Braneworld Models with Broken Bulk Lorentz Invariance

Peter Koroteev, Maxim Libanov

INR RAS, Moscow and UMN

Papers

PK, Maxim Libanov. On Existence of Self-Tuning Solutions in Static Braneworlds without Singularities.
JHEP 02(2008)104. arXiv:0712.1136 [hep-th].

PK, Maxim Libanov. Field Fluctuations in Braneworld Models with Broken Lorentz Invariance.
arXiv:08xx.xxxx.

Motivation

- Problem of Generations
- Heirarchy Problem
- Cosmological Constant Problem
- Fine-tuning problem in Standard Model

Outline

- Extra Dimensions: Big vs. Small
- Randall–Sundrum model
- Lorentz Invariance Violation
- Matter and NO-GO Theorem

- Localization/Delocalization of Fields
- Spectrum of Field Perturbations

GR

Hilbert-Einstein action

$$S = \int\limits_{X^D} \sqrt{|g|}\, d^D x \left(M_{Pl}^{D-2} R + \Lambda \right)$$

Einstein equations

$$G_{AB} := R_{AB} - \frac{1}{2} R g_{AB} = T_{AB}$$

$\Lambda^{1/D} \lll M_{Pl}$

Want them comparable? Extra dimesions?

Xdimensions: small ...

Spacetime $\mathbb{R}^{1,3} \times K$ where K is compact. [Kaluza, Klein 1925]
Simplest example $K = S^1$ of radius $1/k$.
Quantum mechanical partice possesses momentum $p_z \sim nk$, $n \in \mathbb{Z}$ (wave function periodicity)

If k big (small Xdim), then hard to check experimentally

Nevetheless may be employed in combination with another mechanisms.

... and large!

for Cartesian product

$$M = N(x) \times K(y)$$

$$ds^2 = ds_N^2(x) + ds_K^2(y)$$

for warped product

$$ds^2 = ds_1^2(x,y) + ds_2^2(x,y)$$

e.g. AdS_5

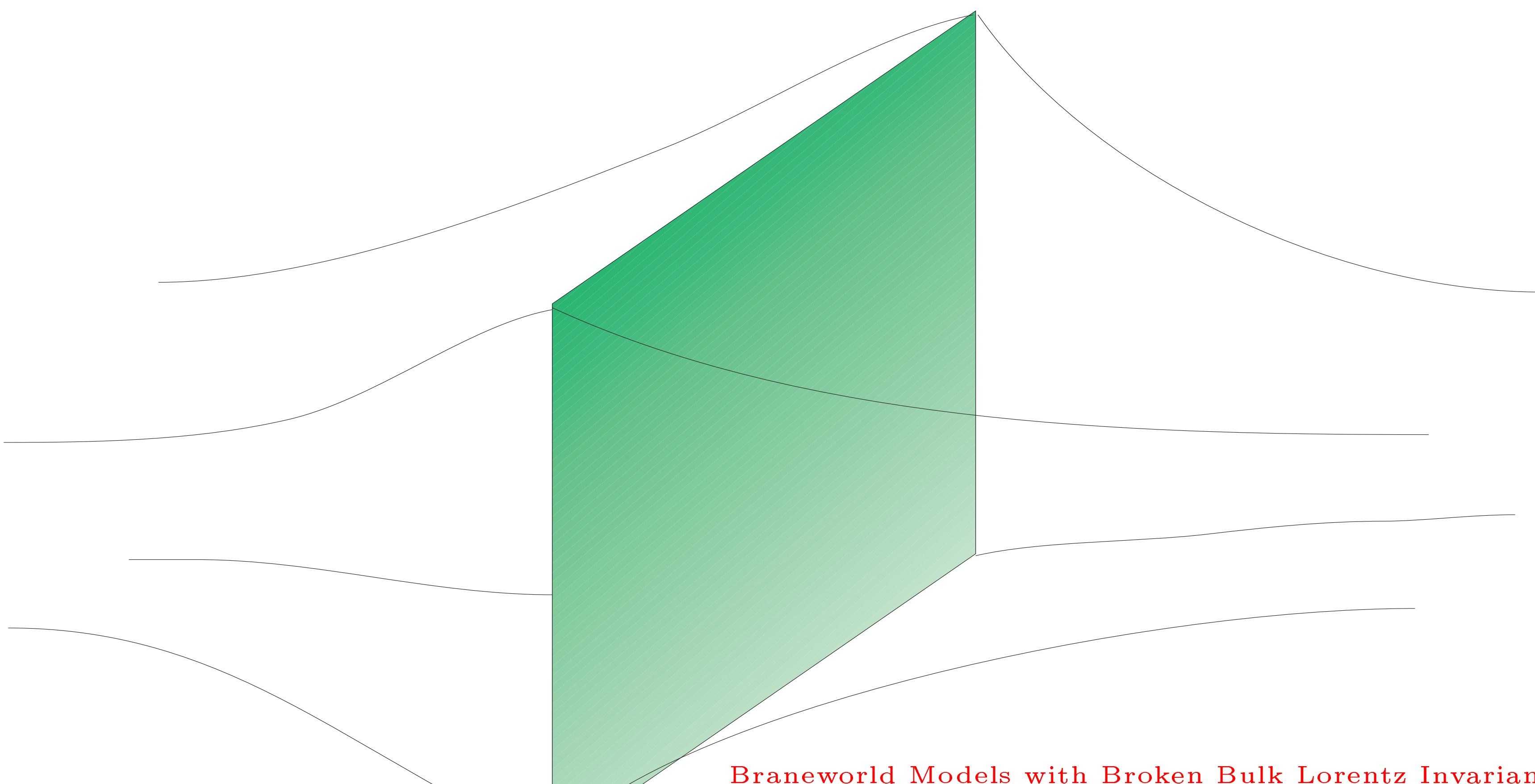
$$ds^2 = \frac{R^2}{y^2}\left(dt^2 - d\mathrm{x}^2 - dy^2\right)$$

