# LIMITS OF THE ELLIPTIC HYPERGEOMETRIC EQUATION AND INTEGRABLE SYSTEMS

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**Abstract.** We describe some new rational degenerations of the elliptic hypergeometric equation and its solutions. They lead to new rational degenerations of the Ruijsenaars and van Diejen elliptic integrable systems.

**Joint work with Gor Sarkissian**

## An elliptic analogue of the Euler-Gauss ${}_2F_1$-function V.S., 2003

$$V(t_1,\ldots,t_8;p,q)=\frac{(p;p)_\infty(q;q)_\infty}{4\pi \mathrm{i}}\int_{\mathbb{T}}\frac{\prod_{j=1}^{8}\Gamma(t_jx^{\pm1};p,q)}{\Gamma(x^{\pm2};p,q)}\frac{dx}{x},$$

where $\prod_{j=1}^{8}t_j=(pq)^2$, $|t_j|<1$ and $(z;p)_\infty=\prod_{j=0}^{\infty}(1-zp^j)$,

$$\Gamma(z;p,q)=\prod_{j,k=0}^{\infty}\frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k},\quad |q|,|p|<1.$$

$$\Gamma(qz;p,q)=\theta(z;p)\Gamma(z;p,q),\qquad \theta(z;p):=(z;p)_\infty(pz^{-1};p)_\infty.$$

## The elliptic beta integral

$$V(t_1,\ldots,t_7,\frac{pq}{t_7};p,q)=\prod_{1\le j<k\le 6}\Gamma(t_jt_k;p,q).$$

V.S., 2000

**The elliptic hypergeometric equation (EHE)** V.S., 2004

$$D(\underline{t};q;p)(U(qt_6,q^{-1}t_7)-U(\underline{t}))+(t_6\leftrightarrow t_7)+U(\underline{t})=0,$$

$$U(\underline{t})=\frac{V(t_1,\ldots,t_8;p,q)}{\Gamma(t_6t_8^{\pm1};p,q)\Gamma(t_7t_8^{\pm1};p,q)}$$

$$D(\underline{t};q;p)=\frac{\theta\left(\frac{t_6}{qt_8};p\right)\theta\left(t_6t_8;p\right)\theta\left(\frac{t_8}{t_6};p\right)}{\theta\left(\frac{t_6}{t_7};p\right)\theta\left(\frac{t_7}{qt_6};p\right)\theta\left(\frac{t_7t_6}{q};p\right)}\prod_{k=1}^{5}\frac{\theta\left(\frac{t_7t_k}{q};p\right)}{\theta(t_8t_k;p)}.$$

Here $U(qt_j,q^{-1}t_k)=U(\underline{t})$ with $t_j,t_k\to qt_j,q^{-1}t_k$, and $(t_6\leftrightarrow t_7)$ means permutation of $t_6$ and $t_7$ in the preceding expression.

From the $p\leftrightarrow q$ symmetry $\Rightarrow$ the second ($p$-difference) equation

After the change $t_6=cz,\ t_7=cz^{-1}\ \Rightarrow$ second order $q$-difference equation in $z$.

Discretization of some parameters $\Rightarrow$ three-term recurrence relation for elliptic biorthogonal rational functions V.S., Zhedanov, 1999

EHE in an $S_6$-symmetric form

$$\frac{\prod_{j=1}^{8}\theta(\varepsilon_j z;p)}{\theta(z^2,qz^2;p)}(\psi(qz)-\psi(z))+\frac{\prod_{j=1}^{8}\theta(\varepsilon_j z^{-1};p)}{\theta(z^{-2},qz^{-2};p)}(\psi(q^{-1}z)-\psi(z))$$

$$+\prod_{k=1}^{6}\theta\left(\frac{\varepsilon_k\varepsilon_8}{q};p\right)\psi(z)=0,$$

where

$$\varepsilon_k=\frac{q}{ct_k},\ k=1,\dots,5,\quad \varepsilon_6=p^4ct_8,\quad \varepsilon_7=\frac{c}{qt_8},\quad \varepsilon_8=\frac{c}{t_8},$$

so that $\prod_{k=1}^{8}\varepsilon_k=p^2q^2$ with $\varepsilon_8=q\varepsilon_7$ and $c^2=t_6t_7=\frac{\varepsilon_6\varepsilon_8}{p^4}$.

A solution:

$$f(z)\propto\frac{V(q/c\varepsilon_1,\dots,q/c\varepsilon_5,cz,c/z,c/\varepsilon_8;p,q)}{\Gamma(c^2z^{\pm1}/\varepsilon_8,z^{\pm1}\varepsilon_8;p,q)}.$$

**The trigonometric limit: $p \to 0$ for fixed $q$.**

For fixed $z$ and $q$:

$$\Gamma(z;0,q) = \frac{1}{(z;q)_\infty}, \quad \lim_{p\to 0} \Gamma(p^\alpha z;p,q) = 1,\ 0<\alpha<1,$$

$$\lim_{p\to 0} \Gamma(pz;p,q) = (qz^{-1};q)_\infty.$$

Cleaning the $V$-balancing condition:

$$t_j = p^{\alpha_j} g_j, \quad \sum_{j=1}^{8} \alpha_j = 2, \quad \prod_{j=1}^{8} g_j = q^2,$$

for fixed $q$ and $g_j$ the limit $p \to 0$ becomes well defined. E.g.,

$$\alpha_j = 0,\ j=1,\ldots,6, \quad \alpha_7 = \alpha_8 = 1, \quad \Rightarrow$$

$$\lim_{p\to 0} V(\underline{t};p,q) = \frac{(q;q)_\infty}{4\pi \mathrm{i}} \int_{\mathbb{T}} \frac{(z^{\pm 2};q)_\infty \prod_{k=7,8}(qg_k^{-1}z^{\pm 1};q)_\infty}{\prod_{k=1}^{6}(g_k z^{\pm 1};q)_\infty} \frac{dz}{z}.$$

EHE $\Rightarrow$ e.g., difference equation for Rahman's $q$-biorthogonal rational functions.

