Integrable Systems and Quantum Deformations

Peter Koroteev

University of Minnesota

In collaboration with N. Beisert, F. Spill, A. Rej

0802.0777, work in progress

MG12, July 13th 2009

Outline

Introduction. Integrability and Symmetry

- What is Integrability
- Coordinate Bethe Ansatz
- AdS/CFT Correspondence

Quantum Deformation of 1D Hubbard Model

- $\mathrm{U}_q(\mathfrak{su}(2|2) \ltimes \mathbb{R}^2)$ algebra
- Hubbard-like Models

Universal R-matrix

- Yangian
- Universal R-matrix

Why Integrability?

- Easy models: Classical Mechanics (ossilator, free point particle)

Why Integrability?

- Easy models: Classical Mechanics (ossilator, free point particle)
- "Complicated", Chaotic models. Almost untractable

Why Integrability?

- Easy models: Classical Mechanics (ossilator, free point particle)
- "Complicated", Chaotic models. Almost untractable
- Integrable models: not necessrily easy/complicated

Why Integrability?

- Easy models: Classical Mechanics (ossilator, free point particle)
- "Complicated", Chaotic models. Almost untractable
- Integrable models: not necessrily easy/complicated

Integrable models can be completely solved

What is Integrability

Exists an infinite set of independent commuting charges

$$\{\mathfrak{Q}_\alpha\}, \quad [\mathfrak{Q}_\alpha, \mathfrak{Q}_\beta] = 0$$

What is Integrability

Exists an infinite set of independent commuting charges

$$\{\mathfrak{Q}_\alpha\}, \quad [\mathfrak{Q}_\alpha, \mathfrak{Q}_\beta] = 0$$

- Finite dimensional: mechanical with n d.o.f.s – exist n-1 local integrals in involution

What is Integrability

Exists an infinite set of independent commuting charges

$$\{\mathfrak{Q}_\alpha\}, \quad [\mathfrak{Q}_\alpha, \mathfrak{Q}_\beta] = 0$$

- Finite dimensional: mechanical with n d.o.f.s – exist n-1 local integrals in involution
- Field Theory: Infinite dimensional symmetry, S–matrix satisfies integrability constraints.

What is Integrability

Exists an infinite set of independent commuting charges

$$\{\mathfrak{Q}_\alpha\}, \quad [\mathfrak{Q}_\alpha, \mathfrak{Q}_\beta] = 0$$

- Finite dimensional: mechanical with n d.o.f.s – exist n-1 local integrals in involution
- Field Theory: Infinite dimensional symmetry, S–matrix satisfies integrability constraints.

We are focused on QFTs and spin chains

Signatures of Integrability

Explicitly find all commuting charges

Signatures of Integrability

Explicitly find all commuting charges
Investigate S-matrix (field theory, spin chains) S-matrix satisfies
Yang-Baxter equation, Unitarity conditions

Signatures of Integrability

Explicitly find all commuting charges
Investigate S-matrix (field theory, spin chains) S-matrix satisfies
Yang-Baxter equation, Unitarity conditions

Some methods

- Analytic Bethe Ansatz. [Leningrad school, 70-80s]

Signatures of Integrability

Explicitly find all commuting charges
Investigate S-matrix (field theory, spin chains) S-matrix satisfies
Yang-Baxter equation, Unitarity conditions

Some methods

- Analytic Bethe Ansatz. [Leningrad school, 70-80s]
- Coordinate Bethe Ansatz. Mostly for spin chains

Signatures of Integrability

Explicitly find all commuting charges
Investigate S-matrix (field theory, spin chains) S-matrix satisfies Yang-Baxter equation, Unitarity conditions

Some methods

- Analytic Bethe Ansatz. [Leningrad school, 70-80s]
- Coordinate Bethe Ansatz. Mostly for spin chains
- Algebraic integrability

XXX Spin Chain

Hilbert space

$$|\psi\rangle = |\uparrow\downarrow\downarrow\uparrow\ldots\rangle$$

XXX Spin Chain

Hilbert space

$$|\psi\rangle = |\uparrow\downarrow\downarrow\uparrow \ldots\rangle$$

Hamiltonian

$$\mathcal{H} = \sum_{k=1}^{L} \mathcal{H}_{k,k+1},$$

$$\mathcal{H}_{k,k+1} = \mathcal{I}_{k,k+1} - \mathcal{P}_{k,k+1} = \frac{1}{2}\Big(1 - \vec{\sigma}_k \otimes \vec{\sigma}_{k+1}\Big)$$

XXX Spin Chain

Hilbert space

$$|\psi\rangle = |\uparrow\downarrow\downarrow\uparrow \ldots\rangle$$

Hamiltonian

$$\mathcal{H} = \sum_{k=1}^{L} \mathcal{H}_{k,k+1},$$

$$\mathcal{H}_{k,k+1} = \mathcal{I}_{k,k+1} - \mathcal{P}_{k,k+1} = \tfrac{1}{2}\Big(1 - \vec{\sigma}_k \otimes \vec{\sigma}_{k+1}\Big)$$

Symmetry $\mathfrak{su}(2)$ interchanges $|\uparrow\rangle$ and $|\downarrow\rangle$. $\mathcal{H}$ commutes with all generators

XXX Spin Chain

Hilbert space

$$|\psi\rangle = |\uparrow\downarrow\downarrow\uparrow \ldots\rangle$$

Hamiltonian

$$\mathcal{H} = \sum_{k=1}^{L} \mathcal{H}_{k,k+1},$$

$$\mathcal{H}_{k,k+1} = \mathcal{I}_{k,k+1} - \mathcal{P}_{k,k+1} = \tfrac{1}{2}\Big(1 - \vec{\sigma}_k \otimes \vec{\sigma}_{k+1}\Big)$$

Symmetry $\mathfrak{su}(2)$ interchanges $|\uparrow\rangle$ and $|\downarrow\rangle$. $\mathcal{H}$ commutes with all generators

What is the spectrum of the linear operator $\mathcal{H}$?

XXX Spin Chain

Hilbert space

$$|\psi\rangle = |\uparrow\downarrow\downarrow\uparrow\ldots\rangle$$

Hamiltonian

$$\mathcal{H} = \sum_{k=1}^{L} \mathcal{H}_{k,k+1},$$

$$\mathcal{H}_{k,k+1} = \mathcal{I}_{k,k+1} - \mathcal{P}_{k,k+1} = \tfrac{1}{2}\Big(1 - \vec{\sigma}_k \otimes \vec{\sigma}_{k+1}\Big)$$

Symmetry $\mathfrak{su}(2)$ interchanges $|\uparrow\rangle$ and $|\downarrow\rangle$. $\mathcal{H}$ commutes with all generators

What is the spectrum of the linear operator $\mathcal{H}$?

Brute force: list all states with given $n_\uparrow, n_\downarrow$, evaluate $\mathcal{H}$ in this basis, diagonalize $\mathcal{H}$. Straightforward but hard (L=20, basis about 10000)

Bethe Ansatz

Consider infinite chain. Vacuum (ferromagnetic)

$$|0\rangle = |\downarrow\downarrow\downarrow\downarrow\downarrow\ldots\rangle$$

Bethe Ansatz

Consider infinite chain. Vacuum (ferromagnetic)

$$|0\rangle = |\downarrow\downarrow\downarrow\downarrow\downarrow \ldots\rangle$$

Find $\mathcal{H}_{12}|\downarrow\downarrow\rangle = 0$ hence $\mathcal{H}|0\rangle = 0$

Bethe Ansatz

Consider infinite chain. Vacuum (ferromagnetic)

$$|0\rangle = |\downarrow\downarrow\downarrow\downarrow\downarrow \ldots\rangle$$

Find $\mathcal{H}_{12}|\downarrow\downarrow\rangle = 0$ hence $\mathcal{H}|0\rangle = 0$
Spin flip $|k\rangle = |\downarrow\downarrow\downarrow\uparrow_k\downarrow\downarrow \ldots\rangle$
Hamiltonian homogeneous, eigenvectors are plane waves

Bethe Ansatz

Consider infinite chain. Vacuum (ferromagnetic)

$$|0\rangle = |\downarrow\downarrow\downarrow\downarrow\downarrow \ldots\rangle$$

Find $\mathcal{H}_{12}|\downarrow\downarrow\rangle = 0$ hence $\mathcal{H}|0\rangle = 0$
Spin flip $|k\rangle = |\downarrow\downarrow\downarrow\uparrow_k\downarrow\downarrow \ldots\rangle$
Hamiltonian homogeneous, eigenvectors are plane waves
Momentum eigenstate

$$|p\rangle = \sum e^{ipk}|k\rangle$$

Bethe Ansatz

Consider infinite chain. Vacuum (ferromagnetic)

$$|0\rangle = |\downarrow\downarrow\downarrow\downarrow\downarrow \ldots\rangle$$

