What are vector spaces?

In our study of vectors in $\mathbb{R}^n$, we would routinely add them, and multiply by scalars.

When manipulating vector expressions, we make — perhaps subconsciously — use of the fact that the addition and multiplication behave as they do for usual numbers.

The notion of vector space axiomatizes the validity of these properties. In other words, a vector space is something whose elements can be algebraically manipulated like vectors in $\mathbb{R}^n$.

What are vector spaces?

Definition

The data of an $\mathbb{R}$ vector space is a set V, equipped with a distinguished element $\mathbf{0} \in V$ and two maps

$$+ : V \times V \to V \qquad \cdot : \mathbb{R} \times V \to V$$

This data determines a vector space if it obeys the following rules.

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \qquad \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} \qquad \mathbf{v} + \mathbf{0} = \mathbf{v}$$

$$c \cdot (d \cdot \mathbf{v}) = (cd) \cdot \mathbf{v} \qquad 1 \cdot \mathbf{v} = \mathbf{v} \qquad 0 \cdot \mathbf{v} = \mathbf{0}$$

$$(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v} \qquad c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$$

Is $\mathbb{R}^n$ a vector space?

It depends what you mean by $\mathbb{R}^n$. If you just meant the set $\mathbb{R}^n$, then you haven't named a vector space — you haven't given the required data of a zero vector, or the addition law, or the scalar multiplication law.

To meet the criteria in the definition, we need to define:

- A zero vector: $\mathbf{0} = (0, 0, \ldots, 0)$
- Addition: $(v_1, \ldots, v_n) + (w_1, \ldots, w_n) = (v_1 + w_1, \ldots, v_n + w_n)$
- Multiplication: $c \cdot (v_1, \ldots, v_n) = (cv_1, \ldots, cv_n)$

And then check the seven axioms before.

Is $\mathbb{R}^n$ a vector space?

Is $\mathbb{R}^n$, equipped with the data of

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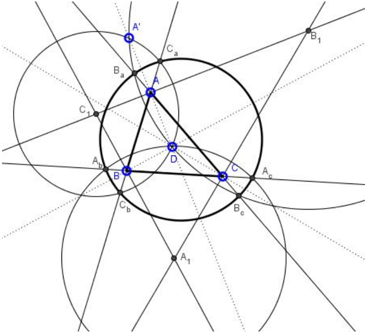
a vector space? Yes. (We checked the axioms previously.)

When, as in this case, everyone understands what zero vector, addition, and multiplication are meant to be, they may not be mentioned explicitly. So: $\mathbb{R}^n$ is a vector space.

The Euclidean plane

In ancient Greece, they didn't think in coordinates as we now understand them. $\mathbb{R}^n$ would not have been a natural notion.

Instead they had geometry: space, planes, lines, points, circles, etc.



The Euclidean plane

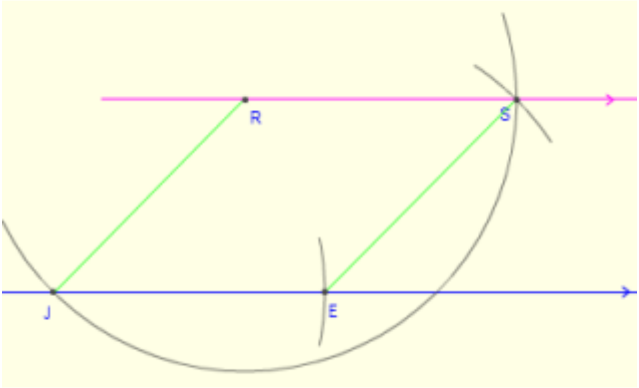
Could you get Euclid to think of a plane as a vector space?

Well, first you need to choose a point to call zero. That's no problem, just pick one and declare it so.

Now you need to say how to add two points. We do this by the parallelogram rule. This is a coordinate-free geometric notion.

The Euclidean plane

Constructing a parallelogram with compass and straightedge.



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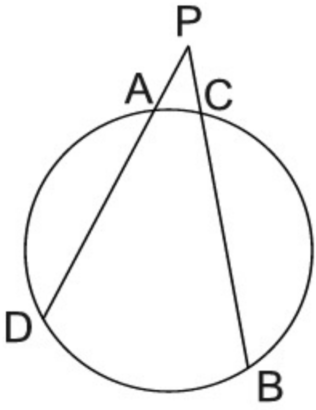
And you need to say how to scale vectors.

