\sigma_2+\sigma_4+\rho_2 & S_2 &= (N_1+N_2-M_2)/2 \\ R_3 &= -\sigma_3-\sigma_4-\rho_2-\mathrm{i} & U_3 &= 0 & S_3 &= -(N_3+N_4+M_2)/2 \\ R_4 &= \sigma_3-\sigma_4-\rho_2-\mathrm{i} & U_4 &= 2\rho_2 & S_4 &= (N_3-N_4-M_2)/2 \end{aligned}$$

$$T_1 = (-M_1-N_2+N_4+M_2)/2, \quad T_2 = (M_1-N_2+N_4+M_2)/2, \quad T_3 = 0, \quad T_4 = M_2,$$

$$A_1 = \frac{N_1-N_2+M_2}{2} \quad A_2 = \frac{N_2-N_3+M_1}{2} \quad A_3 = \frac{-N_3-N_4+M_2}{2} \quad A_4 = \frac{N_1+N_4+M_1}{2}$$

Constraints $\sum_{k=1}^{4}(R_k+U_k) = -4\mathrm{i}, \quad \sum_{k=1}^{4}(S_k+T_k) = 0.$

Elliptic hypergeometric equation V.S., 2004

$$A(x)\,(f(qx)-f(x))+A(x^{-1})\left(f(q^{-1}x)-f(x)\right)+\lambda f(x)=0,$$

$$A(x)=\frac{\prod_{k=1}^{8}\theta(\varepsilon_k x;p)}{\theta(x^2,qx^2;p)},\quad \lambda=\prod_{k=1}^{6}\theta\left(\frac{\varepsilon_k\varepsilon_8}{q};p\right),\quad \prod_{k=1}^{8}\varepsilon_k=p^2q^2,\quad \varepsilon_7=\frac{\varepsilon_8}{q},$$

$$f(x)\propto\frac{V(q/c\varepsilon_1,\ldots,q/c\varepsilon_5,cx,c/x,c/\varepsilon_8;p,q)}{\Gamma(c^2x^{\pm1}/\varepsilon_8,x^{\pm1}\varepsilon_8;p,q)},\qquad c=\frac{\sqrt{\varepsilon_6\varepsilon_8}}{p^2}.$$

Other solutions: multiply $\varepsilon_1,\ldots,\varepsilon_5,x$ by $p^{\#}$ or permute $\varepsilon_1,\ldots,\varepsilon_5$ with ε_6, etc.

Degeneration to the complex $6j$-symbols level

$$B(s_2,N_2;s_3,N_3)(U(s_2-\mathrm{i},N_2-1)-U)+B(s_3,N_3;s_2,N_2)(U(s_2+\mathrm{i},N_2+1)-U)+U=0,$$

$$B=\frac{(s_2-s_4+\mathrm{i}(2+N_2-N_4))(s_4-s_2+\mathrm{i}(N_4-N_2))}{(s_3-s_2+\mathrm{i}(2+N_3-N_2))(s_2-s_3+\mathrm{i}(N_2-N_3))}\prod_{k=1}^{4}\frac{s_3+t_k+\mathrm{i}(2+N_3+M_k)}{s_4+t_k+\mathrm{i}(N_4+M_k)}$$

$$U=\frac{\sum_{N\in\mathbb{Z}}\int_{-\infty}^{\infty}\prod_{a=1}^{4}\mathbf{\Gamma}(s_a-y,N_a-N)\mathbf{\Gamma}(t_a+y,M_a+N)dy}{\mathbf{\Gamma}(s_2-s_4,N_2-N_4)\mathbf{\Gamma}(s_3-s_4,N_3-N_4)},$$

$\sum_{a=1}^{4}(N_a+M_a)=0,\ \sum_{a=1}^{4}(s_a+t_a)=-4\mathrm{i},$
$U(s_2-\mathrm{i},N_2-1):=U(s_2-\mathrm{i},N_2-1;s_3+\mathrm{i},N_3+1).$
$\Rightarrow$ q-difference equation becomes mixed recurrence + difference equation

Derkachov, Sarkissian, V.S., in preparation.

New rational degeneration of the hyperbolic Ruijsenaars N-body problem.

Rational hypergeometric identities Sarkissian, V.S., 2020

$b=\sqrt{\frac{\omega_1}{\omega_2}}=1+\mathrm{i}\delta$, $\delta\to 0^+$ limit of the Faddeev modular dilogarithm ($c_{CFT}=25$)

$$\gamma^{(2)}\Big(\sqrt{\omega_1\omega_2}(n+1+y\delta);\omega\Big) \underset{\delta\to 0^+}{=} (4\pi\delta)^n e^{-\frac{\pi\mathrm{i}}{2}n^2}\Big(\frac{1-n-\mathrm{i}y}{2}\Big)_n,$$

where $n\in\mathbb{Z}$, $y\in\mathbb{C}$ and $(a)_n$ is the standard Pochhammer symbol

$$(a)_n=\begin{cases} a(a+1)\cdots(a+n-1), & \text{for } n>0,\\ \frac{1}{(a-1)(a-2)\cdots(a+n)}, & \text{for } n<0.\end{cases}$$

Degeneration of the hyperbolic beta integral. For $N_k\in\mathbb{Z}$, $a_k\in\mathbb{C}$ and

$$\sum_{k=1}^{6}N_k=2,\qquad \sum_{k=1}^{6}a_k=0,$$

$$\frac{1}{8\pi\mathrm{i}}\sum_{N\in\mathbb{Z}+\nu}\int_{C_N}(y^2-N^2)\prod_{k=1}^{6}\Big(1+\frac{a_k-N_k\pm(y-N)}{2}\Big)_{N_k-1\pm N}dy$$
$$=\prod_{1\le j<k\le 6}\Big(1+\frac{a_j+a_k-N_j-N_k}{2}\Big)_{N_j+N_k-1},$$

contour C_N separates the poles emerging from $\pm$ signs Pochhammer symbols.

New formula for rational hypergeometric identities.