q-Opers, QQ-System and Bethe Ansatz

w/ D Sage, A Zeitlin $(SL(n), q)$-opers
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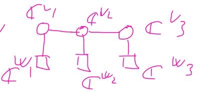
Motivation 1) BPS/CFT correspondence, dualities between integrable systems and gauge/string theory
In particular, quantum/classical duality [PK, Gaiotto 2013]
quantum/quantum

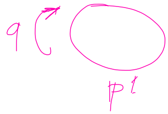
XXZ / tRS or 3d $\mathcal{N}=4$ quiver gauge theory / 4d $\mathcal{N}=2^*$ theory w BPS boundary conditions

2) Recent progress in enumerative AG (Okounkov et al [AFO], [AO]) [PSZ], [KPSZ]
Quantum equivariant K-theory of Nakajima quiver varieties

In physics terms $V_{\vec p} = \left(\text{equiv K-th push forward of moduli space of quasimaps from } \mathbb{P}^1 \text{ to } X\right) = Z^{3d}_{\text{vort}}$ of $\mathcal{N}=2^*$ of quiver gauge theory corresponding to X

X 

k-th vertex function of X

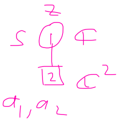


f → X (fixed pts of max torus); $\hbar$ - automorphisms of fibers

$V \underset{q\to 1}{\sim} e^{\frac{\widetilde{W}_{YY}(\vec a,\vec z)}{\log q}}$ for Yang-Yang function

tautological bundles over X → $K_T(X)$, $T = \mathbb{C}^*_\hbar \times \mathbb{C}^*_{a_1} \cdots \mathbb{C}^*_{a_n}$

Quantum version of the duality $H_i^{tRS} V_{\vec p} = e_i(a_1 \ldots a_n) V_{\vec p}$



a_1, a_2

$X = T^*\mathbb{F}l_{n+1}$ in type A, $n=2$

$H_1 = \hbar\frac{z_1 - z_2}{z_1 - z_2} p_1 + \hbar\frac{z_2 - z_1}{z_2 - z_1} p_2$

quantum multiplication by V_1; $\widehat{V_1^*}$

$p_1 f(z_1,z_2) = f(qz_1, z_2)$
$p_2 f(z_1,z_2) = f(z_1, qz_2)$

$X = T^*\mathbb{P}^1$, $z = \frac{z_1}{z_2}$

$p = p_2^{-1} = p$

Classically $(q \to 1)$ $V_{\vec p} \to e^{\frac{\widetilde{W}_{YY}(\vec a,\vec z,\vec s)}{\log q}} \Rightarrow \left\{ \frac{\partial \widetilde{W}(\vec s)}{\partial s} = 0 \right\} \Leftrightarrow \{ H_i = e_i \}$

- Note ~~that~~ this only works for type-A (albeit one can use 'orbifolding' tricks to get some insight about types B and C)
- Also spin chains exist for all possible weights, no constraints on ranks of bundles V_i, W_i

$\mathfrak{sl}_N$ Need to extend the duality $X = T^* Gr_{k,n}$, $k \le n$

It turns out that the proper language to formulate such duality is geometric q-Langlands correspondence

3) quantum Langlands Correspondence. Categorical equivalence $D_\varkappa(Bun\, G) \simeq D_{-\frac{1}{m\varkappa}}(Bun\, {}^L G)$ on Riemann surface X

we will be working at critical level $\varkappa \to 0$

(Automorphic) Bethe equations A-side Gaudin model $D\text{-mod}(Bun\, G) \simeq Loc_{{}^L G}$ Opers (Galois) B-side

Consider difference version of the above relationship

geometric q-Langlands correspondence

A	B
QQ-system ${}^L\mathfrak{g}$	(G,q)-opers

Categorical conjecture made by [Elliot-Pestun] q-Opers, QQ-system, and Bethe Ansatz — Northeastern U 10/10/19 — w/ D. Sage, A. Zeitlin; w/ + E. Frenkel — $(SL(N),q)$-opers [1811.09937]; (G,q)-opers, G - simply connected complex simple Lie group

0 Motivation

1. Dualities in integrable systems: trig RS / XXZ (classical/quantum duality)
2. Recent progress in enumerative AG ([Aganagic, Frenkel, Okounkov], [AO], [Pushkar, Smirnov, Zeitlin], [PK, Pushkar, Smir, Zeit])
3. Quantum q-Langlands correspondence

1 $(SL(2),q)$-opers

Fix $q\in\mathbb{C}^\times$. Given $E\to\mathbb{P}^1$ - holomorphic vector bundle, denote E^q - pull back of E under $z\mapsto qz$. Assume E is trivializable.