Find $\mathcal{H}_{12}|\downarrow\downarrow\rangle = 0$ hence $\mathcal{H}|0\rangle = 0$
Spin flip $|k\rangle = |\downarrow\downarrow\downarrow\uparrow_k\downarrow\downarrow \ldots\rangle$
Hamiltonian homogeneous, eigenvectors are plane waves
Momentum eigenstate

$$|p\rangle = \sum e^{ipk}|k\rangle$$

Act with $\mathcal{H}$ to find eigenvalue and dispersion relation

$$\begin{aligned}\mathcal{H}|p\rangle &= \sum_{-\infty}^{+\infty} e^{ipk}(|k\rangle - |k-1\rangle + |k\rangle - |k+1\rangle) \\ &= 2(1-\cos p)|p\rangle =: e(p)|p\rangle\end{aligned}$$

Two excitations

Position State $|p<q\rangle = \sum_{k<l} e^{ipk+iql} |\ldots \uparrow_k \ldots \uparrow_l \ldots\rangle$
"Almost" an eigenstate (spin flips far from each other)
Contact term

$$\mathcal{H}|p<q\rangle = (e(p)+e(q))|p<q\rangle + \sum_k e^{i(p+q)k}(e^{ip+iq} - 2e^{iq} + 1)|\uparrow_k \uparrow_{k+1}\rangle,$$

$$\mathcal{H}|q<p\rangle = (e(p)+e(q))|q<p\rangle + \sum_k e^{i(p+q)k}(e^{ip+iq} - 2e^{ip} + 1)|\uparrow_k \uparrow_{k+1}\rangle$$

Construct eigenstate $|p,q\rangle = |p<q\rangle + S|q<p\rangle$ with scattering phase

$$S = -\frac{e^{ip+iq} - 2e^{iq} + 1}{e^{ip+iq} - 2e^{ip} + 1} = e^{2i\phi(p,q)}$$

Eigenvalue

$$\mathcal{H}|p,q\rangle = (e(p)+e(q))|p,q\rangle$$

Scattering

6=3! asymptotic regions
Match up regions at contact terms, find eigenstate

$$|p1, p2, p3\rangle = |p_1 < p_2 < p_3\rangle + S_{12}|p_2 < p_1 < p_3\rangle + S_{23}|p_1 < p_3 < p_2$$

$$+S_{13}S_{12}|p_2 < p_3 < p_1\rangle + S_{13}S_{23}|p_3 < p_1 < p_2\rangle + S_{12}S_{13}S_{23}|p_3 < p_2 < p$$

eigenvalue $e(p_1) + e(p_2) + e(p_3)$
Integrability: Scattering factorizes for any number of particles
Two particles scattering phase enough to construct any eigenstate on infinite chain

Bethe Equations

We have: infinite chain. We want: finite periodic chain

Bethe Equations

We have: infinite chain. We want: finite periodic chain Move one excitation p_k past L sites $e^{ip_k L}$ of the chain and $k-1$ other particles $\prod S_{kj}$. Should end up with the same state

Bethe Equations

We have: infinite chain. We want: finite periodic chain Move one excitation p_k past L sites $e^{ip_k L}$ of the chain and $k-1$ other particles $\prod S_{kj}$. Should end up with the same state Bethe equations

$$1 = e^{-ip_k L} \prod_{j=1}^{k} -\frac{e^{ip_k+ip_j} - 2e^{ip_k} + 1}{e^{ip_k+ip_j} - 2e^{ip_j} + 1}$$

Bethe Equations

We have: infinite chain. We want: finite periodic chain Move one excitation p_k past L sites $e^{ip_k L}$ of the chain and $k-1$ other particles $\prod S_{kj}$. Should end up with the same state Bethe equations

$$1 = e^{-ip_k L} \prod_{j=1}^{k} - \frac{e^{ip_k + ip_j} - 2e^{ip_k} + 1}{e^{ip_k + ip_j} - 2e^{ip_j} + 1}$$

Reparametrise $p_k = 2\text{arccot} 2u_k$ via rapidity

$$1 = \left(\frac{u_k - i/2}{u_k + i/2}\right)^L \prod_{j=1}^{k} - \frac{u_k - u_j + i}{u_k - u_j + 1}$$

Total energy

$$E = \sum_{j=1}^{k} e(p_j) = \sum_{j=1}^{k} 4\sin^2 p_j/2 = \sum_{j=1}^{k} 4\left(\frac{i}{u_j + i/2} - \frac{i}{u_j - i/2}\right)$$

Total momentum $e^{ip} = \prod e^{ip_j} = \prod \frac{u_j + i/2}{u_j - i/2}$

$\mathcal{N} = 4$ 4D Super Yang Mills

$\mathfrak{su}(N)$ gauge connection A, 4 Majorana gluinos as 16 component 10d Majorana-Weyl spinor χ, 6 scalars Φ_i. All adjoint!

$$S = \frac{2}{g_{YM}^2} \int d^4x \, \mathrm{Tr} \left\{ \frac{1}{4} F^2 + \frac{1}{2} (\nabla \Phi_i)^2 - \frac{1}{4} [\Phi_i, \Phi_j]^2 + \frac{1}{2} \bar{\chi} \not{\nabla} \chi - \frac{i}{2} \bar{\chi} \Gamma_i [\phi_i, \chi] \right\}$$

$\mathcal{N} = 4$ 4D Super Yang Mills

$\mathfrak{su}(N)$ gauge connection A, 4 Majorana gluinos as 16 component 10d Majorana-Weyl spinor χ, 6 scalars Φ_i. All adjoint!

$$S = \frac{2}{g_{YM}^2} \int d^4x \, \mathrm{Tr} \left\{ \frac{1}{4} F^2 + \frac{1}{2} (\nabla \Phi_i)^2 - \frac{1}{4} [\Phi_i, \Phi_j]^2 + \frac{1}{2} \bar{\chi} \nabla\!\!\!/ \chi - \frac{i}{2} \bar{\chi} \, \Gamma_i \, [\phi_i, \chi] \right\}$$

- Completely fixed by supersymmetry – two parameters g and N $g \sim \sqrt{\lambda} \sim g_{YM} \sqrt{N}$

$\mathcal{N} = 4$ 4D Super Yang Mills

$\mathfrak{su}(N)$ gauge connection A, 4 Majorana gluinos as 16 component 10d Majorana-Weyl spinor χ, 6 scalars Φ_i. All adjoint!

$$S = \frac{2}{g_{YM}^2} \int d^4x \, \mathrm{Tr} \left\{ \frac{1}{4} F^2 + \frac{1}{2} (\nabla \Phi_i)^2 - \frac{1}{4} [\Phi_i, \Phi_j]^2 + \frac{1}{2} \bar{\chi} \not{\nabla} \chi - \frac{i}{2} \bar{\chi} \Gamma_i [\phi_i, \chi] \right\}$$

- Completely fixed by supersymmetry – two parameters g and N $g \sim \sqrt{\lambda} \sim g_{YM}\sqrt{N}$
- All fileds massless

$\mathcal{N} = 4$ 4D Super Yang Mills

$\mathfrak{su}(N)$ gauge connection A, 4 Majorana gluinos as 16 component 10d Majorana-Weyl spinor χ, 6 scalars Φ_i. All adjoint!

$$S = \frac{2}{g_{YM}^2} \int d^4x \, \mathrm{Tr} \left\{ \frac{1}{4} F^2 + \frac{1}{2} (\nabla \Phi_i)^2 - \frac{1}{4} [\Phi_i, \Phi_j]^2 + \frac{1}{2} \bar{\chi} \not{\nabla} \chi - \frac{i}{2} \bar{\chi} \Gamma_i [\phi_i, \chi] \right\}$$

- Completely fixed by supersymmetry – two parameters g and N $g \sim \sqrt{\lambda} \sim g_{YM} \sqrt{N}$
- All fileds massless
- Finiteness: beta function is exactly zero, no running

$\mathcal{N} = 4$ 4D Super Yang Mills

$\mathfrak{su}(N)$ gauge connection A, 4 Majorana gluinos as 16 component 10d Majorana-Weyl spinor χ, 6 scalars Φ_i. All adjoint!

$$S = \frac{2}{g_{YM}^2} \int d^4x \, \mathrm{Tr} \left\{ \frac{1}{4} F^2 + \frac{1}{2} (\nabla \Phi_i)^2 - \frac{1}{4} [\Phi_i, \Phi_j]^2 + \frac{1}{2} \bar{\chi} \not{\nabla} \chi - \frac{i}{2} \bar{\chi} \Gamma_i [\phi_i, \chi] \right\}$$

- Completely fixed by supersymmetry – two parameters g and N $g \sim \sqrt{\lambda} \sim g_{YM}\sqrt{N}$
- All fileds massless
- Finiteness: beta function is exactly zero, no running
- Unbroken conformal symmetry $\mathfrak{psu}(2,2|4)$

Anomalous Dimensions

Composite gauge invariant operators

$$\mathcal{O}(x) = \mathrm{Tr}(\Phi_i(x)\Phi_j(y)\dots)$$

Anomalous Dimensions

Composite gauge invariant operators

$$\mathcal{O}(x) = \mathrm{Tr}(\Phi_i(x)\Phi_j(y)\ldots)$$

need renormalization $\mathcal{O} \to Z\mathcal{O}$

$$\langle \mathcal{O}(x)\mathcal{O}(y) \rangle = \alpha |x-y|^{-2(D_0+D(\lambda))}$$