lternative to Compactification

Introduce a topological deffect – 3– brane [Rubakov, Shaposhnikov 1983, ADD 1998 Randall Sundrum 1999]

$G_{AB} = T_{AB}^{(bulk)} + T_{AB}^{(brane)}$ for

$T_{AB}^{(brane)} = \mathrm{diag}(\varepsilon_b\, p_b\, p_b\, p_b\, 0)\,\delta(y - y^{(brane)})/\sqrt{g}$



Braneworld

$$S = S_{gravity} + S_{brane} = \int d^5x \sqrt{g} \Big(R - 2M^3\Lambda + \delta(z - z^{(brane)}) \mathscr{L}$$

way to solve

- Solve EE in bulk

$$G_{AB} = T_{AB}^{(bulk)}$$

- Boundary conditions for second fundamental form $\Rightarrow$ Israel junction conds [Israel '66]

$$[K_B^A]|^{(brane)} = S_B^A|^{(brane)} - \frac{1}{3} S^{(brane)} \delta_B^A$$

Randall–Sundrum Model

5D matter $T^A_B = \Lambda \delta^A_B$, then a static soution exists [Randall, Sundrum 1999]

Metric (AdS_5) $ds^2 = e^{-2k|z|}(dt^2 - d\mathrm{x}^2) - dz^2$

Brane with tension σ and

$$\Lambda = -\frac{\sigma^2}{6}M^3, \quad k = \frac{\sigma}{6}$$

Heirarchy between Λ and σ

How the solution will change if we put some matter in the bulk?

5D Lorentz Invariance Violation

Consider space with symmetry $SO(3) \times T^3 \times \mathbb{Z}_2$

Metric (GN gauge)

$$ds^2 = e^{-2a(z)} dt^2 - e^{-2b(z)} d\mathrm{x}^2 - dz^2$$

LI violated, if $a(z) \neq b(z)$. Brane located at $z = 0$ preserves LI.

Null Energy Condition (NEC)

$$T_{AB}\xi^A\xi^B \geq 0, \quad g_{AB}\xi^A\xi^B = 0$$

Need to minimize bilinear form T over hypersurface $g_{AB}\xi^A\xi^B = 0$

Examples $p = w\rho$ NEC reads $w \geq -1$ For vacuum $w = -1$.
NEC implies for adiabatic regime $v_s \leq c$.

It appears that $w < -1$ is not too bad sometimes...(Phantoms...)

LIV Solution

Metric

$$ds^2 = e^{-2k\xi|z|}dt^2 - e^{-2k\zeta|z|}d\mathrm{x}^2 - dz^2$$

Matter – ideal reliativistic fluid

$$T_B^A = u^A u_B \rho - p\delta_B^A + \Lambda, \quad u_A u^A = 1$$

equation of state $p = w\rho$, $p_5 = \omega\rho$. NEC $|w| < 1, |\omega| < 1$

$$\rho = -\Lambda + 6k^2\zeta^2$$

$$w = -1 + \frac{3\zeta^2 - 2\zeta\xi - \xi^2}{\rho}$$

$$\omega = -1 + \frac{3\zeta(\zeta - \xi)}{\rho}$$

No-Go Theorem

Let the spartial curvature be equal to zero. Then if a static solution is singular then ALL its singularities are naked provided that both brane and bulk NECs are satisfied. [Cline, Firouzjahi 2001]

The "exceptional" case is the AdS space

$$ds^2_{(D+1)} = e^{-k|z|} ds^2_{(D)} - dz^2$$

warped sliced space – the solution of braneworld setup with Λ-term in the bulk and λ-term on the brane.
NEC is satisfied: $\omega = w = -1$
and the relation from junction conditions
$F(k, \Lambda, \lambda) = 0$

NO–GO Theorem with Broken LI

Let the following conditions be satisfied

1. 5d NEC $\quad T_{AB}\xi^A\xi^B \geq 0, \quad g_{AB}\xi^A\xi^B = 0$
2. brane NEC $\quad T_{b,\mu\nu}\xi^\mu\xi^\nu \geq 0, \quad g_{b,\mu\nu}\xi^\mu\xi^\nu = 0$
3. $\rho_b + \sigma \geq 0$
4. Spartial brane curvature vanishes $k = 0$
5. LI is broken $a(z) \neq b(z)$.

