DELL talk at UC Berkeley   11/06/19

Plan ① Overview of classical integrable systems of Calogero-RS type

Calogero in 1971 introduced a new integrable system, Moser 1975 proved its integrability using Lax pair

$H_{CM} = \sum_{i=1}^{n} \frac{p_i^2}{2m} + g^2 \sum_{i\neq j} \frac{1}{(q_i-q_j)^2}$. $H_{CM} = H_2$, there are others $H_1 \ldots H_n$, $\{H_i, H_j\}_{PB} = 0$

CM has several generalizations

rCM → tCM → eCM   + spin degrees of freedom

$V \sim \frac{1}{z^2}, \frac{1}{\sinh^2 z}, \wp(z)$

A different generalization is called Ruijsenaars-Schneider systems [RS 86] – Relativistic CM s.t. $H_{CM} = \lim_{c\to\infty} H_{RS} - n m c^2$ (# of part)   + spin d.o.f.

Geometrically [Kazhdan, Kostant, Sternberg 78] described CM as a Hamiltonian reduction of $T^*\mathfrak{gl}_n$

Adjoint action of GL_n on $\mathfrak{gl}_n$ lifts to a Hamiltonian GL_n action on $T^*\mathfrak{gl}_n = \mathfrak{gl}_n \times \mathfrak{gl}_n$ w/ moment map

$\mu: T^*\mathfrak{gl}_n \to \mathfrak{gl}_n^* (\sim \mathfrak{gl}_n)$ Let $\mathcal{O}$ be the coadjoint orbit of $\mathfrak{gl}_n$ of minimal dimension containing $\mathcal{O} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ \vdots & & \ddots & \\ 1 & 1 & \cdots & 0 \end{pmatrix}$

$(X, Y) \longmapsto [X, Y]$ Applying Hamiltonian reduction on $T^*\mathfrak{gl}_n$ wrt $\mathcal{O}$ ⟹ CM phase space

X - diagonal ⟹ Y - Lax matrix of rCM $\mu^{-1}(\mathcal{O})/G$ - scheme $C_n = R(X, G, \mathcal{O})$

$X \leftrightarrow Y$ ⟹ spectral duality

Overall there is the following classification of IS of CM-RS-DELL type.

q\ħ	r	t	e
r	rCM	tCM	eCM
t	rRS	tRS	eRS
e	dual eCM	dual eRS	Dell

These are examples of complex algebraic IS w/ N d.o.f whose phase space is a Lagrangian fibration of complex dimension 2N equipped w/ holomorphic symplectic 2-form over a smooth base whose fibers are Abelian varieties (has a group law). In proper Darboux coordinates

$\Omega = \sum_{i=1}^{N} dR \wedge dr_i$, there are N poisson commuting Hamiltonians $H_1 \ldots H_N$

In action-angle variables Hamiltonian evolution is linearized on the fibers which serve as level sets of the Hamiltonians

From physics (Seiberg-Witten solution [SW 94] – [NPS, NP]) Hitchin IS

For our purposes today we take $\mathcal{N} = 2^*$ (4d), $\mathcal{N} = 1^*$ (5d), $\mathcal{N} = (1,0)^*$ (6d) theory w/ 8 supercharges

Classical integrability [Donagi-Witten], [Gorsky-Nekrasov]

$\mathcal{E} \to \mathcal{M}^{\mathcal{N}=2}_{vac}(\mathbb{R}^3 \times S^1)$ – 3d Coulomb branch (hyper Kahler)

$\downarrow$

$B = \mathcal{M}^{\mathcal{N}=2}_{vac}(\mathbb{R}^4)$ – 4d Coulomb branch (special kahler)

{Spectral curve of eCM} = {SW curve for $\mathcal{N}=2^*$ 4d theory}

coupling ⟷ mass   IR description of the 5d theory

$\exists\, \mathcal{F}(a_1 \ldots a_N)$ – prepotential, holom fn of Coulomb branch VEVs $a_1 \ldots a_N$

Period matrix $\tau_{ij} = \partial_i \partial_j \mathcal{F}$ fibered over B

Abelian varieties → p, q live on $\mathbb{C}$, $\mathbb{C}^\times$ or $\mathcal{E} = \mathbb{C}^\times/q$

Hitchin IS. Liouville tori can be found inside Jacobians of an algebraic curve

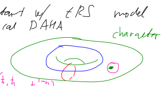
⟹ eCM Hamiltonians

$(\mathcal{E}, \phi)$, $\det(u - \phi) = 0$, $u \in K_{\Sigma_{g=1}}$

② Let us start w/ tRS model (Macdonald operators) Its Hamiltonians form a maximal commutative subalgebra in spherical DAHA