Anomalous Dimensions

Composite gauge invariant operators

$$\mathcal{O}(x) = \mathrm{Tr}(\Phi_i(x)\Phi_j(y)\dots)$$

need renormalization $\mathcal{O} \to Z\mathcal{O}$

$$\langle \mathcal{O}(x)\mathcal{O}(y) \rangle = \alpha |x-y|^{-2(D_0+D(\lambda))}$$

No mixing in scalar sector at 1 loop [Minahan, Zarembo]

Anomalous Dimensions

Composite gauge invariant operators

$$\mathcal{O}(x) = \mathrm{Tr}(\Phi_i(x)\Phi_j(y)\dots)$$

need renormalization $\mathcal{O} \to Z\mathcal{O}$

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \alpha |x-y|^{-2(D_0+D(\lambda))}$$

No mixing in scalar sector at 1 loop [Minahan, Zarembo]
Anomalous dimension matrix (dilatation generator) $D = \frac{d\log Z}{d\log \Lambda}$
interpreted as a spin chain Hamiltonian

$$\mathfrak{D}(g)\mathcal{O} = g^2\mathcal{H}\mathcal{O} = D(g)\mathcal{O}$$

Scalar Sector

$[\Phi]$=1. Length of spin chain = bare scaling dimension. Spectrum of Hamiltonian – anomalous dimension

Scalar Sector

$[\Phi]$=1. Length of spin chain = bare scaling dimension. Spectrum of Hamiltonian – anomalous dimension
Hibert space

$$\mathrm{H} = \mathbb{R}^6 \otimes \cdots \otimes \mathbb{R}^6$$

Scalar Sector

$[\Phi]$=1. Length of spin chain = bare scaling dimension. Spectrum of Hamiltonian – anomalous dimension
Hibert space

$$\mathrm{H} = \mathbb{R}^6 \otimes \cdots \otimes \mathbb{R}^6$$

States – single trace operators

$$|\Phi_i \Phi_j \Phi_k \Phi_l \ldots\rangle = \mathrm{Tr} \Phi_i \Phi_j \Phi_k \Phi_l \ldots$$

Scalar Sector

$[\Phi]$=1. Length of spin chain = bare scaling dimension. Spectrum of Hamiltonian – anomalous dimension
Hibert space

$$\mathrm{H} = \mathbb{R}^6 \otimes \cdots \otimes \mathbb{R}^6$$

States – single trace operators

$$|\Phi_i \Phi_j \Phi_k \Phi_l \ldots\rangle = \mathrm{Tr} \Phi_i \Phi_j \Phi_k \Phi_l \ldots$$

Hamiltonian $\mathcal{H} = \sum \mathcal{H}_{k,k+1}$. Boiled down to $\mathfrak{su}(2)$ sector $XXX_{\frac{1}{2}}$ spin chain

$$\mathcal{H}_{k,k+1} = \mathcal{I}_{k,k+1} - \mathcal{P}_{k,k+1} = \frac{1}{2}(1 - \vec{\sigma}_k \vec{\sigma}_{k+1})$$

Scalar Sector

$[\Phi]$=1. Length of spin chain = bare scaling dimension. Spectrum of Hamiltonian – anomalous dimension
Hibert space

$$\mathrm{H} = \mathbb{R}^6 \otimes \cdots \otimes \mathbb{R}^6$$

States – single trace operators

$$|\Phi_i \Phi_j \Phi_k \Phi_l \ldots\rangle = \mathrm{Tr} \Phi_i \Phi_j \Phi_k \Phi_l \ldots$$

Hamiltonian $\mathcal{H} = \sum \mathcal{H}_{k,k+1}$. Boiled down to $\mathfrak{su}(2)$ sector $XXX_{\frac{1}{2}}$ spin chain

$$\mathcal{H}_{k,k+1} = \mathcal{I}_{k,k+1} - \mathcal{P}_{k,k+1} = \frac{1}{2}(1 - \vec{\sigma}_k \vec{\sigma}_{k+1})$$

Integrable !

SYM Spin Chain Vacuum

Three complex scalars $\mathcal{X} = \Phi_1 + i\Phi_2, \mathcal{Y} = \Phi_3 + i\Phi_4, \mathcal{Z} = \Phi_5 + i\Phi_6$ transform under $\mathfrak{su}(4) \simeq \mathfrak{so}(6)$

SYM Spin Chain Vacuum

Three complex scalars $\mathcal{X} = \mathit{\Phi}_1 + i\mathit{\Phi}_2, \mathcal{Y} = \mathit{\Phi}_3 + i\mathit{\Phi}_4, \mathcal{Z} = \mathit{\Phi}_5 + i\mathit{\Phi}_6$
transform under $\mathfrak{su}(4) \simeq \mathfrak{so}(6)$
Vacuum: $|\mathcal{X}\mathcal{X}\mathcal{X}\ldots\rangle$ breaks superconformal symmetry

SYM Spin Chain Vacuum

Three complex scalars $\mathcal{X} = \Phi_1 + i\Phi_2, \mathcal{Y} = \Phi_3 + i\Phi_4, \mathcal{Z} = \Phi_5 + i\Phi_6$
transform under $\mathfrak{su}(4) \simeq \mathfrak{so}(6)$
Vacuum: $|\mathcal{X}\mathcal{X}\mathcal{X}\ldots\rangle$ breaks superconformal symmetry
Residual symmetry $\mathfrak{u}(1) \ltimes \mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2) \ltimes \mathbb{R}$

SYM Spin Chain Vacuum

Three complex scalars $\mathcal{X} = \Phi_1 + i\Phi_2, \mathcal{Y} = \Phi_3 + i\Phi_4, \mathcal{Z} = \Phi_5 + i\Phi_6$ transform under $\mathfrak{su}(4) \simeq \mathfrak{so}(6)$
Vacuum: $|\mathcal{X}\mathcal{X}\mathcal{X}\ldots\rangle$ breaks superconformal symmetry
Residual symmetry $\mathfrak{u}(1) \ltimes \mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2) \ltimes \mathbb{R}$
Excitations transform in reps of $\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)$
Can work with one copy of $\mathfrak{psu}(2|2)$. Leads us to $\mathfrak{psu}(2|2)$ spin chain

$\mathfrak{su}(2|2)$ Spin Chain

- d.o.f: 2 bosons ϕ^1, ϕ^2, 2 fermions ψ^1, ψ^2

$\mathfrak{su}(2|2)$ Spin Chain

- d.o.f: 2 bosons ϕ^1, ϕ^2, 2 fermions ψ^1, ψ^2
- S-matrix: $V_1 \otimes V_2 \to V_2' \otimes V_1'$ is fully constrained by symmetry up to overall scalar factor (peculiarity of representation theory)

$\mathfrak{su}(2|2)$ Spin Chain

- d.o.f: 2 bosons ϕ^1, ϕ^2, 2 fermions ψ^1, ψ^2
- S-matrix: $V_1 \otimes V_2 \to V_2' \otimes V_1'$ is fully constrained by symmetry up to overall scalar factor (peculiarity of representation theory)
- Central extension to $\mathfrak{su}(2|2) \ltimes \mathbb{R}$ makes spectrum continuous

$\mathfrak{su}(2|2)$ Spin Chain

- d.o.f: 2 bosons ϕ^1, ϕ^2, 2 fermions ψ^1, ψ^2
- S-matrix: $V_1 \otimes V_2 \to V_2' \otimes V_1'$ is fully constrained by symmetry up to overall scalar factor (peculiarity of representation theory)
- Central extension to $\mathfrak{su}(2|2) \ltimes \mathbb{R}$ makes spectrum continuous
- Magnon dispersion relation purely from symmetry [Beisert]

$$E(p) = \sqrt{1 + g^2 \sin^2(\tfrac{1}{2}p)}$$

Quantum Deformations

The Lie superalgebra $\mathfrak{su}(2|2)$ is generated by the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ generators $\mathfrak{R}^a{}_b$, $\mathfrak{L}^\alpha{}_\beta$, the supercharges $\mathfrak{Q}^\alpha{}_b$, $\mathfrak{S}^a{}_\beta$ and the central charge $\mathfrak{C}$.