Let A: $E\to E^q$. Upon picking a trivialization the map A is determined by matrix $A(z)$ giving a linear map $E_z\to E_{qz}$ in a basis.

A change of trivialization by $g(z)$ is $A(z)\longmapsto g(qz)A(z)g^{-1}(z)$ — q-gauge transformation

Let D_q: $E\to E^q$, $S(z)\mapsto S(qz)$ for each section $S(z)$. A solves difference equation $D_q S = AS$

Def 1: A meromorphic $(GL(N),q)$-connection over $\mathbb{P}^1$ is a pair (E,A), where E is a trivializable vector bundle of rank-N over $\mathbb{P}^1$ and A is a meromorphic section of sheaf $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(E,E^q)$ for which $A(z)$ is invertible.
(E,A) is called $(SL(N),q)$ connection if $\exists$ a trivialization w/ $\det A(z)=1$

(Can replace $SL(N)$ by a complex reductive group G)

Now restrict for type A_1

Def 2. A $(GL(2),q)$ [$SL(2)$] -oper on $\mathbb{P}^1$ is a triple $(E,A,\mathcal{L})$, where (E,A) is a $(GL(2),q)$ [$SL(2)$] - connection and $\mathcal{L}$ is a line subbundle such that the induced map: $\bar{A}$. $\mathcal{L}\xrightarrow{\sim}(E/\mathcal{L})^q$ is an isomorphism.

Explicitly, in terms of sections
$$S(qz)\wedge A(z)S(z)\neq 0,$$
where $\mathcal{L}$ is generated by $S(z)$

Def 3. An $(SL(2),q)$-oper w/ regular singularities at $z_1, z_2, \dots z_L \neq 0,\infty$ w/ weights $k_1,\dots,k_L$ is a meromorphic $(SL(2),q)$-oper $(E,A,\mathcal{L})$ for which $\bar{A}$ is an isomorphism everywhere on $\mathbb{P}^1\setminus\{0,\infty\}$ except $z_m, q^{-1}z_m, q^{-2}z_m, \dots q^{-k_m+1}z_m$, $m=1,\dots,L$, where it has simple poles

In other words, $S(qz)\wedge A(z)S(z)$ has simple zeros at $z_m, q^{-1}z_m, \dots q^{-k_m+1}z_m$

Def 4 An $(SL(2),q)$-oper $(E,A,\mathcal{L})$ w/ regular singularities is called Z-twisted q-oper if A is gauge equivalent to Z^{-1}
$$Z=\mathrm{diag}(\zeta,\zeta^{-1})$$

Def 5 Miura q-oper is a quadruple $(E,A,\mathcal{L},\hat{\mathcal{L}})$, where $(E,A,\mathcal{L})$ is a q-oper and $\hat{\mathcal{L}}$ is a line bundle preserved by A

Quantum Wronskian and Bethe Ansatz

Pick a trivialization when $A=Z^{-1}$. $\mathcal{L}$ is trivial on $\mathbb{P}^1\setminus\infty$, so is generated by
$$S(z)=\begin{pmatrix}Q_+(z)\\ Q_-(z)\end{pmatrix},\quad Q_\pm(z) - \text{polynomials w/o common roots}$$
$$S(z)\wedge Z^{-1}S(qz)=\prod_{m=1}^{L}\prod_{j=0}^{k_m-1}(z-q^{-j}z_m)=:\rho(z)\quad\text{(assume monic)}$$
$$\zeta^{-1}Q_+(z)Q_-(qz)-\zeta Q_+(qz)Q_-(z)=\rho(z)\qquad (*)$$
By a diagonal gauge change can make $Q_-(z)$ monic as well $Q_-(z)=\prod_{i=1}^{\ell}(z-w_i)$

Restrict ourselves on nondegenerate q-opers $q^{\mathbb{Z}}z_m\cap q^{\mathbb{Z}}w_i=\emptyset$ $\forall m,i$

Evaluating equation above at $q^{-1}z$: $\rho(q^{-1}z)=\zeta^{-1}Q_+(q^{-1}z)Q_-(z)-\zeta Q_+(z)Q_-(q^{-1}z)$