**EHE rational degeneration no. I:** $q \to 1$

Jackson's $q$-gamma function

$$\Gamma_q(u) := \frac{(q;q)_\infty}{(q^u;q)_\infty}(1-q)^{1-u}, \qquad |q| < 1,$$

Koornwinder (1990): the limit $\lim_{q\to 1}\Gamma_q(u) = \Gamma(u)$ is uniform on compacta

Rains (2006): OK for $|q| \to 1$ from inside $\mathbb{T}$ under fixed angle

Substitute $z = q^u$, $g_j = q^{\alpha_j}$, $\sum_{j=1}^{8}\alpha_j = 2$, infinite products $\to 1/\Gamma_q(u)$
$\Rightarrow$ Mellin-Barnes type integral

$$\lim_{q\to 1}\lim_{p\to 0} V(\underline{t};p,q) = \frac{\kappa(q)}{4\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\prod_{j=1}^{6}\Gamma(\alpha_j \pm u)}{\Gamma(\pm 2u)\prod_{j=7,8}\Gamma(1-\alpha_j \pm u)}du,$$

the diverging factor

$$\kappa(q) = (q;q)^5(1-q)^{-6}\log q^{-1}.$$

**Hyperbolic degeneration**

Parametrize

$$t_j = e^{-2\pi v g_j}, \quad z = e^{-2\pi v u}, \quad p = e^{-2\pi v \omega_1}, \quad q = e^{-2\pi v \omega_2}.$$

In the limit $v \to 0^+$, Ruijsenaars, 1997

$$\Gamma(e^{-2\pi v u}; e^{-2\pi v \omega_1}, e^{-2\pi v \omega_2}) \underset{v\to 0^+}{=} e^{-\pi \frac{2u-\omega_1-\omega_2}{12 v \omega_1 \omega_2}} \gamma^{(2)}(u; \omega_1, \omega_2).$$

Faddeev's (1994) modular dilogarithm, or hyperbolic gamma function

$$\gamma^{(2)}(u;\omega) = \gamma^{(2)}(u;\omega_1,\omega_2) := e^{-\frac{\pi \mathrm{i}}{2} B_{2,2}(u;\omega)} \gamma(u;\omega),$$

second order multiple Bernoulli polynomial

$$B_{2,2}(u;\omega) = \frac{1}{\omega_1\omega_2}\left( (u - \frac{\omega_1+\omega_2}{2})^2 - \frac{\omega_1^2+\omega_2^2}{12} \right),$$

$$\gamma(u;\omega) := \frac{(\tilde{\mathbf{q}} e^{2\pi \mathrm{i} \frac{u}{\omega_1}}; \tilde{\mathbf{q}})_\infty}{(e^{2\pi \mathrm{i} \frac{u}{\omega_2}}; \mathbf{q})_\infty} = \exp\left( - \int_{\mathbb{R}+\mathrm{i}0} \frac{e^{ux}}{(1-e^{\omega_1 x})(1-e^{\omega_2 x})} \frac{dx}{x} \right),$$

$$\mathbf{q} = e^{2\pi \mathrm{i} \frac{\omega_1}{\omega_2}}, \qquad \tilde{\mathbf{q}} = e^{-2\pi \mathrm{i} \frac{\omega_2}{\omega_1}}.$$

It is well defined for $\omega_1, \omega_2 > 0$ (i.e., $|\mathbf{q}| = 1$) and $0 < \mathrm{Re}(u) < \omega_1 + \omega_2$.

Rains (2006): this limit is uniform on compacta!

Then, Rains, 2006

$$V(e^{-2\pi v g_k}; e^{-2\pi v\omega_1}, e^{-2\pi v\omega_2}) \underset{v\to 0^+}{=} e^{\frac{\pi}{4v}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)} I_h(\underline{g}),$$

$$I_h(\underline{g}) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\prod_{j=1}^{8}\gamma^{(2)}(g_j \pm z; \omega_1, \omega_2)}{\gamma^{(2)}(\pm 2z; \omega_1, \omega_2)} \frac{dz}{2\mathrm{i}\sqrt{\omega_1\omega_2}},$$

$$\mathrm{Re}(g_j) > 0, \qquad \sum_{j=1}^{8} g_j = 2(\omega_1 + \omega_2).$$

Hyperbolic hypergeometric equation van de Bult, Rains, Stokman, 2007

$$\mathcal{A}(\underline{g},\omega_2;\omega_1)(Y(g_6+\omega_2,g_7-\omega_2)-Y(\underline{g}))+(g_6\leftrightarrow g_7)+Y(\underline{g})=0,$$