The Lie brackets

$$[\mathfrak{R}^a{}_b, \mathfrak{R}^c{}_d] = \delta^c_b \mathfrak{R}^a{}_d - \delta^a_d \mathfrak{R}^c{}_b, \qquad [\mathfrak{L}^\alpha{}_\beta, \mathfrak{L}^\gamma{}_\delta] = \delta^\gamma_\beta \mathfrak{L}^\alpha{}_\delta - \delta^\alpha_\delta \mathfrak{L}^\gamma{}_\beta,$$

$$[\mathfrak{R}^a{}_b, \mathfrak{Q}^\gamma{}_d] = -\delta^a_d \mathfrak{Q}^\gamma{}_b + \tfrac{1}{2}\delta^a_b \mathfrak{Q}^\gamma{}_d, \qquad [\mathfrak{L}^\alpha{}_\beta, \mathfrak{Q}^\gamma{}_d] = \delta^\gamma_\beta \mathfrak{Q}^\alpha{}_d - \tfrac{1}{2}\delta^\alpha_\beta \mathfrak{Q}^\gamma{}_d,$$

$$[\mathfrak{R}^a{}_b, \mathfrak{S}^c{}_\delta] = \delta^c_b \mathfrak{S}^a{}_\delta - \tfrac{1}{2}\delta^a_b \mathfrak{S}^c{}_\delta, \qquad [\mathfrak{L}^\alpha{}_\beta, \mathfrak{S}^c{}_\delta] = -\delta^\alpha_\delta \mathfrak{S}^c{}_\beta + \tfrac{1}{2}\delta^\alpha_\beta \mathfrak{S}^c{}_\delta$$

$$\{\mathfrak{Q}^\alpha{}_b, \mathfrak{S}^c{}_\delta\} = \delta^c_b \mathfrak{L}^\alpha{}_\delta + \delta^\alpha_\delta \mathfrak{R}^c{}_b + \delta^c_b \delta^\alpha_\delta \mathfrak{C}.$$

Central extension

$$\{\mathfrak{Q}^\alpha{}_b, \mathfrak{Q}^\gamma{}_d\} = \varepsilon^{\alpha\gamma}\varepsilon_{bd}\mathfrak{P}, \qquad \{\mathfrak{S}^a{}_\beta, \mathfrak{S}^c{}_\delta\} = \varepsilon^{ac}\varepsilon_{\beta\delta}\mathfrak{K}.$$

denote

$$\mathfrak{h} := \mathfrak{su}(2|2) \ltimes \mathbb{R}^2 = \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3.$$

Quantum Deformation

Q-number ($q \in \mathbb{C}$)

$$[A]_q = \frac{q^A - q^{-A}}{q - q^{-1}}$$

Commutators

$$[\mathfrak{E}_1, \mathfrak{F}_1] = [\mathfrak{H}_1]_q\,, \qquad \{\mathfrak{E}_2, \mathfrak{F}_2\} = -[\mathfrak{H}_2]_q\,, \qquad [\mathfrak{E}_3, \mathfrak{F}_3] = -[\mathfrak{H}_3]_q\,,$$

Serre relations

$$\begin{aligned} 0 &= [\mathfrak{E}_1, \mathfrak{E}_3] = [\mathfrak{F}_1, \mathfrak{F}_3] = \mathfrak{E}_2\mathfrak{E}_2 = \mathfrak{F}_2\mathfrak{F}_2 \\ &= \mathfrak{E}_1\mathfrak{E}_1\mathfrak{E}_2 - (q + q^{-1})\mathfrak{E}_1\mathfrak{E}_2\mathfrak{E}_1 + \mathfrak{E}_2\mathfrak{E}_1\mathfrak{E}_1 = \mathfrak{E}_3\mathfrak{E}_3\mathfrak{E}_2 - (q + q^{-1})\mathfrak{E}_3\mathfrak{E}_2 \\ &= \mathfrak{F}_1\mathfrak{F}_1\mathfrak{F}_2 - (q + q^{-1})\mathfrak{F}_1\mathfrak{F}_2\mathfrak{F}_1 + \mathfrak{F}_2\mathfrak{F}_1\mathfrak{F}_1 = \mathfrak{F}_3\mathfrak{F}_3\mathfrak{F}_2 - (q + q^{-1})\mathfrak{F}_3\mathfrak{F}_2\mathfrak{F}_3 \end{aligned}$$

Central charges...

Hopf Algebra

Unit element $\eta(1) = 1$ Counit $\varepsilon : \mathrm{U}_q(\mathfrak{h}) \rightarrow \mathbb{C}$ takes the form

$$\varepsilon(1) = 1, \qquad \varepsilon(\mathfrak{H}_j) = \varepsilon(\mathfrak{E}_j) = \varepsilon(\mathfrak{F}_j) = 0.$$

Hopf Algebra

Unit element $\eta(1) = 1$ Counit $\varepsilon : \mathrm{U}_q(\mathfrak{h}) \to \mathbb{C}$ takes the form

$$\varepsilon(1) = 1, \qquad \varepsilon(\mathfrak{H}_j) = \varepsilon(\mathfrak{E}_j) = \varepsilon(\mathfrak{F}_j) = 0.$$

Antipode $S : \mathrm{U}_q(\mathfrak{h}) \to \mathrm{U}_q(\mathfrak{h})$ is uniquely fixed by the compatibility condition

$$\nabla \circ (S \otimes 1) \circ \Delta(\mathfrak{J}) = \nabla \circ (1 \otimes S) \circ \Delta(\mathfrak{J}) = \eta \circ \varepsilon(\mathfrak{J})$$

Hopf Algebra

Unit element $\eta(1) = 1$ Counit $\varepsilon : \mathrm{U}_q(\mathfrak{h}) \to \mathbb{C}$ takes the form

$$\varepsilon(1) = 1, \qquad \varepsilon(\mathfrak{H}_j) = \varepsilon(\mathfrak{E}_j) = \varepsilon(\mathfrak{F}_j) = 0.$$

Antipode $S : \mathrm{U}_q(\mathfrak{h}) \to \mathrm{U}_q(\mathfrak{h})$ is uniquely fixed by the compatibility condition

$$\nabla \circ (S \otimes 1) \circ \Delta(\mathfrak{J}) = \nabla \circ (1 \otimes S) \circ \Delta(\mathfrak{J}) = \eta \circ \varepsilon(\mathfrak{J})$$

For $\mathrm{U}_q(\mathfrak{su}(2|2) \ltimes \mathbb{R}^2)$

$$S(1) = 1, \qquad S(\mathfrak{H}_j) = -\mathfrak{H}_j, \qquad S(\mathfrak{E}_j) = -q^{\mathfrak{H}_j}\mathfrak{E}_j,$$

for central charges

$$S(\mathfrak{C}) = -\mathfrak{C}, \qquad S(\mathfrak{P}) = -q^{-2\mathfrak{C}}\mathfrak{P}, \qquad S(\mathfrak{K}) = -q^{2\mathfrak{C}}\mathfrak{K}$$

Coproduct

$$\Delta(\mathfrak{E}_2) = \mathfrak{E}_2 \otimes 1 + q^{-\mathfrak{H}_2}\mathfrak{U} \otimes \mathfrak{E}_2,$$
$$\Delta(\mathfrak{F}_2) = \mathfrak{F}_2 \otimes q^{\mathfrak{H}_2} + \mathfrak{U}^{-1} \otimes \mathfrak{F}_2,$$
$$\Delta(\mathfrak{P}) = \mathfrak{P} \otimes 1 + q^{2\mathfrak{C}}\mathfrak{U}^2 \otimes \mathfrak{P},$$
$$\Delta(\mathfrak{K}) = \mathfrak{K} \otimes q^{-2\mathfrak{C}} + \mathfrak{U}^{-2} \otimes \mathfrak{K},$$
$$\Delta(\mathfrak{U}) = \mathfrak{U} \otimes \mathfrak{U}.$$

Unbraided for the rest generators (i=1,2,3, j=1,3)

$$\Delta(1) = 1 \otimes 1,$$
$$\Delta(\mathfrak{H}_i) = \mathfrak{H}_i \otimes 1 + 1 \otimes \mathfrak{H}_i,$$
$$\Delta(\mathfrak{E}_j) = \mathfrak{E}_j \otimes 1 + q^{-\mathfrak{H}_j} \otimes \mathfrak{E}_j,$$
$$\Delta(\mathfrak{F}_j) = \mathfrak{F}_j \otimes q^{\mathfrak{H}_j} + 1 \otimes \mathfrak{F}_j,$$

Co-commutativity and R-matrix

Hopf algebra quasi-cocommutative if

$$\Delta_{\mathrm{op}}(\mathfrak{J})\mathcal{R} = \mathcal{R}\Delta(\mathfrak{J}), \quad \Delta_{\mathrm{op}} = \mathcal{P}\Delta\mathcal{P}$$

Fundamental Reresentation

$$
\begin{aligned}
&\mathfrak{H}_2|\phi^1\rangle = -(C-\tfrac{1}{2})|\phi^1\rangle, && \mathfrak{E}_1|\phi^1\rangle = q^{+1/2}|\phi^2\rangle, && \mathfrak{F}_2|\phi^1\rangle = c|\psi^1\rangle,\\
&\mathfrak{H}_2|\phi^2\rangle = -(C+\tfrac{1}{2})|\phi^2\rangle, && \mathfrak{E}_2|\phi^2\rangle = a|\psi^2\rangle, && \mathfrak{F}_1|\phi^2\rangle = q^{-1/2}|\phi^1\rangle,\\
&\mathfrak{H}_2|\psi^2\rangle = -(C+\tfrac{1}{2})|\psi^2\rangle, && \mathfrak{E}_3|\psi^2\rangle = q^{-1/2}|\psi^1\rangle, && \mathfrak{F}_2|\psi^2\rangle = d|\phi^2\rangle,\\
&\mathfrak{H}_2|\psi^1\rangle = -(C-\tfrac{1}{2})|\psi^1\rangle, && \mathfrak{E}_2|\psi^1\rangle = b|\phi^1\rangle, && \mathfrak{F}_3|\psi^1\rangle = q^{+1/2}|\psi^2\rangle.
\end{aligned}
$$