Divide $(*)$ by this at roots of Q_- we get
$$\frac{\rho(w_i)}{\rho(q^{-1}w_i)}=-\zeta^{-2}\frac{Q_-(qw_i)}{Q_-(q^{-1}w_i)}\qquad \text{Setting } k=\sum_{m=1}^{L}k_m:\quad q^{k}\prod_{m=1}^{L}\frac{w_i-q^{-1-k_m}z_m}{w_i-qz_m}=-\zeta^{-2}\prod_{j=1}^{\ell}\frac{qw_i-w_j}{q^{-1}w_i-w_j}$$

or the standard sl_2 XXZ Bethe Ansatz equations
$$\prod_{m=1}^{L}\frac{w_i-q^{k_m}z_m}{w_i-qz_m}=-\zeta^{-2}q^{\ell-k}\prod_{j=1}^{\ell}\frac{qw_i-w_j}{w_i-qw_j}$$
(annotations: k_m — spin; qz_m — anisotropy (shift of spectral parameters); $-\zeta^{-2}q^{\ell-k}$ — twist; w_j — Bethe roots)

Theorem 1: {Set of nondegenerate solutions of sl_2 XXZ BAE with spins $k_1,\dots,k_L$ and anisotropies $z_1,\dots,z_L$} $\underset{1-1}{\Longleftrightarrow}$ {Set of nondegenerate Z-twisted $(SL(2),q)$-opers w/ sing $z_1,\dots,z_L\neq 0,\infty$, weights $k_1,\dots,k_L$}

3 QQ-Miura transform and transfer matrix

Pick $\hat{\mathcal{L}}$ generated by $S(z)$ s.t.
$$g(z)S(z)=\begin{pmatrix}0\\1\end{pmatrix},\quad g(z)=\begin{pmatrix}Q_-(z) & -Q_+(z)\\ 0 & Q_-^{-1}(z)\end{pmatrix}$$

q-connection matrix has the form
$$A(z)=\begin{pmatrix}Q_-(qz)\zeta^{-1} & -\zeta Q_+(qz)\\ 0 & \zeta Q_-^{-1}(qz)\end{pmatrix}\begin{pmatrix}Q_-(z) & -Q_+(z)\\ 0 & Q_-^{-1}(z)\end{pmatrix}=\begin{pmatrix}\zeta^{-1}\frac{Q_-(qz)}{Q_-(z)} & \rho(z)\\ 0 & \zeta\frac{Q_-(z)}{Q_-(qz)}\end{pmatrix}=\begin{pmatrix}a(z) & \rho(z)\\ 0 & a^{-1}(z)\end{pmatrix}$$
(first factor: $g(qz)Z^{-1}g(z)$)

Apply diagonal gauge transformation $\begin{pmatrix}1 & 0\\ \frac{a(z)}{\rho(z)} & 1\end{pmatrix}$ to bring the q-connection to the form
$$\hat{A}(z)=\begin{pmatrix}0 & \rho(z)\\ -\rho^{-1}(z) & T(qz)\rho^{-1}(qz)\end{pmatrix},$$
where $T(z)=\zeta^{-1}\rho(q^{-1}z)\frac{Q_-(qz)}{Q_-(z)}+\zeta\rho(z)\frac{Q_-(q^{-1}z)}{Q_-(z)}$ — eigenvalue of XXZ sl_2 transfer matrix [Reshetikhin]

q-Difference equation
$$D_q\begin{pmatrix}f_1\\f_2\end{pmatrix}=A\begin{pmatrix}f_1\\f_2\end{pmatrix}\Rightarrow\left(D_q^2-T(qz)D_q-\frac{\rho(qz)}{\rho(z)}\right)f_1=0$$

Theorem 2 $\{(SL(2),q)\text{-opers}\}\Longleftrightarrow\{\text{Meromorphic q-connections}\}$

23 Trigonometric Ruijsenaars-Schneider Model w/ 2 degrees of freedom

Take dy $Q_{\pm}(z) = 1$, $\rho(z) = (z-z_+)(z-z_-)$

$$Q_- = z - P_-, \quad Q_+ = C(z - P_+), \quad C = q^{-1}(\zeta - \zeta^{-1})$$

From quantum Wronskian relation

$$z^2 - \frac{z}{q}\left[\frac{\zeta - q\zeta^{-1}}{\zeta - \zeta^{-1}} P_+ + \frac{q\zeta - \zeta^{-1}}{\zeta - \zeta^{-1}} P_-\right] + \frac{P_+ P_-}{q} = (z - z_+)(z - z_-)$$