Then a static solution with symmetry $SO(3)\times T^3 \times \mathbb{Z}_2$ does not exist. [PK, Libanov 2007]

The statement does not depend on the volume $\int_{-\infty}^{+\infty}\sqrt{g}\,dz$ of the extra dimension.

Proof Sketch

Bulk NEC leads to the inequalities

$$b'' - a'' - 3b'^2 + a'^2 + 2a'b' \geq 0$$
$$b'' - b'^2 + b'a' \geq 0$$

On the brane

$$2(b'(0) - a'(0)) = p_b + \rho_b$$

Reshuffle and replace by equalities with $\phi(z) > 0$, e.g.

$$b'' - a'' - 3(a' - b')^2 - 4a'(b' - a') = \phi(z)(b' - a')$$

It follows

$$b'(z) = a'(z) - \frac{\exp\left(4a(z) + 4\int_0^z \phi(y)\,dy\right)}{3\int_0^z dy \exp\left(4a(y) + 4\int_0^y \phi(t)\,dt\right) - C}$$

LIV case

Theorem statement simplifies $b'' \geq 0$. If we additionally impose positiveness of bulk energy, it yields a new condition for b.
In particular the following function fits

$$b'' = \begin{cases} (z - z_0)^2, & 0 \leq z < z_0, \\ 0, & z \geq z_0. \end{cases}$$

i.e. the soution which does not “deviate considerably” from Λ-term solution

Thus we have classified all LIV static solutions.

“Resolution” of the issue – Spherical Brane

Metric

$$ds^2 = dt^2 - \frac{e^{-2k|z|}}{\left(1+\frac{\kappa^2}{4}x_i x^i\right)^2} d\mathrm{x}^2 - dz^2,$$

Solution for anisotropic fluid

$$w = -\frac{(\xi^2 + 2\xi\zeta + 3\zeta^2)e^{-2\zeta kz} - 4\kappa^2/k^2}{6\zeta^2 e^{-2\zeta kz} - 12\kappa^2/k^2}, \quad \omega = -\frac{(\xi\zeta + \zeta^2)e^{-2\zeta kz} - 4\kappa^2/k^2}{2\zeta^2 e^{-2\zeta kz} - 4\kappa^2/k^2}$$

In order to fit NEC we need $\kappa \sim k$, which is unacceptable...

Spectrum of Perturbations

Localization/Delocalization – Sketch

Geodesic equation

$$\ddot{x}^A + \Gamma^A_{BC}\dot{x}^B\dot{x}^C = 0$$

reads

$$\ddot{z} = a' \mathrm{e}^{-2a}\left(\dot{x}^0\right)^2 - b' \mathrm{e}^{-2b}\left(\dot{\mathrm{x}}\right)^2$$

Combined with light cone condition

$$g_{AB}\dot{x}^A\dot{x}^B = 0$$

yields

$$\ddot{z} = -(b' - a')\mathrm{e}^{-2a}\dot{x}^0 + b'\dot{z}^2$$

Which roughly means that $a' < b'$ – localization and $a' > b'$ – delocalization

Localization/Delocalization – Proof

Scalar 5D field

$$S = \int dt d\mathrm{x} \int_{-\infty}^{+\infty} dz \sqrt{g}\, g^{AB} \partial_A \phi \partial_B \phi$$

EOM

$$\left[-\partial_z^2 + (a' + 3b')\partial_z + \mathrm{e}^{2a(z)}\partial_t^2 - \mathrm{e}^{2b(z)}\partial_i^2 \right] \phi = 0$$

Fourrier transform, redefinition of ϕ, and reparametrization of z yield Schrödinger equation

$$\chi'' + (E^2 - V)\chi = 0$$

for the potential

$$V = -\frac{1}{4}a'^2 + \frac{9}{4}b'^2 + p^2 \mathrm{e}^{2(b-a)} - \frac{1}{2}(a'' + 3b'')$$

Localization/Delocalization – Features

- The behavior at infinity which is controlled by the Lorentz invariance violation (the sign of $b-a$). The potential V can increase/decrease as $y \to \infty$.
- The sign of the delta-function term. The potential may have either delta-well or delta-peak depending on this sign.

Localization/Delocalization – Features

- If the momenta-dependent term increases as $y \to \infty$, then one has a discrete spectrum as in the box-type potential. The potential might have local minima and maxima but the behavior of this potential at infinity qualitatively defines the character of the spectrum. On the contrary, if the potential decays at infinity, then we have continuous spectrum of plane waves propagating along y-direction. Some combination of these two scenarios is possible when $V \to V_\infty = const$ as $z \to \infty$. Then those modes with $E^2 < V_\infty$ belong to discrete spectrum and modes with $E^2 > V_\infty$ contribute to continuous spectrum.
- The sign of delta-function term affects zero mode existence. In a delta-well there might be a zero-mode and none in a delta-peak.