Constraints

$$
ad = [C+\tfrac{1}{2}]_q, \qquad bc = [C-\tfrac{1}{2}]_q, \qquad ab = P, \qquad cd = K.
$$

$$
(ad - qbc)(ad - q^{-1}bc) = 1.
$$

Parametrization

$$
a = \sqrt{g}\,\gamma, \quad b = \frac{\sqrt{g}\,\alpha}{\gamma}\,\frac{1}{x^-}\left(x^- - q^{2C-1}x^+\right),
$$

$$
c = \frac{i\sqrt{g}\gamma}{\alpha}\,\frac{q^{-C+1/2}}{x^+}, \quad d = \frac{i\sqrt{g}}{\gamma}\,q^{C+1/2}\left(x^- - q^{-2C-1}x^+\right).
$$

Fundamental R-matrix

$$
\begin{aligned}
\mathcal{R}|\phi^1\phi^1\rangle &= A_{12}|\phi^1\phi^1\rangle \\
\mathcal{R}|\phi^1\phi^2\rangle &= \frac{qA_{12}+q^{-1}B_{12}}{q+q^{-1}}|\phi^2\phi^1\rangle + \frac{A_{12}-B_{12}}{q+q^{-1}}|\phi^1\phi^2\rangle - \frac{q^{-1}C_{12}}{q+q^{-1}}|\psi^2\psi \\
\mathcal{R}|\phi^2\phi^1\rangle &= \frac{A_{12}-B_{12}}{q+q^{-1}}|\phi^2\phi^1\rangle + \frac{q^{-1}A_{12}+qB_{12}}{q+q^{-1}}|\phi^1\phi^2\rangle + \frac{C_{12}}{q+q^{-1}}|\psi^2\psi \\
\mathcal{R}|\phi^2\phi^2\rangle &= A_{12}|\phi^2\phi^2\rangle
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}|\psi^1\psi^1\rangle &= -D_{12}|\psi^1\psi^1\rangle \\
\mathcal{R}|\psi^1\psi^2\rangle &= -\frac{qD_{12}+q^{-1}E_{12}}{q+q^{-1}}|\psi^2\psi^1\rangle - \frac{D_{12}-E_{12}}{q+q^{-1}}|\psi^1\psi^2\rangle + \frac{q^{-1}F_{12}}{q+q^{-1}}|\phi \\
\mathcal{R}|\psi^2\psi^1\rangle &= -\frac{D_{12}-E_{12}}{q+q^{-1}}|\psi^2\psi^1\rangle - \frac{q^{-1}D_{12}+qE_{12}}{q+q^{-1}}|\psi^1\psi^2\rangle - \frac{F_{12}}{q+q^{-1}}|\phi \\
&\dots
\end{aligned}
$$

R-matrix Coefficients

$$A_{12} = R^0_{12}\,\frac{q^{C_1}U_1}{q^{C_2}U_2}\,\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}$$

$$B_{12} = R^0_{12}\,\frac{q^{C_1}U_1}{q^{C_2}U_2}\,\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\left(1 - (q+q^{-1})q^{-1}\frac{x_2^+ - x_1^+}{x_2^+ - x_1^-}\,\frac{x_2^- - s(x_1^+)}{x_2^- - s(x_1^-)}\right)$$

$$C_{12} = R^0_{12}\,(q+q^{-1})\,\frac{ig\alpha^{-1}\gamma_2\gamma_1 q^{C_1}U_1}{q^{2C_2+3/2}U_2^2}\,\frac{ig^{-1}x_2^+ - (q-q^{-1})}{x_2^- - s(x_1^-)}\,\frac{s(x_2^+) - s(x_1^+ \ldots}{x_2^- - x_1^+}$$

$$D_{12} = -R^0_{12}$$

$$E_{12} = -R^0_{12}\left(1 - (q+q^{-1})q^{-2C_2-1}U_2^{-2}\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}\,\frac{x_2^+ - s(x_1^-)}{x_2^- - s(x_1^-)}\right)$$

$$F_{12} = -R^0_{12}\,(q+q^{-1})\,\frac{ig\alpha^{-1}\gamma_2\gamma_1 q^{C_1}U_1}{q^{2C_2+3/2}U_2^2}\,\frac{ig^{-1}x_2^+ - (q-q^{-1})}{x_2^- - s(x_1^-)}\,\frac{s(x_2^+) - s(\ldots}{x_2^- - x_1^+}$$

$$\cdot\,\frac{\alpha^2}{1-g^2(q-q^{-1})^2}\,\frac{U_2 q^{C_2+1/2}(x_2^+ - x_2^-)}{\gamma_2^2}\,\frac{U_1 q^{C_1+1/2}(x_1^+ \ldots}{\gamma_1^2}$$

$$G_{12} = R^0_{12}\,\frac{1}{q^{C_2+1/2}U\ldots}\,\frac{x_2^+ - x_1^+}{\ldots},\qquad H_{12} = R^0_{12}\,\frac{\gamma_1}{\ldots}\,\frac{x_2^+ - x_2^-}{\ldots}$$

Discrete Symmetries of R-matrix

Braiding unitarity $\mathcal{R}_{12}\mathcal{R}_{21} = 1 \otimes 1$ entails

$$A_{12}A_{21} = B_{12}B_{21} + C_{12}F_{21} = G_{12}L_{21} + H_{12}H_{21} = 1,$$
$$A_{12}D_{12} = B_{12}E_{12} - C_{12}F_{12} = H_{12}K_{12} - G_{12}L_{12}.$$

Yang–Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

Matrix Unitarity

$$(\mathcal{R}_{12})^{\dagger}\mathcal{R}_{12} = 1 \otimes 1.$$

Crossing Symmetry

$$(\mathcal{C}^{-1} \otimes 1)\mathcal{R}_{\bar{1}2}^{\mathsf{ST}\otimes 1}(\mathcal{C} \otimes 1)\mathcal{R}_{12} = 1 \otimes 1.$$

imposes relations on scalar factor R^0_{12}

Near Neighbor Hamiltonian

Homogeneous Hamiltonian

$$\mathcal{H} = \sum_{k=1}^{L} \mathcal{H}_{k,k+1}.$$

The pairwise interaction $\mathcal{H}_{12}$ is the following logarithmic derivative of the R-matrix

$$\mathcal{H}_{12} = -i \frac{\left(x^+ - s(x^+)\right)\left(x^- - s(x^-)\right)}{q^{-1} x^+ s(x^+)} \left(\frac{du^*}{du}\right)^{-1/2} \mathcal{R}_{12}^{-1} \frac{d}{du_1} \mathcal{R}_{12}\Big|_{x_{12}^{\pm} = x^{\pm}} .$$

The spectral parameters u_k are defined via $x_k^{\pm}$

$$u_k = q^{-1} u(x_k^+) - \frac{i}{2g} = q u(x_k^-) + \frac{i}{2g} .$$

Bethe Equations and Spectrum

Generic for rank 3 algebra

$$1 = \prod_{j=1}^{K} R^{\mathrm{I},\mathrm{II}}(x_j, y_k) \prod_{\substack{j=1 \\ j\neq k}}^{N} R^{\mathrm{II},\mathrm{II}}(y_j, y_k) \prod_{j=1}^{M} R^{\mathrm{III},\mathrm{II}}(w_j, y_k),$$

$$1 = \prod_{j=1}^{N} R^{\mathrm{I},\mathrm{III}}(x_j, w_k) \prod_{j=1}^{N} R^{\mathrm{II},\mathrm{III}}(y_j, w_k) \prod_{\substack{j=1 \\ j\neq k}}^{M} R^{\mathrm{III},\mathrm{III}}(w_j, y_k),$$

Bethe Equations and Spectrum

Generic for rank 3 algebra

$$1 = \prod_{j=1}^{K} R^{\mathrm{I,II}}(x_j, y_k) \prod_{\substack{j=1 \\ j \neq k}}^{N} R^{\mathrm{II,II}}(y_j, y_k) \prod_{j=1}^{M} R^{\mathrm{III,II}}(w_j, y_k),$$

$$1 = \prod_{j=1}^{N} R^{\mathrm{I,III}}(x_j, w_k) \prod_{j=1}^{N} R^{\mathrm{II,III}}(y_j, w_k) \prod_{\substack{j=1 \\ j \neq k}}^{M} R^{\mathrm{III,III}}(w_j, y_k),$$