Power of a point



$\overline{AP} \cdot \overline{PD} = \overline{CP} \cdot \overline{PB}$

Say P is the zero vector.

To rescale the vector C by λ, construct the segment PA of length 1 and PD of length λ.

Find where the circle through A, C, D meets the line PC.

This point B is the rescaling.

Aside

Actually there is a slight subtlety.

It's true that a straightedge and compass construction can rescale by any given length.

But, how are the lengths given to you in the first place? It turns out that not all real numbers can be constructed with a compass and straight-edge.

If you want to find out more about that, you have to learn a subject called abstract algebra; here at Berkeley it's math 113.

The Euclidean plane

Having said how to define addition and scalar multiplication, we still have to check that they satisfy the axioms.

This can be done geometrically!

An appropriate choice of coordinates on the plane gives an identification with $\mathbb{R}^2$, but note that this can be done in many different ways.

Functions

Consider the set of functions $f : \mathbb{R} \to \mathbb{R}$.

There's a zero function 0 defined by $0(x) = 0$.

We can add: $f + g$ is defined by $(f + g)(x) = f(x) + g(x)$.

We can scale: cf is defined by $(cf)(x) = c \cdot f(x)$.

Is it a vector space? Check the axioms!

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Checking the axioms

To see that, for three functions,

$$f + (g + h) = (f + g) + h$$

it means that we should check that, as functions both sides are the same. In other words, evaluated at any $x \in \mathbb{R}$, they should give the same answer. So we compute

$$(f + (g + h))(x) = f(x) + (g + h)(x) = f(x) + (g(x) + h(x))$$

and similarly

$$((f + g) + h)(x) = (f + g)(x) + h(x) = (f(x) + g(x)) + h(x)$$

The left hand sides are equal by usual associativity of numbers.

Try it yourself!

Check that $f + 0 = f$.

We should show both sides are equal as functions which means that they are equal when evaluated at any $x \in \mathbb{R}$.

$$(f + 0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$$

Try it yourself!

Check that for $c, d \in \mathbb{R}$ and a function f,

$$(c + d)f = cf + df$$

We should show both sides are equal as functions which means that they are equal when evaluated at any $x \in \mathbb{R}$.

$$((c + d)f)(x) = (c + d)f(x) = cf(x) + df(x)$$

$$(cf + df)(x) = (cf)(x) + (df)(x) = cf(x) + df(x)$$

Functions

The remaining axioms can be checked similarly — i.e., by expanding out the definitions and then appealing to the corresponding fact about ordinary numbers.

It is a good exercise to go home and do it yourself.

Given any set S and any vector space V, we could have said all the same words for the set of functions $f : S \to V$.

In this case, instead of appealing to facts about $\mathbb{R}$, you appeal to the vector space axioms for V.

Functions

For instance, if $S = \{1, \ldots, n\}$, then functions $f : S \to \mathbb{R}$ are more or less the same as n-tuples of real numbers.

$$\text{functions: } \{1, \ldots, n\} \to \mathbb{R} \;\cong\; \mathbb{R}^n$$

We'll give a precise meaning to "more or less the same as" soon.

Subspaces

Definition

If V is a vector space, we say a subset W of V is a subspace if:

- Zero is in W.
- The sum of any two elements in W is in W.
- Any scalar multiple of an element in W is in W.

We already saw this notion in case $V = \mathbb{R}^n$.

Subspaces

Fact
If W is a subspace of V, then W is itself a vector space (using the operations inherited from V).

This is because: first, the definition ensures that the addition and scalar multiplication make sense when restricted to W. The axioms hold for elements of W because they held for any elements of V.

Review: Is it a subspace of $\mathbb{R}^3$

Just zero? yes

A single point other than zero? no

A line through the orign? yes

A line not through the orign? no

A union of lines through the orign? no

A circle through the origin? no

A plane through the origin? yes

An intersection of planes through the origin? yes

Subspaces of function space

Some subspaces of the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$.