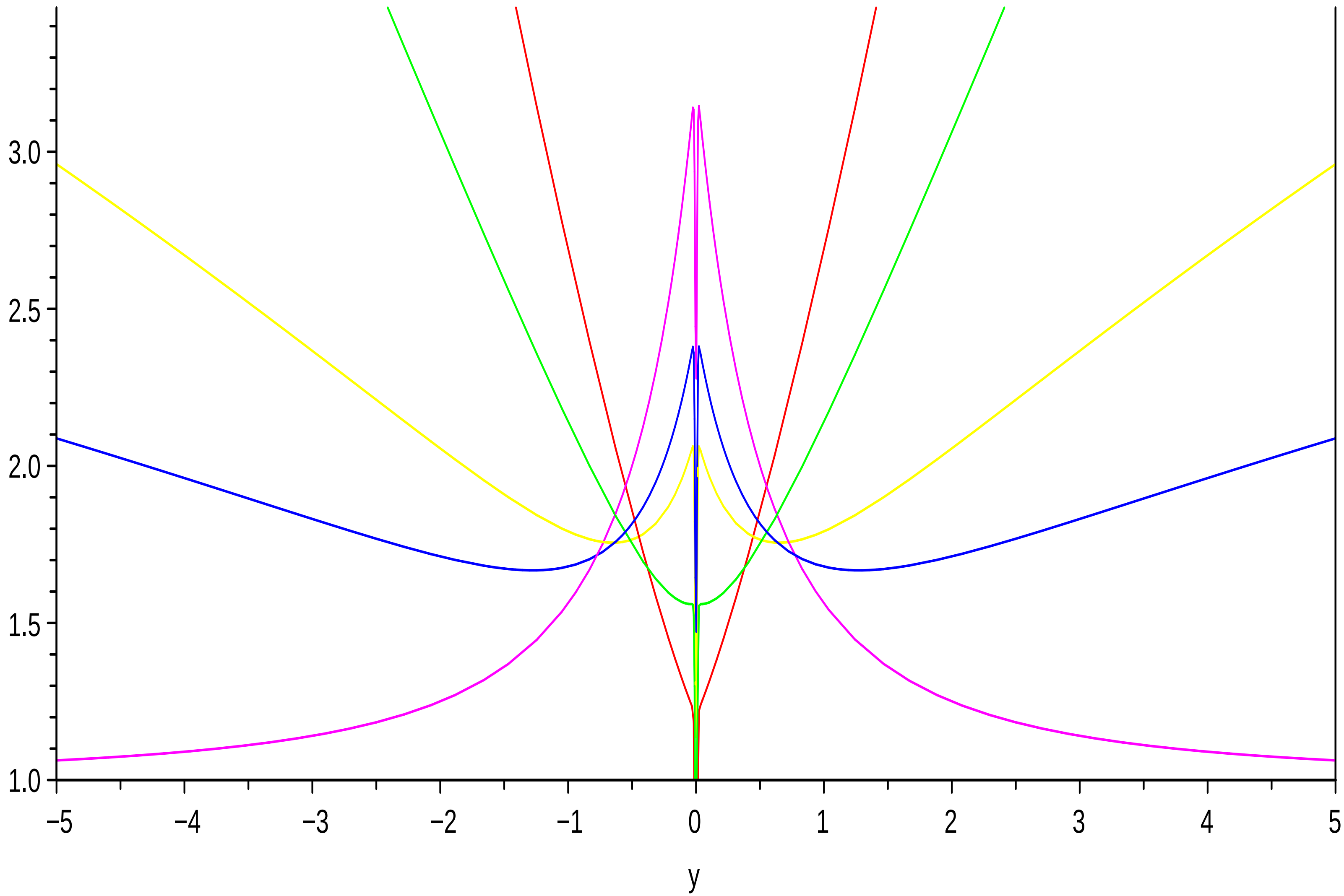
Linear Parameterization

$a(z) = \xi k|z|$, $b(z) = \zeta k|z|$

The potential

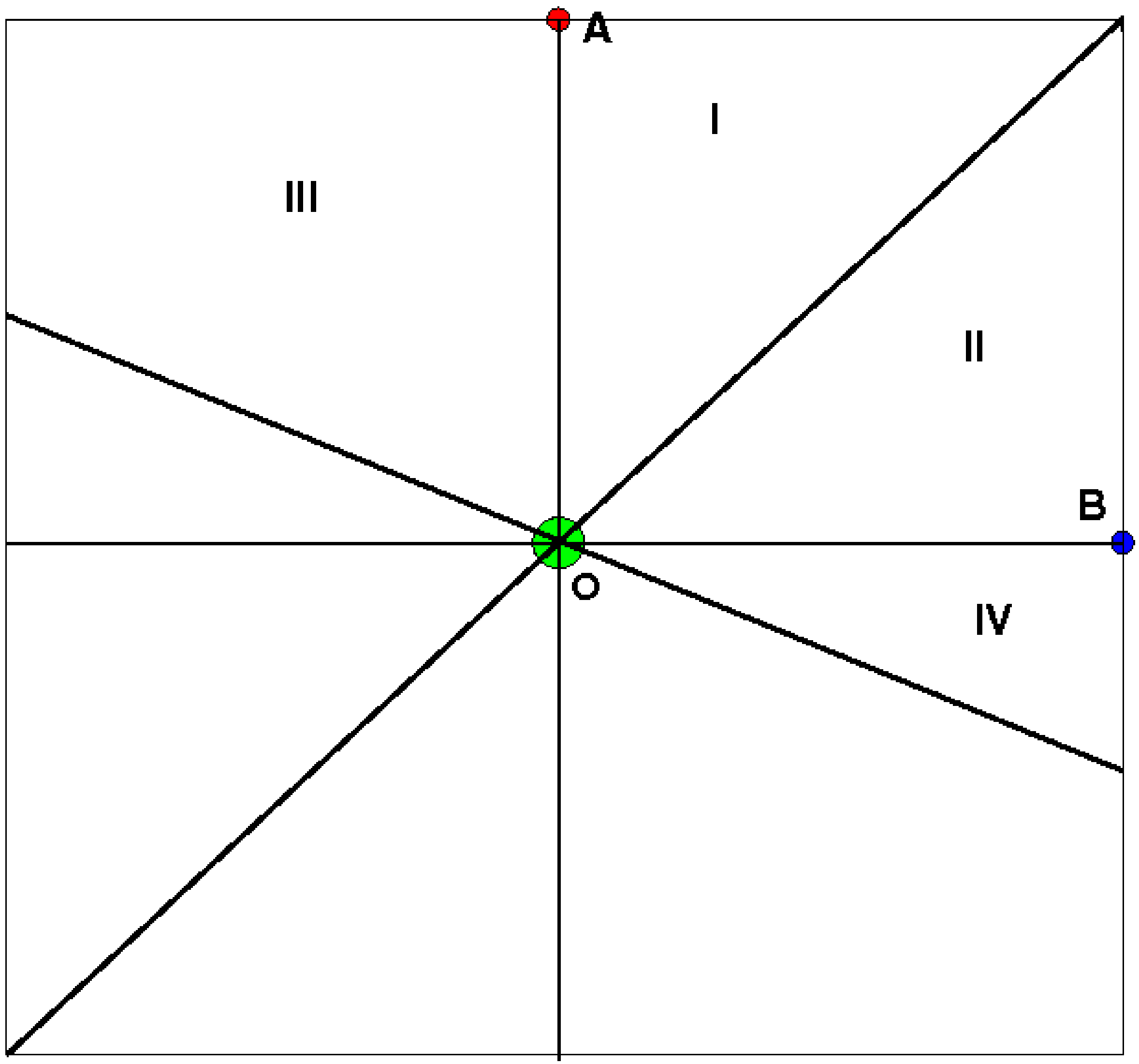
$$V_p(y) = p^2\left(1+\xi k|y|\right)^{2(1-\zeta/\xi)} + \frac{9\zeta^2 k^2}{4\left(1+\xi k|y|\right)^2} - 3\zeta k\delta(y)$$

Family of Potentials



Space of Metrics

$$V_p(y) = p^2\left(1+\xi k|y|\right)^{2(1-\zeta/\xi)} + \frac{9\zeta^2k^2}{4\left(1+\xi k|y|\right)^2} - 3\zeta k\delta(y)$$