$A B A^{-1} B^{-1} = C$, $C = \mathrm{diag}(\hbar, \hbar, \ldots \hbar^{1-n})$

character variety



$\mathcal{A}_\hbar = \mathbb{C}_\hbar\left[\overbrace{\mathcal{M}_{flat}(Gl(N), T^2\setminus\{pt\})}\right]$ geometric quantization of the moduli space of flat connections on a single-punctured torus

$\{H_1 \ldots H_n\} \subset \mathcal{A}_\hbar$

Ex $N=2$

$$\begin{cases} \hat{H}_1 = \frac{\hbar z_1 - z_2}{z_1 - z_2}\hat{p}_1 + \frac{\hbar z_2 - z_1}{z_2 - z_1}\hat{p}_2 \\ \hat{H}_2 = \hat{p}_1 \hat{p}_2 \end{cases}$$

$p_i z_j = q^{\delta_{ij}} z_j p_i$ $p_1 f(z_1) = f(q z_1)$

In the limit $q \to 1$ $H_1 \ldots H_N$ – eigenvalues of B monodromy matrix $H_r \sim \mathrm{Tr}_{\Lambda^r} B$, $e_r(\vec{a}) = \mathrm{Tr}_{\Lambda^r} A$

Quantum tRS spectrum $H_r^{tRS}(z, p)\, V = e_r(a_1 \ldots a_n)\, V$

Q What is V? (RT, AG, etc)

answering this question will help us to understand eRS, eCM and DELL (!)

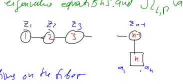
"S-duality" 3d mirror

Notice the symmetry $A \leftrightarrow B$, $\hbar \leftrightarrow \hbar^{-1}$ The equation looks like $(\mathrm{Tr}_{\Lambda^r} B)\cdot V = (\mathrm{Tr}_{\Lambda^r} A)\cdot V$ [Gaiotto, PK]

$\mathcal{M} = \mathcal{M} \times \mathcal{M}^\vee$ $\mathcal{L}_a \subset \mathcal{M}$ – Lagrangian given by specifying eigenvalues of A

then $\mathcal{L}_a$ is defined by tRS eigenvalue equations and $\Omega_{I,P}$ vanishes on $\mathcal{L}_a$

③ AG interpretation of tRS IS   Let $X = T^* \mathbb{F}l_n$



$\hbar$ acts by dilations on the fiber

[Okounkov]
[PK Smirnov Pushkar Zeitlin]

Notice that X is self-mirror $X \leftrightarrow X^\vee$, $a \leftrightarrow z$, $\hbar \leftrightarrow \hbar^{-1}$

Moduli space of quasi-maps

$q \circlearrowright \mathbb{P}^1 \overset{f}{\dashrightarrow} X$ in order to construct quantum equivariant K-theory of X. $K_T(X)$ -algebra with quantum multiplication

$T = \Pi(Gl(n)) \times \mathbb{C}^\times_\hbar$

Theorem (PK) Let V_p be a K-theory vertex function of X (coefficient of the vertex function, possibly w/ descendant) Then it is the

p ← described by fixed pts of maximal torus

eigenfunction of the tRS integrable system

$H_r(z_i, p_i)\, V_p = e_r(a_i)\, V_p$, $H_r = \sum_{\substack{\mathcal{I}\subset\{1,\ldots,n\} \\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I} \\ j\notin\mathcal{I}}} \frac{\hbar \zeta_i - \zeta_j}{\zeta_i - \zeta_j} \prod_{i\in\mathcal{I}} p_i$

" $V_p = \sum_{\vec{d}} z_1^{d_1} \ldots z_{n-1}^{d_{n-1}} \int_{QM^{\vec{d}}_{reg}} 1$ "

tRS momenta have geometrical meaning $p_i \sim \Lambda^i V_i \otimes \Lambda^{i+1} V_{i+1}^*$ - quantum multiplication by this class

$V^{(\tau)}(z) = \sum_{\vec{d}} \vec{z}^{\vec{d}}\, ev_{p_2,*}\left(QM^{\vec{d}}_{nonsing, p_2}, \hat{\mathcal{O}}_{vir}\, \tau(V_i/p_i)\right) \in K_{T_q}(X)_{loc}[[z]]$

Ex $X = T^*\mathbb{P}^1$

$V_p^{(1)} = \sum_{d\geq 0} z^d \prod_{i=1}^{2} \frac{\left(\frac{q}{\hbar}\frac{a_2}{a_1} \middle| q\right)_d}{\left(\frac{a_2}{a_1}, q\right)_d} = {}_2\phi_1\left(\hbar, \hbar\frac{a_2}{a_1}, q\frac{a_2}{a_1} \middle| q, \frac{q}{\hbar} z\right)$ - q hypergeometric series

If $\frac{a_2}{a_1} = q^\lambda \hbar$, $\lambda \in \mathbb{Z}_+$ then $V_p^{(1)}\big| = P_\lambda(z \,|\, q, \hbar)$ symmetric Macdonald poly

③ Elliptic RS model $H_r^{eRS} = \sum_{\substack{\mathcal{I}\subset\{1\ldots n\} \\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I} \\ j\notin\mathcal{I}}} \frac{\theta_1\left(\hbar \frac{z_i}{z_j} \middle| p\right)}{\theta_1\left(\frac{z_i}{z_j} \middle| p\right)} \prod_{i\in\mathcal{I}} p_i$,   where p – modular parameter of the torus $p \to 0$ eRS → tRS

Eigenvalue problem

$H_r^{eRS}(\vec{z}, \vec{p}, p)\, Z = E_r(\vec{a}, p, \hbar)\, Z$ Q What is Z, what are E_r?