where

$$\mathcal{A}(\underline{g},\omega_2;\omega_1):=\frac{\sin\frac{\pi}{\omega_1}(g_6-g_8-\omega_2)\sin\frac{\pi}{\omega_1}(g_6+g_8)\sin\frac{\pi}{\omega_1}(g_8-g_6)}{\sin\frac{\pi}{\omega_1}(g_6-g_7)\sin\frac{\pi}{\omega_1}(g_7-g_6-\omega_2)\sin\frac{\pi}{\omega_1}(g_7+g_6-\omega_2)}$$

$$\times\prod_{k=1}^{5}\frac{\sin\frac{\pi}{\omega_1}(g_7+g_k-\omega_2)}{\sin\frac{\pi}{\omega_1}(g_8+g_k)}$$

and a solution

$$Y(\underline{g};\omega_1,\omega_2):=\frac{I_h(\underline{g};\omega_1,\omega_2)}{\gamma^{(2)}(g_6\pm g_8,g_7\pm g_8;\omega_1,\omega_2)}.$$

From the symmetry $\omega_1\leftrightarrow\omega_2\Rightarrow$ second hyperbolic equation

$$\mathcal{A}(\underline{g},\omega_1;\omega_2)(Y(g_6+\omega_1,g_7-\omega_1)-Y(u))+(u_6\leftrightarrow u_7)+Y(u)=0.$$

## From hyperbolic to the rational limit no. I

Ruijsenaars (1997):

$$\gamma^{(2)}(\omega_1 x;\omega) \underset{\omega_1\to 0,\,\omega_2\,\text{fixed}}{=} \frac{\Gamma(x)}{\sqrt{2\pi}}\left(\frac{\omega_2}{2\pi\omega_1}\right)^{\frac{1}{2}-x}, \qquad q=e^{2\pi \mathrm{i}\frac{\omega_1}{\omega_2}}\to 1, \quad \tilde{q}=e^{-2\pi\mathrm{i}\frac{\omega_2}{\omega_1}}\to 0.$$

Shift $g_{a,b}\to g_{a,b}+\omega_2$ (cleans the balancing condition: $\sum_{j=1}^{8} g_j = 2\omega_1$) and apply the inversion formula

$$\gamma^{(2)}(x+\omega_2;\omega)=\gamma^{(2)}(\omega_1-x;\omega)^{-1}.$$

Then set $g_k := \omega_1\alpha_k$, $z:=\omega_1 u \Rightarrow$

$$I_h(\underline{g};\omega_1,\omega_2) \underset{\omega_1\to 0}{=} \left(\frac{1}{2\pi}\sqrt{\frac{\omega_2}{\omega_1}}\right)^5 I_{rI}(\underline{\alpha}),$$

where

$$I_{rI}(\underline{\alpha})=\frac{1}{4\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\prod_{j=1,\neq a,b}^{8}\Gamma(\alpha_j\pm u)}{\Gamma(\pm 2u)\prod_{j=a,b}\Gamma(1-\alpha_j\pm u)}du, \quad \sum_{k=1}^{8}\alpha_k=2.$$

Fix $a=4$, $b=5 \Rightarrow$ an analytic difference equation

$$\mathcal{B}(\underline{\alpha})(\mathcal{I}_r(\alpha_6+1,\alpha_7-1)-\mathcal{I}_r(\underline{\alpha}))+(\alpha_6\leftrightarrow\alpha_7)+\mathcal{I}_r(\underline{\alpha})=0,$$

where

$$\mathcal{B}(\underline{\alpha})=\frac{(\alpha_6-\alpha_8-1)(\alpha_6+\alpha_8)(\alpha_8-\alpha_6)}{(\alpha_6-\alpha_7)(\alpha_7-\alpha_6-1)(\alpha_7+\alpha_6-1)}\prod_{k=1}^{5}\frac{\alpha_7+\alpha_k-1}{\alpha_8+\alpha_k},$$

$$\mathcal{I}_r(\underline{\alpha})=\frac{I_{rI}(\underline{\alpha})}{\Gamma(\alpha_6\pm\alpha_8)\Gamma(\alpha_7\pm\alpha_8)}.$$

The symmetry $\omega_1\leftrightarrow\omega_2$ is broken $\Rightarrow$ second hyperbolic hypergeometric equation yields the identity

$$\frac{\sin\pi(\alpha_8\pm\alpha_6)\prod_{k=4}^{5}\sin\pi(\alpha_7+\alpha_k)}{\sin\pi(\alpha_6-\alpha_7)\prod_{k=1}^{3}\sin\pi(\alpha_8+\alpha_k)}(\tilde{\mathcal{I}}_r(\underline{\alpha})-\mathcal{I}_r(\underline{\alpha}))+(\alpha_6\leftrightarrow\alpha_7)+\mathcal{I}_r(\underline{\alpha})=0,$$

where

$$\tilde{\mathcal{I}}_r(\underline{\alpha})=\frac{1}{4\pi \mathrm{i}}\frac{\Gamma(1-\alpha_6\pm\alpha_8)}{\Gamma(\alpha_7\pm\alpha_8)}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\prod_{k=1}^{3}\Gamma(\alpha_k\pm u)\prod_{k=7}^{8}\Gamma(\alpha_k\pm u)}{\Gamma(\pm 2u)\prod_{k=4}^{6}\Gamma(1-\alpha_k\pm u)}\,du.$$

## Complex hypergeometric functions

Euler's beta integral:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0.$$

Take $\alpha, \alpha' \in \mathbb{C}$, $\alpha - \alpha' = n_\alpha \in \mathbb{Z}$ and

$$[z]^\alpha := z^\alpha \bar{z}^{\alpha'} = |z|^{2\alpha'} z^{n_\alpha}, \qquad \int_{\mathbb{C}} d^2 z := \int_{\mathbb{R}^2} d(\mathrm{Re}\, z)\, d(\mathrm{Im}\, z),$$

$\bar{z}$ is a complex conjugate of $z$.