Spectrum of Model A

Put $\xi = 0$, $\zeta = 1$. The potential

$$V(z) = p^2 e^{2k|z|} + \frac{9}{4}k^2 - 3k\delta(z)$$

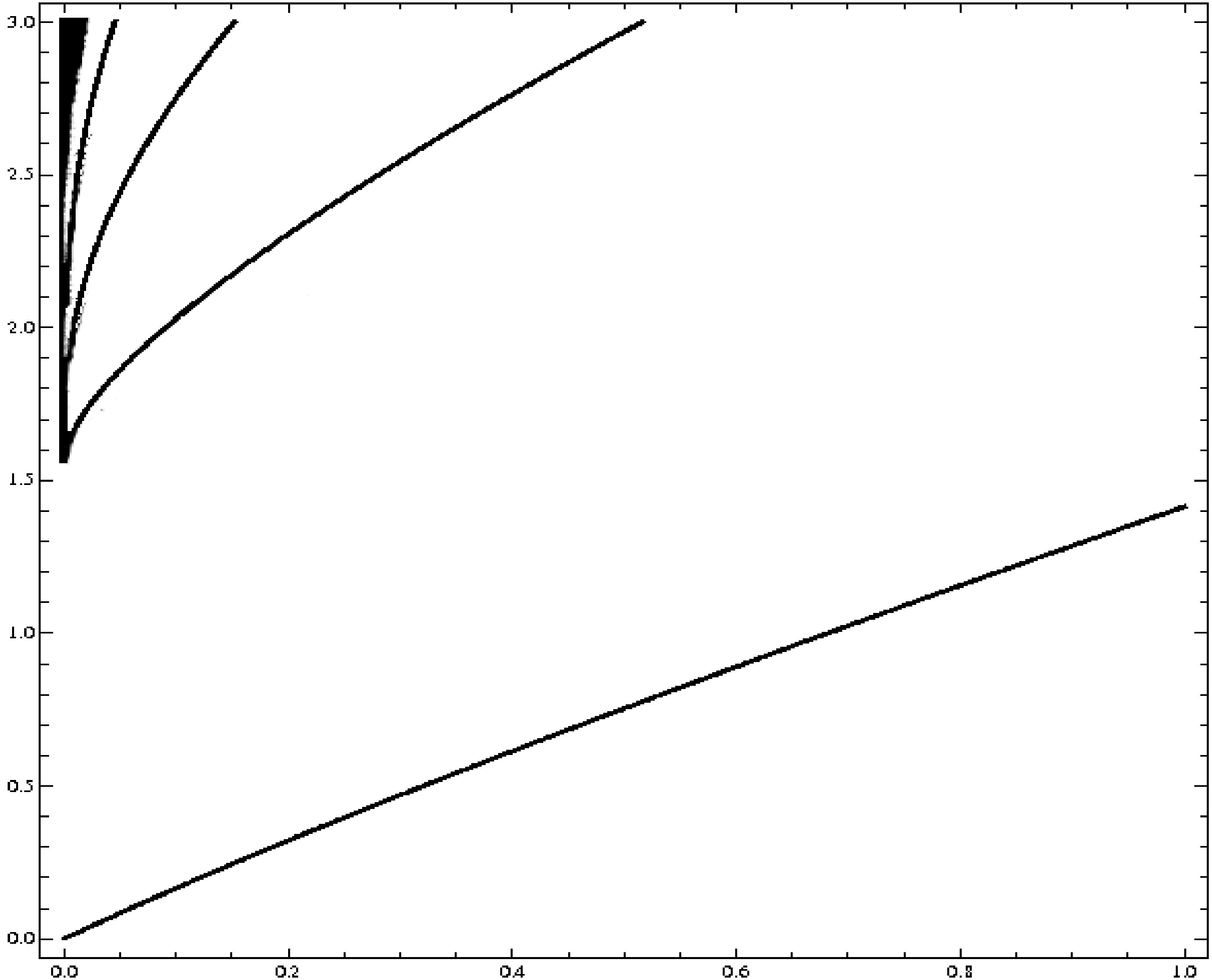
Generic solution

$$\chi(z) = N\,\mathrm{K}_\nu\left(\frac{p}{k}e^{k|z|}\right)$$

Matching

$$\frac{p}{k}\frac{\mathrm{K}_{\nu+1}\left(\frac{p}{k}\right)}{\mathrm{K}_\nu\left(\frac{p}{k}\right)} = \frac{3}{2} + \nu$$

Spectrum of Model A



Dispersion relations

- zero mode at small momenta

$$E^2 = 3p^2\left(1 - \frac{p}{k} + \mathcal{O}(p^2)\right)$$

- higher modes at low momenta

$$E_n^2 = \frac{9}{4}k^2 + \frac{\pi^2 k^2 n^2}{4\log^2\frac{p}{k}}$$

- large momenta – everything $E = p + \ldots$

Green Function and Static Potential

Propagator

$$\left[-\partial_z^2 - E^2 + p^2 \mathrm{e}^{2k|z|} + \frac{9}{4}k^2 - 3k\delta(z)\right]\Delta_p(E,p,z) = \delta(z)\,.$$

Put $E = 0$ to find the Green function. On the brane $z = 0$

$$G_p(0,p,0) = \frac{k}{2p^2} + \frac{1}{2p}$$

or in coordinate representation

$$G_p(r) = \frac{k}{4\pi r}\left(1 + \frac{2}{\pi k r}\right)$$

Brane static potential

$$V(r) = \int_{-\infty}^{+\infty} dz\,\mathrm{e}^{-2k|z|}\,G(r) = \frac{1}{4\pi r}\left(1 + \frac{2}{\pi k r}\right)$$

Spin–$\frac{1}{2}$ Fermionic Perturbations

Fermions

Action

$$S = \int dt dx \int_{-\infty}^{+\infty} dz \sqrt{g} \left(i\bar{\Psi} \not{\nabla} \Psi + m_\psi \bar{\Psi} \Psi \right)$$

Dirac equation

$$i\Gamma^A \nabla_A \Psi(x,z) + m_\psi \Psi(x,z) = 0, \quad m_\psi = m \,\mathrm{sign}(z)$$

Rescaling spinor

$$\left(E\gamma^0 - \mathrm{e}^{k|z|} \gamma^i p_i - \gamma^5 \partial_z \right) \psi - m_\psi \psi = 0$$