from where tRS Hamiltonians

$$H_1 = q(z_+ + z_-)$$
$$H_2 = z_+ z_-$$

$$qWr \Leftrightarrow \det\left(z\mathbb{1} - L_{tRS}\right) = \rho(z)$$

24 SL(N) Generalization

25 Opers hierarchy

26 Enumerative geometry

5 (SL(N), q) − opers $SL(2) \longrightarrow SL(N)$

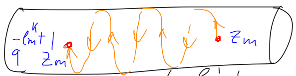
Def 6: A $(GL(N), q)$ − oper on $\mathbb{P}^1$ is a triple $(\mathcal{E}, A, \mathcal{L}_\bullet)$, where $(\mathcal{E}, A)$ is a $(GL(N), q)$ − connection and $\mathcal{L}_\bullet$ is a complete flag of subbundles $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \dots \subset \mathcal{L}_{N-1} \subset \mathcal{E}$ such that A maps $\mathcal{L}_i$ to $\mathcal{L}^q_{i+1}$ and induced maps $\bar{A}_i : \mathcal{L}_i / \mathcal{L}_{i-1} \xrightarrow{\sim} \mathcal{L}^q_{i+1} / \mathcal{L}^q_i$ are isomorphisms, $i = 1, N-1$. We get $(SL(N), q)$ − oper if $(\mathcal{E}, A)$ is an $(SL(N), q)$ − connection.

Explicitly pick $s(z)$ − local section of $\mathcal{L}_1$

$$\mathcal{W}(z) \sim \left(s(q^{j-1} z) \wedge A(q^{j-2} z)\, s(q^{j-2} z) \wedge \dots \wedge \left(\prod_{j=0}^{i-2} A(q^{i-2-j} z) \right) s(z) \right) \Big|_{\Lambda^i \mathcal{L}_i^{q^{i-1}}} \neq 0$$

Def 7. $(SL(N), q)$ oper w/ regular singularities at $z_1, \dots, z_L \neq 0, \infty$ w/ weights $\lambda_1, \dots, \lambda_L$ is a meromorphic $(SL(N), q)$ − oper s.t. each A_i is an iso everywhere except $q^{-\ell_m^{i-1}} z_m \quad q^{-\ell_m^i + 1} z_m$, $\lambda_m = \sum \ell_m^i \omega_i$, where ω_i − dominant integral weights

$$\ell_m^i = \sum_{j=1}^{i} \ell_m^j$$



Define $\Lambda_1 = \prod_{m=1} \prod_{j=\ell_m^{i-1}}^{\ell_m^i - 1} (z - q^{-j} z_m)$, $\quad P_i = \Lambda_1 \dots \Lambda_i$

q-difference equation $\quad f^{(j)}(z) = D_q^j (f)(z) = f(q^j z)$

$$W_k(s) = \Lambda_1 (\Lambda_1^{(1)} \Lambda_2^{(1)}) \dots (\Lambda_1^{(k-2)} \dots \Lambda_{k-1}^{(k-2)}) = P_1 P_2^{(1)} \dots P_{k-1}^{(k-2)}$$

Def 8 An $(SL(N), q)$ − oper $(\mathcal{E}, A, \mathcal{L}_\bullet)$ is Z − twisted if A is gauge equivalent to $Z^{-1} = \mathrm{diag}(\zeta_1, \zeta_2, \dots, \zeta_N)$, $\prod_{a=1}^{N} \zeta_a = 1$

Def 9 q − Miura structure $(\mathcal{E}, A, \mathcal{L}_\bullet, \hat{\mathcal{L}}_\bullet)$, where $\hat{\mathcal{L}}_\bullet$ is preserved by A (some genericity conditions)

Let $s(z) = (s_1(z) \dots s_N(z))$ − section generating $\mathcal{L}_1$. Express in terms of Wronskians

$$\mathcal{D}_k(s) = e_1 \wedge \dots \wedge e_{N-k} \wedge s(z) \wedge Z s(qz) \wedge \dots \wedge Z^{k-1} s(q^{k-1} z) \qquad \text{(+ nondegenerate conditions)}$$

$$\mathcal{V}_k(z) = \prod_{a=1}^{r_k} (z - v_{k,a})$$

$$\mathrm{Det}\, \mathcal{D}_k(s) = \alpha_k W_k \mathcal{V}_k \qquad \text{in index notation} \quad \det_{i,j} \left[\zeta_{N-k+i}^{j-1} s_{N-k+i}^{(j-1)} \right] = \alpha_k W_k \mathcal{V}_k$$