Conjecture [PK] (proven earlier by Negut for quantum cohomology):   " $Z = \sum_{\vec{d}} \vec{q}^{\,\vec{d}} \int_{\mathcal{L}^{aff}_{\vec{d}}} 1$ " is K-theoretic holomorphic equivariant Euler characteristic of the affine Laumon space $\mathcal{L}^{aff}_{\vec{d}}$, $\vec{q} = (q_1 \ldots q_n)$ – maximal torus of $\mathcal{L}^{aff}_{\vec{d}}$. Eigenvalues E_r are equivariant Chern characters of bundles $\Lambda^r W$, W– constant bundle of the ADHM space $K_{\mathbb{C}^\times_q \times \mathbb{C}^\times_\hbar}(\mathrm{Hilb}^k \mathbb{C}^2)$

$\mathbb{C}[x_1 \ldots x_n] \otimes \mathbb{C}(q, \hbar)$ classes of fixed points are Macdonald polynomials P_λ

ADHM quiver

$E_r = e_r + \sum_{e=1}^{\infty} p^e E_r^{(e)}$ $p = q_1 \ldots q_n$

In gauge-theoretic terms Z - instanton partition function of 5d $\mathcal{N}=1^*$ theory w/ 3d defect along $\mathbb{R}^2_q \times S^1$ in the Ω-background on $\mathbb{R}^2 \times \mathbb{R}^2_q \times S^1$

$E_r = \mathrm{VeV}\, \langle W_{\Lambda^r \mathbf{n}} \rangle$ along S^1

In the limit $p \to 0$ $\mathcal{L}^{aff}_{\vec{d}} \longrightarrow \mathcal{L}_{\vec{d}}$ and $Z \longrightarrow V$

Conjecture (PK, Shakirov) $E_r(\vec{a}) = (-1)^r \frac{\theta_1(\hbar^N)}{\theta_1(\hbar)} \frac{Z(\vec{a}\, \hbar^{\vec{\rho}} q^{\omega_i})}{Z(\vec{a}, \hbar^{\vec{\rho}})}$

$\vec{\rho}$ – $SU(N)$ Weyl vector
ω_i – i-th fundamental weight of rep of $SU(N)$

Ex $N=2$ $Z = {}_2\phi_1(\;) + \sum_{n=1}^{\infty} \left(\frac{p}{z}\right)^n f_n(\vec{a}, \hbar, q)$

④ From eRS to DELL: Hamiltonians $H_a = \mathcal{O}_0^{-1}\mathcal{O}_a$, $a = 1, \ldots, n-1$

2nd elliptic parameter $w \in \mathcal{E}$

$$\mathcal{O}(\zeta) = \sum_{n\in\mathbb{Z}} \mathcal{O}_n \zeta^n = \sum_{n_1 \ldots n_n = -\infty}^{+\infty} (-\zeta)^{\sum n_i} w^{\sum \frac{n_i(n_i-1)}{2}} \prod_{i<j} \theta_1\left(\hbar^{n_i - n_j} \frac{z_i}{z_j} \middle| p\right) p_1^{n_1} \ldots p_N^{n_N}$$

Conjecture 1 (checked on a computer) $[H_a, H_b] = 0$

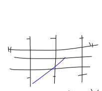
Conjecture 2 $\mathcal{O}(\zeta)\, Z(p, \vec{z}) = \lambda(\zeta, \vec{a}, w, p)\, \mathcal{O}_0\, Z(p, \vec{z})$,   where $Z(p, \vec{z})$ – equivariant elliptic genus of the affine Laumon space

or, after expanding in ζ we get $H_n Z = E_n Z$

Ex $N=2$ w/o center of mass $H = \mathcal{O}_0^{-1}\mathcal{O}_1$, $\mathcal{O}_0 = \Theta\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}\left(\frac{z_2}{z_1}, \frac{p_2}{p_1}\right)$, $\mathcal{O}_1 = \Theta\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\left(\frac{z_2}{z_1}, \frac{p_2}{p_1}\right)$

theta function of genus 2

Phys sics interpretation



spin version

$g = N+1$ surface

As a byproduct we discovered elliptic Macdonald functions!