For our case

$$1 = \left(q^{-C-1/2} U^{-1} \frac{y_k - x^+}{y_k - x^-} \right)^K \prod_{j=1}^{M} q^{-1} \frac{q u(y_k) - w_j + \frac{i}{2} g^{-1}}{q^{-1} u(y_k) - w_j - \frac{i}{2} g^{-1}} ,$$

$$1 = \prod_{j=1}^{N} q \, \frac{w_k - q^{-1} u(y_j) + \frac{i}{2} g^{-1}}{w_k - q u(y_j) - \frac{i}{2} g^{-1}} \prod_{\substack{j=1 \\ j \neq k}}^{M} \frac{q^{-1} w_k - q w_j - \frac{i}{2}(q + q^{-1}) g^{-1}}{q w_k - q^{-1} w_j + \frac{i}{2}(q + q^{-1}) g^{-1}} .$$

Energy

Energy

$$E = E_0 K + \sum_{j=1}^{N} E(y_j).$$

$$E_0 = A, \qquad E(y_k) = H + K - 2A + Ge^{ip_k} + Le^{-ip_k},$$

1D Hubbard Model

Hamiltonian

$$\mathcal{H}_{j,k}^{\mathrm{Hub}} = \sum_{\alpha=1,2} \left(c_{\alpha,j}^{\dagger} c_{\alpha,k} + c_{\alpha,k}^{\dagger} c_{\alpha,j} \right) + U n_{1,j} n_{2,j}.$$

1D Hubbard Model

Hamiltonian

$$\mathcal{H}_{j,k}^{\mathrm{Hub}} = \sum_{\alpha=1,2} \left(c_{\alpha,j}^{\dagger} c_{\alpha,k} + c_{\alpha,k}^{\dagger} c_{\alpha,j} \right) + U n_{1,j} n_{2,j}.$$

exhibits $\mathfrak{su}(2) \times \mathfrak{su}(2) \in \mathfrak{su}(2|2)$ symmetry

1D Hubbard Model

Hamiltonian

$$\mathcal{H}^{\mathrm{Hub}}_{j,k} = \sum_{\alpha=1,2} \left(c^{\dagger}_{\alpha,j} c_{\alpha,k} + c^{\dagger}_{\alpha,k} c_{\alpha,j} \right) + U n_{1,j} n_{2,j} .$$

exhibits $\mathfrak{su}(2) \times \mathfrak{su}(2) \in \mathfrak{su}(2|2)$ symmetry

$$|\phi^1_k\rangle = |\circ\rangle, \qquad |\phi^2_k\rangle = \kappa c^{\dagger}_{1,k} c^{\dagger}_{2,k} |\circ\rangle,$$
$$|\psi^1_k\rangle = c^{\dagger}_{1,k} |\circ\rangle, \qquad |\psi^2_k\rangle = c^{\dagger}_{2,k} |\circ\rangle$$

Alcaraz and Bariev Chain

$$
\begin{aligned}
\mathcal{H}_{j,k}^{\mathrm{AB}} = &(c_{1,j}^{\dagger} c_{1,k} + c_{1,k}^{\dagger} c_{1,j})(1 + t_{11} n_{2,j} + t_{12} n_{2,k} + t_1' n_{2,j} n_{2,k}) \\
&+ (c_{2,j}^{\dagger} c_{2,k} + c_{2,k}^{\dagger} c_{2,j})(1 + t_{21} n_{1,j} + t_{22} n_{1,k} + t_2' n_{1,j} n_{1,k}) \\
&+ J(c_{1,j}^{\dagger} c_{2,k}^{\dagger} c_{2,j} c_{1,k} + c_{1,k}^{\dagger} c_{2,j}^{\dagger} c_{2,k} c_{1,j}) \\
&+ t_{\mathrm{p}}(c_{1,j}^{\dagger} c_{2,j}^{\dagger} c_{2,k} c_{1,k} + c_{1,j}^{\dagger} c_{2,j}^{\dagger} c_{2,k} c_{1,k}) \\
&+ V_{11} n_{1,j} n_{1,k} + V_{12} n_{1,j} n_{2,k} + V_{21} n_{2,j} n_{1,k} + V_{22} n_{2,j} n_{2,k} + U n_{1,j} n_{2,} \\
&+ V_3^{(1)} n_{2,j} n_{1,k} n_{2,k} + V_3^{(2)} n_{1,j} n_{1,k} n_{2,k} \\
&+ V_3^{(3)} n_{1,j} n_{2,j} n_{2,k} + V_3^{(4)} n_{1,j} n_{2,j} n_{1,k} \\
&+ V_4 n_{1,j} n_{2,j} n_{1,k} n_{2,k},
\end{aligned}
$$

where

$$
\begin{aligned}
t_{11} &= t_4 - 1, & t_{12} &= t_3 - 1, & t_1' &= t_5 - t_3 - t_4 + 1, \\
t_{21} &= t_1 - 1, & t_{22} &= t_2 - 1, & t_2' &= t_5 - t_1 - t_2 + 1.
\end{aligned}
$$

Relationship to Condensed Matter Notation

Four d.o.f. for each site

$$|\circ\rangle, \quad |\uparrow\rangle \sim c_1^\dagger|\circ\rangle, \quad |\downarrow\rangle \sim c_2^\dagger|\circ\rangle, \quad |\updownarrow\rangle \sim c_1^\dagger c_2^\dagger|\circ\rangle$$

or

$$|\phi_k^1\rangle = |\circ\rangle, \quad |\phi_k^2\rangle = \kappa c_{1,k}^\dagger c_{2,k}^\dagger|\circ\rangle, \quad |\psi_k^1\rangle = c_{1,k}^\dagger|\circ\rangle, \quad |\psi_k^2\rangle = c_{2,k}^\dagger|\circ\rangle.$$

anticommutators

$$\{c_{\alpha,k}, c_{\beta,l}^\dagger\} = \delta_{\alpha\beta}\delta_{kl}, \qquad \{c_{\alpha,k}, c_{\beta,l}\} = \{c_{\alpha,k}^\dagger, c_{\beta,l}^\dagger\} = 0.$$

number operators

$$n_{\alpha,k} = c_{\alpha,k}^\dagger c_{\alpha,k}$$

It is a subsector of $\mathrm{U}_q(\mathfrak{su}(2|2))$ Hamiltonian!

It is a subsector of $\mathrm{U}_q(\mathfrak{su}(2|2))$ Hamiltonian!

Possible transformations: twist, add central elements... and change spectrum in controllable way

$$
\begin{aligned}
\mathcal{H}'_{12} = {} & a_0 \, \mathcal{T} \mathcal{H}_{12} \mathcal{T}^{-1} + \tfrac{1}{2} a_1 \Delta(\mathfrak{H}_1) + a_2 \Delta(1) + \tfrac{1}{2} a_3 \Delta(\mathfrak{H}_3) \\
& + \tfrac{1}{2} b_1 (\mathfrak{H}_1 \otimes 1 - 1 \otimes \mathfrak{H}_1) + b_2 (\mathfrak{H}_1 \mathfrak{H}_1 \otimes 1 - 1 \otimes \mathfrak{H}_1 \mathfrak{H}_1) \\
& + \tfrac{1}{2} b_3 (\mathfrak{H}_3 \otimes 1 - 1 \otimes \mathfrak{H}_3)
\end{aligned}
$$

It is a subsector of $\mathrm{U}_q(\mathfrak{su}(2|2))$ Hamiltonian!

Possible transformations: twist, add central elements... and change spectrum in controllable way

$$\mathcal{H}'_{12} = a_0\,\mathcal{T}\mathcal{H}_{12}\mathcal{T}^{-1} + \tfrac{1}{2}a_1\Delta(\mathfrak{H}_1) + a_2\Delta(1) + \tfrac{1}{2}a_3\Delta(\mathfrak{H}_3) + \tfrac{1}{2}b_1(\mathfrak{H}_1 \otimes 1 - 1 \otimes \mathfrak{H}_1) + b_2(\mathfrak{H}_1\mathfrak{H}_1 \otimes 1 - 1 \otimes \mathfrak{H}_1\mathfrak{H}_1) + \tfrac{1}{2}b_3(\mathfrak{H}_3 \otimes 1 - 1 \otimes \mathfrak{H}_3)$$

Twist [Reshetikhin]

$$\mathcal{T} = \exp\left(if_1 \sum_{j=1}^{K}(j-1)\mathfrak{H}_{1,j} + \tfrac{i}{2}f_2 \sum_{j<k=1}^{K}(\mathfrak{H}_{1,j}\mathfrak{H}_{3,k} - \mathfrak{H}_{3,j}\mathfrak{H}_{1,k}) + if_3 \sum_{j=1}^{K}(j-1)\mathfrak{H}_{3,k} \right)$$