Theorem 3 $\left\{ \begin{array}{c} sl_N \text{ XXZ} \\ \text{Bethe equations} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} Z\text{-twisted} \\ (SL(N), q) \text{ − opers} \end{array} \right\}$

In $Q\tilde{Q}$ form

$$\xi_{k+1} Q_k^{(-\frac{1}{2})} \tilde{Q}_k^{(\frac{1}{2})} - \xi_k Q_k^{(\frac{1}{2})} \tilde{Q}_k^{(-\frac{1}{2})} = (\xi_{k+1} - \xi_k) Q_{k-1} Q_{k+1} \Pi_k$$

twists $\zeta \sim \xi$

$$Q_k = \prod_{a=1}^{r_k} (z - u_{k,a}), \qquad \Pi_k = \# \prod_{s=1} \prod_{j = \ell_s^{k-1}}^{\ell_s^k - 1} (z - q^{1 - \frac{k}{2} - j} z_s)$$

Bethe roots $u \sim \upsilon$

5 (SL(3), q) example (q − DS reduction)

$A \sim \mathrm{diag}(\zeta_1^{-1}, \zeta_2^{-1}, \zeta_1 \zeta_2)$ After diagonal gauge transform reduce q − connection to $A(z) = \begin{pmatrix} a_1 & \Lambda_2 & 0 \\ 0 & a_2 & \Lambda_1 \\ 0 & 0 & a_3 \end{pmatrix}$

Difference equation

$$D_q \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = A \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \Rightarrow \Lambda_1 \Lambda_2 \Lambda_2^{(1)} f_1^{(3)} - \Lambda_2 T_1 f_1^{(2)} + \Lambda_2^{(2)} T_2 f_1^{(1)} - \Lambda_1^{(1)} \Lambda_2^{(1)} \Lambda_2^{(2)} f_1 = 0$$

T_1, T_2: eigenvalues of SL(3) XXZ transfer matrix

6 Scaling limits

XYZ	XXX	Gaudin model
(SL(N), q) opers on $\mathbb{C}^\times$	Twisted discrete opers	Opers

(8) (G,q)-opers for arbitrary simply-connected simple Lie group G

Let B_- - Borel subgroup, $N_- = [B_-, B_-]$ - unipotent radical, $H \subset B_-$ - maximal torus, $\{e_i, f_i, \check{\alpha}_i\}_{i=1,\dots,r}$ - Chevalley triple

generated by $\{f_i, \check{\alpha}_i\}$ $\langle \check{\alpha}_j, \omega_i \rangle = \delta_{ij}$, $a_{ji} = \langle \check{\alpha}_j, \alpha_i \rangle$

$W = N(H)/H$ - Weyl group

w_i - simple reflections, s_i - their lifts to $N(H)$ - normalizer

<u>Def</u> (G,q)-oper on $\mathbb{P}^1$ is a triple $(\mathcal{F}_G, \mathcal{F}_{B_-}, A)$, $\mathcal{F}_G$ - principal G-bundle, $\mathcal{F}_{B_-}$ its Borel reduction, $A \in Hom_{\mathcal{O}(\mathbb{P}^1)}(\mathcal{F}_G, \mathcal{F}_G^{(q)})$ - q-connection such that for any $C \in Hom_{\mathcal{O}(\mathbb{P}^1)}(\mathcal{F}_{B_-}, \mathcal{F}_{B_-}^{(q)})$ the expression $C^{-1}A \in Hom_{\mathcal{O}(\mathbb{P}^1)}(\mathcal{F}_G, \mathcal{F}_G)$ takes values in $M^s = B_- s B_-$, $s = \prod s_i$
(C - q-connection preserving $\mathcal{F}_{B_-}$) (particular order of simple reflections)

Locally, on a Zariski open subset of $\mathbb{P}^1$ $A(z) = n'(z) \prod_i (\phi_i^{\check{\alpha}_i} s_i)\, n(z)$, $n(z), n'(z) \in N_-(z)$

Now add singularities. polynomials $\{\Lambda_i(z)\}_{i=1,\dots,r}$ where
$$A(z) = n'(z) \prod_i (\Lambda_i^{\check{\alpha}_i}(z) s_i)\, n(z)$$

+ add Z-twisted (G,q)-oper on $\mathbb{P}^1$ which is q-gauge equivalent to a constant element $Z \in H(z)$