Using special rep os γ-matrices

$$E\chi - ip\sigma_3 \phi \mathrm{e}^{k|z|} + \sigma_1 \phi' - m_\psi \phi = 0$$

$$E\phi + ip\sigma_3 \chi \mathrm{e}^{k|z|} - \sigma_1 \chi' - m_\psi \chi = 0$$

Represent

$$\chi(z) = \chi_+(z)X_+ + \chi_-(z)X_-, \quad \sigma_2 X_\pm = \pm X_\pm, \quad X_\pm = \begin{pmatrix} \mp i \\ 1 \end{pmatrix}.$$

Can be brought into

$$\chi''_\pm + \left(E^2 - m^2 - p^2 \mathrm{e}^{2k|z|} \mp pk\,\mathrm{sign}(z)\mathrm{e}^{k|z|}\right)\chi_\pm = 0$$

Normalizable solution

$$\chi^>_\pm(z) = C^>_\pm \xi_\pm\left(\frac{p}{k}\mathrm{e}^{k|z|}\right)$$

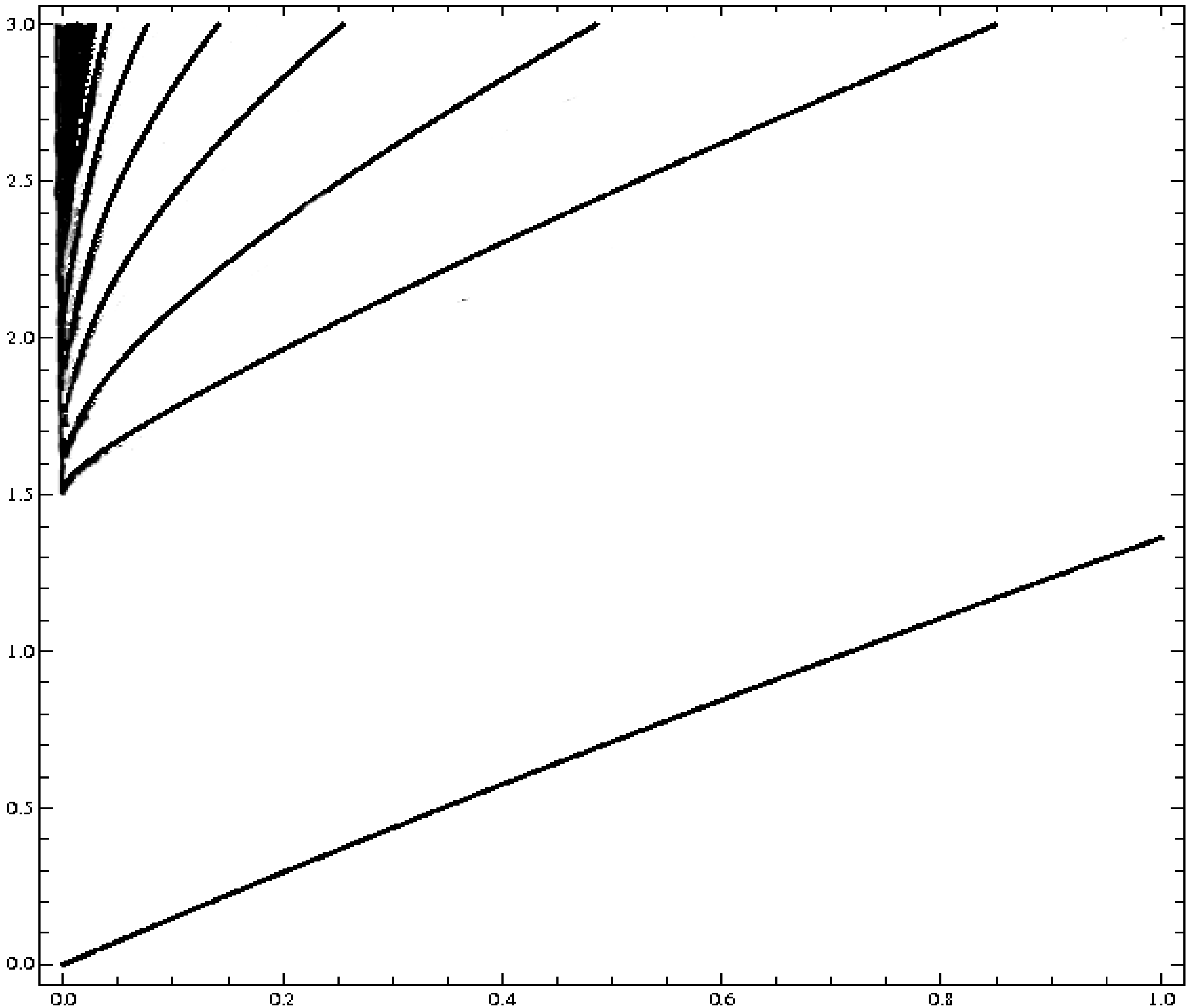
where

$$\xi_\pm(x) = \sqrt{x}\left[\mathrm{K}_{\nu+\frac{1}{2}}(x) \mp \mathrm{K}_{\nu-\frac{1}{2}}(x)\right]$$