Other Hubbard-like Models

Q-deformation of the Hubbard model limit [Beisert PK]

$$
\begin{aligned}
\mathcal{H}'_{j,k} = A' \sum_{\ell=j,k} &\left((1-n_{1,\ell})(1-n_{2,\ell}) + n_{1,\ell}n_{2,\ell} - \tfrac{1}{2}\right) \\
&+ iq^{+1/2} c^\dagger_{1,j} c_{1,k} \big(1-(1-q^{+1/2})n_{2,j}\big)\big(1-(1-q^{-3/2})n_{2,k}\big) \\
&+ iq^{+1/2} c^\dagger_{2,j} c_{2,k} \big(1-(1-q^{-1/2})n_{1,j}\big)\big(1-(1-q^{-1/2})n_{1,k}\big) \\
&- iq^{-1/2} c^\dagger_{1,k} c_{1,j} \big(1-(1-q^{+3/2})n_{2,j}\big)\big(1-(1-q^{-1/2})n_{2,k}\big) \\
&- iq^{-1/2} c^\dagger_{2,k} c_{2,j} \big(1-(1-q^{+1/2})n_{1,j}\big)\big(1-(1-q^{+1/2})n_{1,k}\big).
\end{aligned}
$$

Outlook so far

- We found embedding of $\mathfrak{su}(2|2)$ in quantum group setup (some issue are not clear though)

Outlook so far

- We found embedding of $\mathfrak{su}(2|2)$ in quantum group setup (some issue are not clear though)
- All known deformations of 1D Hubbard model are shown to be embedded into quantum Super Yang Mill chain

Outlook so far

- We found embedding of $\mathfrak{su}(2|2)$ in quantum group setup (some issue are not clear though)
- All known deformations of 1D Hubbard model are shown to be embedded into quantum Super Yang Mill chain
- Quantum deformations uncover hidden symmetries

Outlook so far

- We found embedding of $\mathfrak{su}(2|2)$ in quantum group setup (some issue are not clear though)
- All known deformations of 1D Hubbard model are shown to be embedded into quantum Super Yang Mill chain
- Quantum deformations uncover hidden symmetries
- Quantum Deformed gauge theory is not known

Outlook so far

- We found embedding of $\mathfrak{su}(2|2)$ in quantum group setup (some issue are not clear though)
- All known deformations of 1D Hubbard model are shown to be embedded into quantum Super Yang Mill chain
- Quantum deformations uncover hidden symmetries
- Quantum Deformed gauge theory is not known
- Gravity dual theory is not known

Universal R-matrix

Yangian in RTT realization

- Consider Lie (super)algebra $\mathfrak{gl}(n|m)$ and its vector representation

Yangian in RTT realization

- Consider Lie (super)algebra $\mathfrak{gl}(n|m)$ and its vector representation
- Yangian $Y(\mathfrak{gl}(n|m))$ is isomorphic to associative algebra $U(R)$ generated by 1 and the matrices

$$T_{ij}^{(k)}, \quad i,j = \overline{1, n+m}, \quad k \in \mathbb{Z}_{\geq 0}$$

Yangian in RTT realization

- Consider Lie (super)algebra $\mathfrak{gl}(n|m)$ and its vector representation
- Yangian $Y(\mathfrak{gl}(n|m))$ is isomorphic to associative algebra $U(R)$ generated by 1 and the matrices

$$T_{ij}^{(k)}, \quad i,j=\overline{1,n+m}, \quad k\in\mathbb{Z}_{\geq 0}$$

  It is convenient to gather them in the formal series

$$T(\lambda)=\sum_{i,j=1}^{n}\sum_{n=0}^{+\infty}T_{ij}^{(n)}\lambda^{-n}E_{i,j}\,,$$

Yangian in RTT realization

- Consider Lie (super)algebra $\mathfrak{gl}(n|m)$ and its vector representation
- Yangian $Y(\mathfrak{gl}(n|m))$ is isomorphic to associative algebra $U(R)$ generated by 1 and the matrices

$$T_{ij}^{(k)}, \quad i,j = \overline{1, n+m}, \quad k \in \mathbb{Z}_{\geq 0}$$

It is convenient to gather them in the formal series

$$T(\lambda) = \sum_{i,j=1}^{n} \sum_{n=0}^{+\infty} T_{ij}^{(n)} \lambda^{-n} E_{i,j} \,,$$

$T(\lambda)$ satisfy the so-called RTT relations

$$R^{(n)}(\lambda - \mu)(T(\lambda) \otimes 1)(1 \otimes T(\mu)) = (1 \otimes T(\mu))(T(\lambda) \otimes 1)R^{(n)}(\lambda - \mu$$

$$\mathrm{qdet}(T(\lambda)) = 1 \,,$$

where qdet is the quantum determinant and the Yang matrix is given by

$$R^{(n)}(\lambda) = 1 \otimes 1 + \sum_{1 \leq i,j \leq n} \lambda^{-1} E_{i,j} \otimes E_{j,i}$$

- Commutation relations for $T(\lambda)$

$$(\lambda-\mu)[T_{ij}(\lambda),T_{kl}(\mu)]=T_{kj}(\mu)T_{il}(\lambda)-T_{kj}(\lambda)T_{il}(\mu)$$

- Commutation relations for $T(\lambda)$

$$(\lambda - \mu)[T_{ij}(\lambda), T_{kl}(\mu)] = T_{kj}(\mu)T_{il}(\lambda) - T_{kj}(\lambda)T_{il}(\mu)$$

- $T_{ij}(\lambda)$ is a generating function for the Yangian $Y(\mathfrak{gl}(n|m))$ generators. Expansion around $\lambda = \infty$ gives these generators and commutation relations on $T_{ij}(\lambda)$ give defining relations on Yangian generators as well as Serre relations. Coproduct for Yangian generators follow from coproduct of $T_{ij}(\lambda)$.

- Commutation relations for $T(\lambda)$

$$(\lambda-\mu)[T_{ij}(\lambda), T_{kl}(\mu)] = T_{kj}(\mu)T_{il}(\lambda) - T_{kj}(\lambda)T_{il}(\mu)$$

- $T_{ij}(\lambda)$ is a generating function for the Yangian $Y(\mathfrak{gl}(n|m))$ generators. Expansion around $\lambda=\infty$ gives these generators and commutation relations on $T_{ij}(\lambda)$ give defining relations on Yangian generators as well as Serre relations. Coproduct for Yangian generators follow from coproduct of $T_{ij}(\lambda)$.
- Call the diagonal and upper/lower triangular part of $T_{ij}^{(k)}$ $\mathfrak{H}_i^{(k)}, \mathfrak{E}_i^{(k)}, \mathfrak{F}_i^{(k)} | i = \overline{1, n+m-1}, k \in \mathbb{Z}_>$, then from RTT defining relations it follows

$$[\mathfrak{H}_i^{(0)}, \mathfrak{E}_j^{(l)}] = A_{ij}\mathfrak{E}_j^{(l)}, \quad [\mathfrak{H}_i^{(0)}, \mathfrak{F}_j^{()l}] = -A_{ij}\mathfrak{F}_j^{(l)} \dots$$

Quantum Double

- Algebraically, R-matrix is the canonical element of the Hopf Algebra tensored with its dual (similar to a Casimir)

Quantum Double

- Algebraically, R-matrix is the canonical element of the Hopf Algebra tensored with its dual (similar to a Casimir)
- Classical analogy: Lie algebra $\mathfrak{g}$ with generators $[\mathfrak{J}^a, \mathfrak{J}^b] = f_c^{ab}\mathfrak{J}^c$ extends to loop algebra (Kac-Moody algebra without central charge) $\mathfrak{g}[\lambda, \lambda^{-1}]$ with generators $[\mathfrak{J}_n^a, \mathfrak{J}_m^b] = f_c^{ab}\mathfrak{J}_{n+m}^c$, i.e. $\mathfrak{J}_n^a = \lambda^n \mathfrak{J}^a$. Then Killing form $\kappa^{ab} \propto str(\mathfrak{J}^a, \mathfrak{J}^b)$ is extended by $(\mathfrak{J}_n^a, \mathfrak{J}_m^b) = \kappa^{ab}\delta_{n,-m-1}$. This form splits $\mathfrak{g}[\lambda, \lambda^{-1}] = \mathfrak{g}[\lambda] + \lambda^{-1}\mathfrak{g}[\lambda^{-1}]$ into positive and negative degrees.