$\exists\, g(z) \in G(z)$ s.t. $A(z) = g(qz) Z g^{-1}(z)$, $Z = \prod_{i=1}^r \zeta_i^{\check{\alpha}_i}$, $\zeta_i \in \mathbb{C}^\times$
defined up to an action of W

Using Bruhat decomposition $G(z) = \bigsqcup_{w \in W} B_-(z) w B_-(z)$ we can prove the following

<u>Theorem</u> $\forall$ Z-twisted (G,q)-oper $\tilde{A}(z)$ we can choose the element $g(z)$ to be in $B_-(z) w_0 B_-(z)$, where w_0 is the maximal element of W

<u>Corollary</u> $\forall$ Z-twisted (G,q)-oper $A(z)$ $\exists\, n(z) \in N_-(z)$ s.t. $A(z) = n(qz)\tilde{A}(z) n^{-1}(z) \in B_+(z)$ has the form $A(z) = v(qz) Z' v^{-1}(z)$, $v(z) \in B_+(z)$,
where Z' is in W-orbit of Z. Thus $\mathcal{F}_G$ carries a B_+-reduction which is preserved by the oper's q-connection

<u>Def</u> Z-twisted Miura q-oper on $\mathbb{P}^1$ is the oper as above which can be written as
$A(z) = v(qz) Z v^{-1}(z)$, $v(z) \in B_+(z)$

<u>Explicit form of Miura q-opers</u> Using the following proposition about double Bruhat cells

Map $(\mathbb{C}(z)^\times)^r \longrightarrow G(z)$ gives an isomorphism to a Zariski-open subset of $B_+(z) \cap N_-(z) \prod_i (\Lambda^{\check{\alpha}_i} s_i) N_-(z)$

$(g_1, \dots, g_r) \longmapsto \prod_i g_i^{\check{\alpha}_i} e^{\frac{\Lambda_i}{g_i} e_i}$

We can write the q-connection$_{\text{Miura}}$ as follows $A(z) = \prod_i g_i^{\check{\alpha}_i}(z)\, e^{\frac{\Lambda_i(z)}{g_i(z)} e_i}$

<u>Main idea!</u> Attach to a collection of $(SL(2), q)$-opers $A_i(z)$ to our q-oper

V_i - fin dim rep of G with weight ω_i (highest wrt B_+)

W_i - 2d subspace spanned by $\{v_{\omega_i}, f_i v_{\omega_i}\}$ (highest weight vector)

Then W_i is B_+-invariant,
so we have a rank-2 subbundle $\mathcal{W}_i$ associated with W_i
$\mathcal{G}_i$

<u>Lemma</u> q-connection $A(z)$ acting on $\mathcal{V}_i$ preserves the subbundle $\mathcal{W}_i$ and
$$A_i(z) = A(z)\big|_{\mathcal{W}_i} = \begin{pmatrix} g_i & \Lambda_i \prod_{j>i} g_j^{-a_{ji}} \\ 0 & g_i^{-1} \prod_{j \neq i} g_j^{-a_{ji}} \end{pmatrix}$$

Let $\mathcal{L}_i \subset \mathcal{W}_i$ be a line subbundle spanned by $f_i v_{\omega_i}$ w/ weight $\omega_i - \alpha_i$

By Lemma we have that $A(z)$ gives rise to q-conn on A_i on $\mathcal{W}_i$, $i = 1, \dots, r$ s.t.

$\mathcal{L}_i \longrightarrow \mathcal{W}_i \rightarrow \mathcal{W}_i/\mathcal{L}_i$ induced by A_i is an isomorphism except finite # pts on $\mathbb{P}^1$

<u>Main Theorem</u>

Nondegenerate Miura (G,q) opers	$\Longleftrightarrow$	w_0-nondegenerate solutions of QQ-equations
$g_i(z) = \zeta_i \dfrac{Q_+^i(qz)}{Q_+^i(z)}$		$\Lambda_i(z) \prod_{j>i} \left(Q_+^j(qz)\right)^{-a_{ji}} \prod_{j<i} \left[Q_+^j(z)\right]^{-a_{ji}} = \tilde{\xi}_i Q_-^i(z) Q_+^i(qz) - \xi_i Q_-^i(qz) Q_+^i(z)$, $\tilde{\xi}_i = \zeta_i \prod_{j>i} \zeta_j^{a_{ji}}$, $\xi_i = \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}}$

the same as in Reshetikhin's paper for ADE, different otherwise