Then, the complex beta integral is — Gelfand, Graev, Vilenkin, 1962

$$\int_{\mathbb{C}} [w-z_1]^{\alpha-1}[z_2-w]^{\beta-1}\frac{d^2 w}{\pi} = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\alpha'-\beta')}{\Gamma(1-\alpha')\Gamma(1-\beta')\Gamma(\alpha+\beta)}[z_2-z_1]^{\alpha+\beta-1}.$$

Complex gamma function

$$\mathbf{\Gamma}(x,n) = \mathbf{\Gamma}(\alpha|\alpha') := \frac{\Gamma(\alpha)}{\Gamma(1-\alpha')} = \frac{\Gamma(\frac{n+\mathrm{i}x}{2})}{\Gamma(1+\frac{n-\mathrm{i}x}{2})}, \quad x \in \mathbb{C},\ n \in \mathbb{Z}.$$

Applying linear fractional transformation to all variables
$\Rightarrow$ the star-triangle relation:

$$\int_{\mathbb{C}} [z_1 - w]^{\alpha-1} [z_2 - w]^{\beta-1} [z_3 - w]^{\gamma-1} \frac{d^2 w}{\pi}$$
$$= \frac{\mathbf{\Gamma}(\alpha|\alpha')\mathbf{\Gamma}(\beta|\beta')\mathbf{\Gamma}(\gamma|\gamma')}{[z_3 - z_2]^{\alpha}[z_1 - z_3]^{\beta}[z_2 - z_1]^{\gamma}}, \quad \alpha + \beta + \gamma = 1.$$

Mellin-Barnes form of this identity Derkachov, Manashov, Valinevich, 2018

$$\frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_{j=1}^{3} \mathbf{\Gamma}(s_j + y, n + n_j) \mathbf{\Gamma}(t_j - y, n - m_j) dy = \prod_{j,k=1}^{3} \mathbf{\Gamma}(s_j + t_k, n_j + m_k),$$

$$\sum_{j=1}^{3} (n_j + m_j) = 0, \qquad \sum_{j=1}^{3} (s_j + t_j) = -2\mathrm{i}.$$

Nice applications to Feynman diagrams and non-compact spin chains:
Vasil'ev, Pismak, Khonkonen, Derkachov, Manashov, Valinevich, 1981 - 2020

Complex hypergeometric functions:
Emergence in 2d conformal field theory: Dotsenko, Fateev, 1985
Complex Selberg integral: Aomoto, 1987
$3j$-symbols of $SL(2,\mathbb{C})$ for unitary principal series (real $x$): Naimark, 1959
$6j$-symbols of $SL(2,\mathbb{C})$ and star-triangle relations: Ismagilov, 2006; Derkachov, V.S., 2017; Sarkissian, V.S., 2020; Derkachov, Manashov, 2020; Derkachov, Sarkissian, V.S., 2021
Rigorous spectral analysis: Molchanov, Neretin, 2018

**From hyperbolic integrals to complex hypergeometric functions**

Formally, special cases: Bazhanov, Mangazeev, Sergeev, 2008; Kels, 2014

Rigorous general consideration: Sarkissian, V.S, 2019

Take

$$\gamma(u;\omega_1,\omega_2) = \frac{(e^{2\pi \mathrm{i}\frac{u}{\omega_1}}e^{-2\pi \mathrm{i}\frac{\omega_2}{\omega_1}}; e^{-2\pi \mathrm{i}\frac{\omega_2}{\omega_1}})_\infty}{(e^{2\pi \mathrm{i}\frac{u}{\omega_2}}; e^{2\pi \mathrm{i}\frac{\omega_1}{\omega_2}})_\infty}$$

and set

$$\text{“}b\text{”} = \sqrt{\frac{\omega_1}{\omega_2}} = \mathrm{i} + \delta, \quad \delta \to 0^+ \qquad\qquad c_{CFT} = 1 + 6(b + b^{-1})^2 \to 1.$$

Special choice of the argument $u$:

$$u = \mathrm{i}\sqrt{\omega_1\omega_2}(n + x\delta), \quad n \in \mathbb{Z},\ x \in \mathbb{C}.$$

Then, for $q = e^{2\pi \mathrm{i}\frac{\omega_1}{\omega_2}} \to 1$

$$(e^{2\pi \mathrm{i}\frac{u}{\omega_2}}; e^{2\pi \mathrm{i}\frac{\omega_1}{\omega_2}})_\infty = (e^{-2\pi\delta(n+\mathrm{i}x+\delta x)}; q)_\infty \underset{\delta\to 0^+}{=} \frac{(q;q)_\infty (1-q)^{1-\frac{n+\mathrm{i}x}{2}+O(\delta)}}{\Gamma_q\left(\frac{n+\mathrm{i}x}{2} + O(\delta)\right)}.$$