- Classical r-matrix:

$$r = \sum_{n=0}^{\infty} \kappa_{ab} \mathfrak{J}^a_n \otimes \mathfrak{J}^b_{-n-1}$$

- Classical r-matrix:

$$r = \sum_{n=0}^{\infty} \kappa_{ab} \mathfrak{J}_n^a \otimes \mathfrak{J}^b_{-n-1}$$

- Quantum R-matrix of Yangian: $\mathcal{R} = \sum_{J \in \mathcal{Y}(\mathfrak{g})} J \otimes J^*$, where J^* is the dual of J

- Classical r-matrix:

$$r = \sum_{n=0}^{\infty} \kappa_{ab} \mathfrak{J}_n^a \otimes \mathfrak{J}_{-n-1}^b$$

- Quantum R-matrix of Yangian: $\mathcal{R} = \sum_{J \in \mathcal{Y}(\mathfrak{g})} J \otimes J^*$, where J^* is the dual of J
- Invariant form for Yangian:

$$\langle \mathfrak{E}_{i,k}^+, \mathfrak{E}_{j,l}^- \rangle = -\delta_{ij}\delta_{k,-l-1}$$
$$\langle \mathfrak{E}_{i,k}^-, \mathfrak{E}_{j,l}^+ \rangle = -(-1)^{|i|}\delta_{ij}\delta_{k,-l-1}$$
$$\langle \mathfrak{H}_{i,k}, \mathfrak{H}_{j,-l-1} \rangle = -2\left(\frac{A_{ij}}{2}\right)^{n-m}\binom{n}{m}, \quad n \geq m,$$

- Classical r-matrix:

$$r = \sum_{n=0}^{\infty} \kappa_{ab} \mathfrak{J}_n^a \otimes \mathfrak{J}^b_{-n-1}$$

- Quantum R-matrix of Yangian: $\mathcal{R} = \sum_{J \in \mathcal{Y}(\mathfrak{g})} J \otimes J^*$, where J^* is the dual of J
- Invariant form for Yangian:

$$\langle \mathfrak{E}^+_{i,k}, \mathfrak{E}^-_{j,l} \rangle = -\delta_{ij}\delta_{k,-l-1}$$
$$\langle \mathfrak{E}^-_{i,k}, \mathfrak{E}^+_{j,l} \rangle = -(-1)^{|i|}\delta_{ij}\delta_{k,-l-1}$$
$$\langle \mathfrak{H}_{i,k}, \mathfrak{H}_{j,-l-1} \rangle = -2 \left(\frac{A_{ij}}{2} \right)^{n-m} \binom{n}{m}, \quad n \geq m,$$

- For explicit form of R-matrix need to diagonalize this form

R-matrix

- For a simple Lie superalgebra $\mathfrak{g}$ with symmetrized Cartan matrix $A^{\mathfrak{g}}$ define its quantum counterpart

$$A^{\mathfrak{g}}_{ij} \to A^{\mathfrak{g}}_{ij}(q) := \left[A^{\mathfrak{g}}_{ij}\right]_q$$

R-matrix

- For a simple Lie superalgebra $\mathfrak{g}$ with symmetrized Cartan matrix $A^{\mathfrak{g}}$ define its quantum counterpart

$$A^{\mathfrak{g}}_{ij} \to A^{\mathfrak{g}}_{ij}(q) := \left[A^{\mathfrak{g}}_{ij}\right]_q$$

- Construct a matrix

$$C^{\mathfrak{g}}_{ij}(q) = \ell^{\mathfrak{g}}(q)\left(A^{\mathfrak{g}}(q)\right)^{-1}_{ij}$$

R-matrix

- For a simple Lie superalgebra $\mathfrak{g}$ with symmetrized Cartan matrix $A^{\mathfrak{g}}$ define its quantum counterpart

$$A^{\mathfrak{g}}_{ij} \to A^{\mathfrak{g}}_{ij}(q) := \left[A^{\mathfrak{g}}_{ij}\right]_q$$

- Construct a matrix

$$C^{\mathfrak{g}}_{ij}(q) = \ell^{\mathfrak{g}}(q)\left(A^{\mathfrak{g}}(q)\right)^{-1}_{ij}$$

- The constant $\ell^{\mathfrak{g}}(q)$ is defined as the *minimal* proportionality factor that makes $C^{\mathfrak{g}}(q)$ polynomial in q and q^{-1}. It is usually proportional to the dual Coxeter number.

- Triangular decomposition of $\mathfrak{g}$ into subalgebras of positive roots, Cartan and negative roots

$$\mathfrak{g} = \mathfrak{e}^{+} \oplus \mathfrak{h} \oplus \mathfrak{e}^{-} \,,$$

one has $[\mathfrak{e}_{\pm}, \mathfrak{h}] \subset \mathfrak{e}_{\pm}$

- Triangular decomposition of $\mathfrak{g}$ into subalgebras of positive roots, Cartan and negative roots

$$\mathfrak{g} = \mathfrak{e}^{+} \oplus \mathfrak{h} \oplus \mathfrak{e}^{-} \,,$$

  one has $[\mathfrak{e}_{\pm}, \mathfrak{h}] \subset \mathfrak{e}_{\pm}$
- induces triangular decomposition of R-matrix

$$\mathcal{R}_{12} = \mathcal{R}_{+}\mathcal{R}_{H}\mathcal{R}_{-} .$$

- Triangular decomposition of $\mathfrak{g}$ into subalgebras of positive roots, Cartan and negative roots

$$\mathfrak{g} = \mathfrak{e}^{+} \oplus \mathfrak{h} \oplus \mathfrak{e}^{-} \,,$$

  one has $[\mathfrak{e}_{\pm}, \mathfrak{h}] \subset \mathfrak{e}_{\pm}$
- induces triangular decomposition of R-matrix

$$\mathcal{R}_{12} = \mathcal{R}_{+}\mathcal{R}_{H}\mathcal{R}_{-}.$$

$$\mathcal{R}_{+} = \prod_{\alpha\in\Xi^{+}}^{\rightarrow} \exp(-(-1)^{\theta(\alpha)} a(\alpha) \mathfrak{E}^{+}_{\alpha} \otimes \mathfrak{E}^{-}_{\alpha}),$$

$$\mathcal{R}_{-} = \prod_{\alpha\in\Xi^{+}}^{\leftarrow} \exp(-(-1)^{\theta(\alpha)} a(\alpha) \mathfrak{E}^{-}_{\alpha} \otimes \mathfrak{E}^{+}_{\alpha}) \,,$$

$\theta(\alpha)$ is parity of $\mathfrak{E}^{\pm}_{\alpha}$.

- The set of positive roots

$$\Xi^{+} := \{\gamma + n\delta | \gamma \in \Delta^{+}\},$$

δ is affine root

- The set of positive roots

$$\Xi^{+} := \{\gamma + n\delta | \gamma \in \Delta^{+}\},$$

δ is affine root

$$[\mathfrak{E}_{\alpha}^{+}, \mathfrak{E}_{\alpha}^{-}] = a(\alpha)^{-1}\mathfrak{H}_{\gamma}, \quad \alpha = \gamma + n\delta, \quad \gamma \in \Delta_{+}(\mathfrak{g})$$

- The set of positive roots

$$\Xi^{+} := \{\gamma + n\delta | \gamma \in \Delta^{+}\} ,$$

δ is affine root

$$[\mathfrak{E}_{\alpha}^{+}, \mathfrak{E}_{\alpha}^{-}] = a(\alpha)^{-1}\mathfrak{H}_{\gamma}, \quad \alpha = \gamma + n\delta, \quad \gamma \in \Delta_{+}(\mathfrak{g})$$

Cartan part of the Yangain $\mathcal{R}_H$

$$\prod_{n=0}^{\infty} \exp\left(\left(\mathfrak{K}'_{i,+}(\lambda)\right)_m \otimes \left(C_{i,j}^{\mathfrak{g}}(T^{1/2})\mathfrak{K}_{j,-}(\tilde{\lambda} + \ell^{\mathfrak{g}}(n+1))\right)_{m+1} \right)$$

$\mathfrak{gl}(n|m)$ R-matrix

Inverse of q-Cartan matrix $\left(A^{\mathfrak{gl}(n|m)}(q)\right)^{-1}$ [PK Rej Spill to appear]

$$\begin{pmatrix} a_{n+m-1,1} & \cdots & \cdots & \cdots & \text{upper} & \text{elements} & \text{are} \\ \vdots & \ddots & \cdots & \cdots & \cdots & \text{obtained} & \text{by} \\ a_{m+1,1} & \cdots & a_{m+1,n-1} & \cdots & \cdots & \cdots & i \leftrightarrow j \\ b_{m,1} & \cdots & \cdots & b_{m,n} & \cdots & \cdots & \cdots \\ \vdots & \ddots & \cdots & \vdots & c_{m-1,n+1} & \cdots & \cdots \\ b_{2,1} & b_{2,2} & \cdots & \vdots & \vdots & \ddots & \cdots \\ b_{1,1} & b_{1,2} & \cdots & b_{1,n} & c_{1,n+1} & \cdots & c_{1,n+m-1} \end{pmatrix}$$

with

$$a_{i,j} = -\frac{[2m-i]_q[j]_q}{[n-m]_q}$$

$$b_{i,j} = -\frac{[i]_q[j]_q}{[n-m]_q}$$

$$c_{i,j} = -\frac{[i]_q[2n-j]_q}{[n-m]_q}$$

Outlook

- Calculate Universal R-matrix for other superalgebras

Outlook

- Calculate Universal R-matrix for other superalgebras
- Study different representations (not necessarily highest or lowest weight) May help to understand how more than one spectral parameter may appear in R (S)-matrix

Outlook

- Calculate Universal R-matrix for other superalgebras
- Study different representations (not necessarily highest or lowest weight) May help to understand how more than one spectral parameter may appear in R (S)-matrix
- Use it for amplitudes in $\mathcal{N} = 4$