- Continuous functions
- Differentiable functions
- Differentiable functions whose derivative is continuous
- Differentiable functions whose derivative is differentiable, and the derivative of that is differentiable, and so on forever
- Polynomial functions
- Polynomial functions of degree at most n.

Try it yourself!

Which are subspaces of the vector space of functions $f : \mathbb{R} \to \mathbb{R}$?

Just zero? yes

A single function other than zero? no

Functions which are zero at all integers? yes

Functions satisfying $f(x) = f(x+1)$? yes

Functions satisfying $f''(x) = xf(x)$? yes

Functions satisfying $f(x) = x + 1$? no

Functions equal to their own square ? no

Functions you know how to write down? yes

Span

Given a vector space V and $\mathbf{v}_1, \dots \mathbf{v}_n \in V$ and $c_1, \dots, c_n \in R$, we can name an element

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \in V$$

This is said to be a linear combination of the $\mathbf{v}_i$.

The set of linear combinations of the $\mathbf{v}_i$ is the linear span of the $\mathbf{v}_i$.

We met these notions already for $V = \mathbb{R}^n$. Just like for $\mathbb{R}^n$, linear spans are subspaces.

Linear transformations

Definition

If V and W are vector spaces, a function $T : V \to W$ is said to be a linear transformation if

$$T(c\mathbf{v} + c'\mathbf{v}') = cT(\mathbf{v}) + c'T(\mathbf{v}')$$

for all c, c' in $\mathbb{R}$ and all $\mathbf{v}, \mathbf{v}'$ in V.

We already met this notion for $\mathbb{R}^n$.

Subspaces from linear transformations

The range (a.k.a image) is a subspace.

Given $T : V \to W$, if if $\mathbf{w}, \mathbf{w}'$ are in the range, this means that there must have been $\mathbf{v}, \mathbf{v}'$ with $T(\mathbf{v}) = \mathbf{w}$ and $T(\mathbf{v}') = \mathbf{w}'$. But then for any $c, d \in \mathbb{R}$,

$$c\mathbf{w} + d\mathbf{w}' = cT(\mathbf{v}) + dT(\mathbf{v}') = T(c\mathbf{v} + d\mathbf{v}') \text{ is in the range}$$

We already saw this argument for $\mathbb{R}^n$.

Subspaces from linear transformations

The null space (a.k.a. kernel) is a subspace.

Given $T : V \to W$, if if $T(\mathbf{v}) = 0 = T(\mathbf{v}')$ then for any $c, d \in \mathbb{R}$,

$$T(c\mathbf{v} + d\mathbf{v}') = cT(\mathbf{v}) + dT(\mathbf{v}') = 0$$

so $c\mathbf{v} + d\mathbf{v}'$ is also in the kernel.

We already saw this argument for $\mathbb{R}^n$.

Example

The derivative

$$\begin{aligned} \frac{d}{dx} : (\text{differentiable functions } \mathbb{R} \to \mathbb{R}) &\to (\text{functions } \mathbb{R} \to \mathbb{R}) \\ f(x) &\mapsto f'(x) \end{aligned}$$

is a linear transformation: for functions f, g and constants a, b,

$$\frac{d}{dx}(af(x) + bg(x)) = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$$

The kernel of $\frac{d}{dx}$ is the constant functions.

Its image is not so easy to describe.

Example

The definite integral

$$\begin{aligned} \int_r^s : (\text{continuous functions } \mathbb{R} \to \mathbb{R}) &\to \mathbb{R} \\ f(x) &\mapsto \int_r^s f(x)dx \end{aligned}$$

is a linear transformation: for functions f, g and constants a, b,

$$\int_r^s (af(x) + bg(x))dx = a\int_r^s f(x) + b\int_r^s g(x)dx$$

The kernel are the functions whose average value over $[a, b]$ is zero.

The image is $\mathbb{R}$.

Example

The evaluation at zero

$$\begin{aligned}(\text{functions } \mathbb{R} \to \mathbb{R}) &\to \mathbb{R} \\ f(x) &\mapsto f(0)\end{aligned}$$

is a linear transformation: for functions f, g and constants a, b,

$$(af + bg)(0) = af(0) + bg(0)$$

The kernel are the functions vanishing at zero.

The image is $\mathbb{R}$.

The physicists write this a strange way: "$\int f(x)\delta(x)dx$"