Similarly for $\tilde{q} = e^{-2\pi\mathrm{i}\frac{\omega_2}{\omega_1}} \to 1$

$$(e^{2\pi\mathrm{i}\frac{u}{\omega_1}} e^{-2\pi\mathrm{i}\frac{\omega_2}{\omega_1}}; e^{-2\pi\mathrm{i}\frac{\omega_2}{\omega_1}})_\infty \underset{\delta\to 0^+}{=} \frac{(\tilde{q};\tilde{q})_\infty (1-\tilde{q})^{\frac{-n+\mathrm{i}x}{2}+O(\delta)}}{\Gamma_{\tilde{q}}\left(1+\frac{n-\mathrm{i}x}{2}+O(\delta)\right)}.$$

As a result,

$$\gamma(u;\omega_1,\omega_2) \underset{\delta\to 0^+}{=} \frac{\Gamma_q\left(\frac{n+\mathrm{i}x}{2}+O(\log q)\right)}{\Gamma_{\tilde{q}}\left(1+\frac{n-\mathrm{i}x}{2}+O(\log\tilde{q})\right)} \frac{(\tilde{q};\tilde{q})_\infty}{(q;q)_\infty} \frac{(1-\tilde{q})^{\frac{-n+\mathrm{i}x}{2}+O(\log\tilde{q})}}{(1-q)^{1-\frac{n+\mathrm{i}x}{2}+O(\log q)}}.$$

$$\frac{(\tilde{q};\tilde{q})_\infty}{(q;q)_\infty} = e^{\frac{\pi\mathrm{i}}{12}\left(\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}\right)} \left(-\mathrm{i}\frac{\omega_1}{\omega_2}\right)^{\frac{1}{2}} \underset{\delta\to 0^+}{=} e^{\frac{\pi\mathrm{i}}{12}}.$$

Finally, for $\sqrt{\frac{\omega_1}{\omega_2}} = \mathrm{i}+\delta$, uniformly on the compacta

$$\gamma^{(2)}(\mathrm{i}\sqrt{\omega_1\omega_2}(n+x\delta);\omega_1,\omega_2) \underset{\delta\to 0^+}{\approx} (4\pi\delta)^{\mathrm{i}x-1} e^{\frac{\pi\mathrm{i}}{2}n^2}\,\mathbf{\Gamma}(x,n).$$

**Degeneration to the complex hypergeometric functions**

Sarkissian, V.S., 2020

Parametrization in $I_h(\underline{g};\omega_1,\omega_2)$

$$z=\mathrm{i}\sqrt{\omega_1\omega_2}(n+y\delta),\quad g_j=\mathrm{i}\sqrt{\omega_1\omega_2}(n_j+s_j\delta),\quad \sqrt{\frac{\omega_1}{\omega_2}}=\mathrm{i}+\delta,$$

where $n, n_j \in \mathbb{Z}+\nu$, $\nu=0,1/2$ (only the combinations $n_j\pm n$ should be integer). This yields the diverging asymptotics

$$I_h(\underline{g};\omega_1,\omega_2) \underset{\delta\to 0^+}{=} \frac{1}{(4\pi\delta)^5}\mathbf{F}(\underline{s},\underline{n}),$$

where

$$\mathbf{F}(\underline{s},\underline{n})=\frac{1}{8\pi}\sum_{n\in\mathbb{Z}+\nu}\int_{-\infty}^{\infty}(y^2+n^2)\prod_{k=1}^{8}\mathbf{\Gamma}(s_k\pm y,n_k\pm n)dy,$$

with the balancing condition

$$\sum_{j=1}^{8}s_j=-4\mathrm{i},\qquad \sum_{j=1}^{8}n_j=0.$$

In particular,

$$\gamma^{(2)}(\pm 2z;\omega)\to\frac{(-1)^{2\nu}}{(4\pi\delta)^2}\frac{\Gamma(n+\mathrm{i}y)}{\Gamma(1+n-\mathrm{i}y)}\frac{\Gamma(-n-\mathrm{i}y)}{\Gamma(1-n+\mathrm{i}y)}=\frac{(4\pi\delta)^{-2}}{y^2+n^2}.$$

**EHE rational limit no. II** Sarkissian, V.S., 2021

The $\delta \to 0^+$ limit of the first hyperbolic hypergeometric equation

$$\mathcal{B}(\underline{\alpha})(\mathcal{F}(s_6 - \mathrm{i}, n_6 - 1, s_7 + \mathrm{i}, n_7 + 1) - \mathcal{F}(\underline{s}, \underline{n})) + (s_6, n_6 \leftrightarrow s_7, n_7) + \mathcal{F}(\underline{s}, \underline{n}) = 0,$$

where $\mathcal{B}(\underline{\alpha})$ is the same potential as in the rational limit no. I with

$$\alpha_k = \frac{1}{2}(\mathrm{i}s_k - n_k)$$

and

$$\mathcal{F}(\underline{s}, \underline{n}) = \frac{\mathbf{F}(\underline{s}, \underline{n})}{\mathbf{\Gamma}(s_6 \pm s_8, n_6 \pm n_8)\mathbf{\Gamma}(s_7 \pm s_8, n_7 \pm n_8)}.$$

This is NOT an analytic difference equation in terms of the complex variables $\alpha_k$ !