Matching $t = p/k$

$$t^2\left(\frac{\partial}{\partial t}(\xi_+(t)\xi_-(t))\right)^2 = \frac{4m^2}{k^2}(\xi_+\xi_-)^2,$$

Spectrum



Dispersion Relations

- At $m > k/2$ the dispersion relation is

$$E = p\left(\frac{2m}{2m-k}\right) + \mathcal{O}(p^2)$$

- At $m = k/2$

$$E = -\frac{p}{k}\log\frac{p\,\mathrm{e}^{\gamma_E}}{2k} + \mathcal{O}(p\log p)^2$$

- At $0 < m < k/2$

$$E = 2m\frac{\Gamma\left(\frac{1}{2}-\frac{m}{k}\right)}{\Gamma\left(\frac{1}{2}+\frac{m}{k}\right)}\left(\frac{p}{\sqrt{2}k}\right)^{\frac{2m}{k}}(1+\mathcal{O}(p))$$

- The case $m = 0$ is very neat

$$E_n = -\frac{\pi k(2n+1)}{4\log\frac{p}{2k}} + \mathcal{O}(p), \quad n \in \mathbb{N}$$

Higher modes at low momenta

$$E_n = m\sqrt{1+\left(\frac{\pi n k}{k-2\Psi\left(\frac{1}{2}\right)m+2m\log\frac{p}{2k}}\right)^2}$$

All modes at high momenta

$$E = p + \mathscr{O}(p^{1/3})$$

Chirality

$$\psi_{\rm R}(E,{\rm p},z) = -\frac{i-1}{2}\xi_+(0)\begin{pmatrix} -\tilde{C}_+(E,{\rm p}) \\ \tilde{C}_+(E,{\rm p}) \\ C_+(E,{\rm p}) \\ C_+(E,{\rm p}) \end{pmatrix}\left(\frac{\xi_+(z)}{\xi_+(0)} + \frac{\xi_-(z)}{\xi_-(0)}\right) \equiv u_{\rm R}(E,{\rm p})\, f_{\rm R}(z)$$

$$\psi_{\rm L}(E,{\rm p},z) = -\frac{i-1}{2}\xi_+(0)\begin{pmatrix} \tilde{C}_+(E,{\rm p}) \\ \tilde{C}_+(E,{\rm p}) \\ -C_+(E,{\rm p}) \\ C_+(E,{\rm p}) \end{pmatrix}\left(\frac{\xi_+(z)}{\xi_+(0)} - \frac{\xi_-(z)}{\xi_-(0)}\right) \equiv u_{\rm L}(E,{\rm p})\, f_{\rm L}(z)$$

Effective action

$$\psi_{\mathrm{R,L}}(E,\mathrm{p},z) = \sum_n \alpha^{(n)}_{\mathrm{R,L}} u^{(n)}_{\mathrm{R,L}}(E,\mathrm{p}) f^{(n)}_{\mathrm{R,L}}(z),$$

and

$$\begin{aligned}
\mathscr{L}^{(n)} &= i\overline{u}^{(n)}_{\mathrm{R}} \gamma^0 E u^{(n)}_{\mathrm{R}} \left| f^{(n)}_{\mathrm{R}} \right|^2 + i\overline{u}^{(n)}_{\mathrm{R}} \gamma^i p_i u^{(n)}_{\mathrm{R}} \left| f^{(n)}_{\mathrm{R}} \right|^2 e^{k|z|} \\
&+ i\overline{u}^{(n)}_{\mathrm{L}} \gamma^0 E u^{(n)}_{\mathrm{L}} \left| f^{(n)}_{\mathrm{L}} \right|^2 + i\overline{u}^{(n)}_{\mathrm{L}} \gamma^i p_i u^{(n)}_{\mathrm{L}} \left| f^{(n)}_{\mathrm{L}} \right|^2 e^{k|z|} \\
&- i\overline{u}^{(n)}_{\mathrm{R}} \gamma^5 u^{(n)}_{\mathrm{L}} \left(f^{(n)}_{\mathrm{R}} \right)^* \partial_z f^{(n)}_{\mathrm{L}} - i\overline{u}^{(n)}_{\mathrm{L}} \gamma^5 u^{(n)}_{\mathrm{R}} \left(f^{(n)}_{\mathrm{L}} \right)^* \partial_z f^{(n)}_{\mathrm{R}} \\
&+ m_\psi \overline{u}^{(n)}_{\mathrm{R}} u^{(n)}_{\mathrm{L}} \left(f^{(n)}_{\mathrm{R}} \right)^* f^{(n)}_{\mathrm{L}} + m_\psi \overline{u}^{(n)}_{\mathrm{L}} u^{(n)}_{\mathrm{R}} \left(f^{(n)}_{\mathrm{L}} \right)^* f^{(n)}_{\mathrm{R}}
\end{aligned}$$

Some Outlook

- Study Cosmology. Maybe evade NO-GO.
- Exotic: Holography. How will the principle will change if the conformal theory is broken? "AdS"/"CFT" "correpondence"
- Study hihger codimensions