Degeneration of the second hyperbolic equation

$$\mathcal{B}(\underline{\alpha})(\mathcal{F}(s_6 - \mathrm{i}, n_6 + 1, s_7 + \mathrm{i}, n_7 - 1) - \mathcal{F}(\underline{s}, \underline{n})) + (s_6, n_6 \leftrightarrow s_7, n_7) + \mathcal{F}(\underline{s}, \underline{n}) = 0,$$

the same $\mathcal{B}(\underline{\alpha})$, but now

$$\alpha_k = \frac{1}{2}(\mathrm{i}s_k + n_k).$$

**EHE rational limit no. III** Sarkissian, V.S., 2020-2021

$$b=\sqrt{\frac{\omega_1}{\omega_2}}=1+\mathrm{i}\delta,\quad \delta\to 0^+,\qquad\qquad c_{CFT}=1+6(b+b^{-1})^2\to 25.$$

For generic values of the argument $\gamma^{(2)}(u;\omega)$ is finite, but

$$\gamma^{(2)}\Big(\sqrt{\omega_1\omega_2}(n+1+y\delta);\omega\Big)\underset{\delta\to 0^+}{=}(4\pi\delta)^n e^{-\frac{\pi \mathrm{i}}{2}n^2}\Big(\frac{1-n-\mathrm{i}y}{2}\Big)_n,$$

where $n\in\mathbb{Z}$, $y\in\mathbb{C}$ and $(a)_n$ is the standard Pochhammer symbol

$$(a)_n=\begin{cases} a(a+1)\cdots(a+n-1), & \text{for } n>0,\\ \dfrac{1}{(a-1)(a-2)\cdots(a+n)}, & \text{for } n<0.\end{cases}$$

Parametrization of the integration variable and parameters in $I_h(\underline{g})$:

$$z=\sqrt{\omega_1\omega_2}(n+y\delta),\qquad g_j=\sqrt{\omega_1\omega_2}(n_j+s_j\delta),$$

where $y,s_j\in\mathbb{C}$ and $n,n_j\in\mathbb{Z}+\nu,\ \nu=0,\frac{1}{2}$. Then

$$I_h(\underline{g})\underset{\delta\to 0^+}{=}\frac{\mathrm{i}}{(4\pi\delta)^5}\mathbf{R}(\underline{s},\underline{n}).$$

where $\mathbf{R}(\underline{s},\underline{n})$ is a rational function of the form

$$\mathbf{R}(\underline{s},\underline{n}) = \frac{1}{8\pi}\sum_{n\in\mathbb{Z}+\nu}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}(y^2+n^2)\prod_{j=1}^{8}\left(1-\frac{n_k+\mathrm{i}s_k\pm(n+\mathrm{i}y)}{2}\right)_{n_k\pm n-1} dy,$$

with the balancing condition $\sum_{j=1}^{8} s_j = 0,\ \sum_{j=1}^{8} n_j = 4.$

First hyperbolic equation $\Rightarrow$

$$\mathcal{B}(\underline{\alpha})(\mathcal{R}(s_6-\mathrm{i},n_6+1,s_7+\mathrm{i},n_7-1)-\mathcal{R}(\underline{s},\underline{n}))+(s_6,n_6\leftrightarrow s_7,n_7)+\mathcal{R}(\underline{s},\underline{n})=0,$$

with $\mathcal{B}(\underline{\alpha})$ as in the $b\to\mathrm{i}$ limit of the second hyperbolic equation $\alpha_k=\frac{1}{2}(\mathrm{i}s_k+n_k)$,

$$\mathcal{R}(\underline{s},\underline{n}) = \frac{\mathbf{R}(\underline{s},\underline{n})}{\prod_{k=6,7}\left(1-\frac{n_k+\mathrm{i}s_k\pm(n_8+\mathrm{i}s_8)}{2}\right)_{n_k\pm n_8-1}},$$

contour $C_N$ separates poles from different $\pm$ signs.

Second hyperbolic equation $\Rightarrow$

$$\mathcal{B}(\underline{\alpha})(\mathcal{R}(s_6+\mathrm{i},n_6+1,s_7-\mathrm{i},n_7-1)-\mathcal{R}(\underline{s},\underline{n}))+(s_6,n_6\leftrightarrow s_7,n_7)+\mathcal{R}(\underline{s},\underline{n})=0,$$

for the potential of the same shape $\mathcal{B}(\underline{\alpha})$, but with $\alpha_k=\frac{1}{2}(-\mathrm{i}s_k+n_k)$.

**Elliptic Ruijsenaars and van Diejen $N$-body problems**

The elliptic Ruijsenaars model Hamiltonian

$$\mathcal{H}_R = \sum_{j=1}^{N} \prod_{k=1,\neq j}^{N} \frac{\theta(tz_j z_k^{-1};p)}{\theta(z_j z_k^{-1};p)} T_j,$$

where $z_j, t, q \in \mathbb{C}^\times, p \in \mathbb{C}, |p| < 1$, and $T_j$ are the $q$-shift operators

$$T_j \psi(z_1, \ldots, z_j, \ldots, z_N) = \psi(z_1, \ldots, qz_j, \ldots, z_N).$$

The van-Diejen model

$$\mathcal{H}_{vD} = \sum_{j=1}^{N} \Big( A_j(\underline{z}) T_j + A_j(\underline{z}^{-1}) T_j^{-1} \Big) + v(\underline{z}),$$

where

$$A_j(\underline{z}) = \frac{\prod_{m=1}^{8} \theta(t_m z_j;p)}{\theta(z_j^2, qz_j^2;p)} \prod_{\substack{k=1 \\ k\neq j}}^{N} \frac{\theta(tz_j z_k, tz_j z_k^{-1};p)}{\theta(z_j z_k, z_j z_k^{-1};p)}, \quad t_m \in \mathbb{C}^\times.$$

Under the balancing condition

$$t^{2N-2}\prod_{m=1}^{8} t_m = p^2q^2,$$

$A_j(\ldots, pz_k, \ldots) = A_j(\underline{z})$, and up to some inessential additive constant

$$\mathcal{H}_{vD} = \sum_{j=1}^{N}\Big(A_j(\underline{z})(T_j-1) + A_j(\underline{z}^{-1})(T_j^{-1}-1)\Big).$$

For $N = 1$ the eigenvalue problem $\mathcal{H}\psi(\underline{z}) = \lambda\psi(\underline{z})$ is

$$\frac{\prod_{j=1}^{8}\theta(t_jz;p)}{\theta(z^2, qz^2;p)}(\psi(qz)-\psi(z)) + \frac{\prod_{j=1}^{8}\theta(t_jz^{-1};p)}{\theta(z^{-2}, qz^{-2};p)}(\psi(q^{-1}z)-\psi(z)) = \lambda\psi(z),$$

with $\prod_{m=1}^{8} t_m = p^2q^2$. For

$$t_8 = qt_7, \qquad \lambda = -\prod_{k=1}^{6}\theta\Big(\frac{t_kt_8}{q};p\Big)$$

this is the elliptic hypergeometric equation, i.e. we know the general solution.

Symmetric EHE rational reduction no. II (complex hypergeometric level)

$$\frac{\prod_{j=1}^{8}(\beta_j+z)}{2z(2z+1)}(\Psi(u-\mathrm{i},m-1)-\Psi(u,m))+\frac{\prod_{j=1}^{8}(\beta_j-z)}{2z(2z-1)}(\Psi(u+\mathrm{i},m+1)-\Psi(u,m))$$
$$+\prod_{k=1}^{6}(\beta_k+\beta_7)\,\Psi(u,m)=0, \qquad \sum_{k=1}^{8}\beta_k=2, \quad \beta_8=\beta_7+1.$$

where

$$\beta_j=\frac{\mathrm{i}\gamma_j-r_j}{2}, \quad z=\frac{\mathrm{i}u-m}{2},$$

$$r_8=r_7+1, \quad \gamma_8=\gamma_7-\mathrm{i}, \quad \sum_{j=1}^{8}r_j=0, \quad \sum_{j=1}^{8}\gamma_j=-4\mathrm{i}, \quad \gamma_j\in\mathbb{C}, \quad r_j\in\mathbb{Z}+\mu,\ \mu=0,\frac{1}{2}.$$

Removing the constraint $\beta_8=\beta_7+1$ and replacing $\prod_{k=1}^{6}(\beta_k+\beta_7)\to-\lambda\Rightarrow$ balanced $N=1$ van Diejen Hamiltonian eigenvalue problem.

## Scalar product degenerations

General elliptic level (no balancing condition):

$$\langle \varphi, \psi \rangle_e = \frac{(p;p)_\infty^N (q;q)_\infty^N}{(4\pi \mathrm{i})^N N!} \int_{\mathbb{T}^N} \Delta(\underline{x}, \underline{t})\ \varphi(\underline{x}) \psi(\underline{x})\ \frac{dx_1}{x_1} \cdots \frac{dx_N}{x_N},$$

where

$$\Delta(\underline{x}, \underline{t}) = \prod_{1 \le j < k \le N} \frac{\Gamma(t x_j^{\pm 1} x_k^{\pm 1}; p, q)}{\Gamma(x_j^{\pm 1} x_k^{\pm 1}; p, q)} \prod_{j=1}^N \frac{\prod_{k=1}^8 \Gamma(t_k x_j^{\pm 1}; p, q)}{\Gamma(x_j^{\pm 2}; p, q)}.$$

No singularities in the annulus $|q| < |x_k| < |q|^{-1} \Rightarrow$ hermicity

$$\langle \varphi, \mathcal{H}_{vD} \psi \rangle_e = \langle \mathcal{H}_{vD} \varphi, \psi \rangle_e.$$

Complex hypergeometric level

$$\langle \varphi, \psi \rangle_{rII} = \frac{1}{(8\pi)^N N!} \sum_{m_j \in \mathbb{Z} + \mu} \int_{u_j \in \mathbb{R}} \varphi(\underline{u}, \underline{m}) \psi(\underline{u}, \underline{m}) \prod_{1 \le j < k \le N} \frac{\mathbf{\Gamma}(\gamma \pm u_j \pm u_k, n \pm m_j \pm m_k)}{\mathbf{\Gamma}(\pm u_j \pm u_k, \pm m_j \pm m_k)}$$

$$\times \prod_{j=1}^N \left[ \prod_{\ell=1}^8 \mathbf{\Gamma}(\gamma_\ell \pm u_j, r_\ell \pm m_j) \right] (u_j^2 + m_j^2)\, du_j.$$

No balancing condition.

Rational degeneration no. II of the Hamiltonians.
Ruijsenaars model:

$$\mathcal{H}_R = \sum_{j=1}^{N} \prod_{k=1,\neq j}^{N} \frac{\beta + z_j - z_k}{z_j - z_k} T_{u_j,m_j},$$

van Diejen model:

$$\mathcal{H}_{rII} = \sum_{j=1}^{N} \Big( B_j(\underline{u}, \underline{m})(T_{u_j,m_j} - 1) + B_j(-\underline{u}, -\underline{m})(T^{-1}_{u_j,m_j} - 1) \Big),$$

$$T^{\pm 1}_{u_j,m_j} f(\underline{u}, \underline{m}) = f(\ldots, u_j \mp \mathrm{i}, \ldots, m_j \mp 1, \ldots),$$

$$B_j(\underline{u}, \underline{m}) = \prod_{\substack{k=1 \\ k\neq j}}^{N} \frac{(\beta + z_j + z_k)(\beta + z_j - z_k)}{(z_j + z_k)(z_j - z_k)} \frac{\prod_{l=1}^{8}(\beta_l + z_j)}{2z_j(2z_j + 1)}.$$

Here

$$z_j = \frac{\mathrm{i}u_j - m_j}{2}, \qquad \beta = \frac{\mathrm{i}\gamma - r}{2}, \qquad \beta_k = \frac{\mathrm{i}\gamma_k - r_k}{2},$$

where $u_j, \gamma, \gamma_k \in \mathbb{C}$, $r \in \mathbb{Z}$, $m_j, r_k \in \mathbb{Z} + \mu$, $\mu = 0, \frac{1}{2}$, with the balancing condition (otherwise no factorization)

$$(2N-2)\gamma + \sum_{k=1}^{8} \gamma_k = -4\mathrm{i}, \qquad (2N-2)r + \sum_{k=1}^{8} r_k = 0.$$

No singularities for $\mathrm{Im}(\gamma), \mathrm{Im}(\gamma_k) < -1 \Rightarrow$ the hermiticity condition

$$\langle \varphi, \mathcal{H}_{rII}\psi \rangle_{rII} = \langle \mathcal{H}_{rII}\varphi, \psi \rangle_{rII}.$$

## Special scalar product ⇒ Selberg integrals

The elliptic Selberg integral (van Diejen, V.S., 2000, ...):

$$\langle 1,1\rangle\big|_{t_7t_8=pq} = \frac{(p;p)_\infty^N(q;q)_\infty^N}{(4\pi\mathrm{i})^N N!}\int_{\mathbb{T}^N}\prod_{1\le j<k\le N}\frac{\Gamma(tx_j^{\pm1}x_k^{\pm1};p,q)}{\Gamma(x_j^{\pm1}x_k^{\pm1};p,q)}$$

$$\times\prod_{j=1}^N\frac{\prod_{\ell=1}^6\Gamma(t_\ell x_j^{\pm1};p,q)}{\Gamma(x_j^{\pm2};p,q)}\frac{dx_j}{x_j} = \prod_{j=1}^N\Big(\frac{\Gamma(t^j;p,q)}{\Gamma(t;p,q)}\prod_{1\le\ell<s\le6}\Gamma(t^{j-1}t_\ell t_s;p,q)\Big),$$

where $t^{2N-2}\prod_{\ell=1}^6 t_\ell = pq$.

**Complex rational level**

$$\langle 1,1\rangle_{rat}\Big|_{\substack{\gamma_7+\gamma_8=-2\mathrm{i}\\ n_7+n_8=0}} = \frac{(-1)^{2\mu N}}{(8\pi)^N N!}\sum_{m_j\in\mathbb{Z}+\mu}\int_{u_j\in\mathbb{R}}\prod_{1\le j<k\le N}\frac{\boldsymbol{\Gamma}(\gamma\pm u_j\pm u_k, r\pm m_j\pm m_k)}{\boldsymbol{\Gamma}(\pm u_j\pm u_k,\pm m_j\pm m_k)}$$

$$\times\prod_{j=1}^{N}\left[\prod_{\ell=1}^{6}\boldsymbol{\Gamma}(\gamma_\ell\pm u_j, n_\ell\pm m_j)\right](u_j^2+m_j^2)\,du_j$$

$$= (-1)^{N(2\mu+r\frac{N-1}{2})}\prod_{j=1}^{N}\frac{\boldsymbol{\Gamma}(j\gamma, jr)}{\boldsymbol{\Gamma}(\gamma,r)}\prod_{1\le\ell<s\le 6}\boldsymbol{\Gamma}((j-1)\gamma+\gamma_\ell+\gamma_s,(j-1)r+r_\ell+r_s),$$

where $r\in\mathbb{Z}, n_\ell\in\mathbb{Z}+\mu,\ \gamma,\gamma_\ell\in\mathbb{C}$, and the balancing

$$(2N-2)\gamma+\sum_{\ell=1}^{6}\gamma_\ell=-2\mathrm{i},\quad (2N-2)r+\sum_{\ell=1}^{6}r_\ell=0.$$

A new complex hypergeometric generalization of the Selberg integral.
Should contain the Mellin-Barnes form of the Dotsenko-Fateev and Aomoto results.