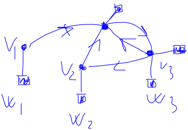
0) Framed quiver w/ set of vertices $I = \{V_i\}$ — internal vertices



$\{W_i\}$ — external vertices (framing)

Morphisms between vertices

$\mathrm{Rep}(\vec{v}, \vec{w})$ — space of quiver representations

$v_i = \dim V_i$
$w_i = \dim W_i$

Moment map

$$\mu \quad T^*\mathrm{Rep}(\vec{v}, \vec{w}) \to \mathrm{Lie}(G)^*$$

$$G = \prod_i GL(v_i)$$

$$L(\vec{v}, \vec{w}) = \mu^{-1}(0)$$

Nakajima quiver variety

- algebraic symplectic resolution

$$X = L(\vec{v}, \vec{w}) /\!\!/_\theta G = L(\vec{v}, \vec{w})_{ss} / G$$

stability parameter

semi-stable locus

$$\mathrm{Aut}(X) = \prod GL(v_i) \times \prod GL(w_i) \times \mathbb{C}^\times_\hbar$$ — automorphism group

inadmissible / framing / scales cotangent directions

$$T = \pi(G)$$

$AB = 0$

B - full rank

$K_T(T^*G_r)$ - generated by fixed points of the action of maximal torus

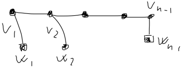
Ah seminar 05/17/18 A-type quiver varieties and ADHM moduli spaces

0) Definition of a quiver variety

1) Formulation of result

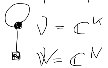
2 types of quiver varieties

A-type X_n



and

Kronecker/ADHM type



We shall demonstrate the equivalence between

$$K_T(\underbrace{QM}_{\text{moduli space quasimaps}}(\mathbb{P}^1, X_n)) \simeq \bigoplus_{\ell=1}^{K} K_{q,t_1}(\mathcal{M}^{\ell}_{ADHM})$$

X_n – basic example of quiver variety

$\mathcal{M}^{\ell}_{ADHM}$ – moduli space of torsion-free sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$

$X^n = T^* \mathbb{F}\ell_n$

$w_1 \ \ w_{n-2} = 0, \ w_{n-1} = n$

$v_1 = 1$

$\mathbb{C}^\times_q \times \mathbb{C}^\times_{t_1}$ acts on $\mathbb{C}^2$ by dilations

Assume $\ell < n$

torus $T = \underbrace{\mathbb{C}^\times_{t_1} \times T(Gl(n, \mathbb{C}))}_{a_1 \ \ a_n} \times \mathbb{C}^\times_q$

$\mathbb{C}^\times_q$ acts only on base curve

Acts w/ automorphisms on X_n, $\mathbb{C}^\times_{t_1}$ scales the cotangent dir w/ weight t_1. We evaluate parameters $\vec{a}$ on a special locus

Then we take $n \to \infty$ limit

$$K_T(QM(\mathbb{P}^1, X_\infty)) \longleftarrow \bigoplus_{\ell=1}^{\infty} K_{q,t_1}(\mathcal{M}_{ADHM})$$

Highly nontrivial identification of equivariant parameters

2) Quasimaps to Nak quiver varieties

Def: $f: C \dashrightarrow X$ – collection of vector bundles $\mathcal{V}_i$ on C of ranks v_i together w/ a section of the bundle

$$f \in H^0(C, \mathcal{M} \oplus \mathcal{M}^* \otimes \hbar)$$

satisfying $\mu = 0$, where $\mathcal{M} = \sum_{i \in I} \mathrm{Hom}(\mathcal{W}_i, \mathcal{V}_i) \oplus \sum_{i,j \in J} Q_{ij} \otimes \mathrm{Hom}(\mathcal{V}_i, \mathcal{V}_j)$

trivial bundle of rank v_i

$\vec{d} \overset{\text{degree of } f}{=} \deg \mathcal{V}_i$

$QM^{\vec{d}}$ – moduli space of QM when we are allowed to vary f and $\mathcal{V}_i$

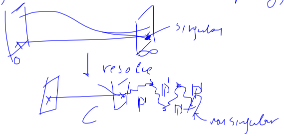
$ev_p \; QM^d \to [X]$

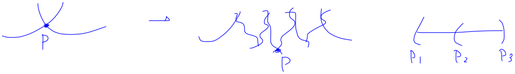
The quotient stack contains X as an open subset corresponding to locus of semistable points

$$X = \mu^{-1}_{ss}(0)/G \subset L(v,w)/G$$

Space is still singular, we need to resolve (compactify)

$QM^d_{\text{relative } p}$





Quantum K-theory of X $\quad F \otimes F' \longrightarrow F \otimes F' + \sum z^d F \circledast_d F'$

Can show that it is an associative unital algebra $\quad$ using gluing operator

Classical K-th of X is generated by tensorial polynomials of tautological bundles $\mathcal{V}_i, \mathcal{W}_i$

eg $T^*Gr_{k,N}$

V – taut bundle

$\tau(V) = V^{\otimes 2} - \Lambda^3 V^*$

$$\tau = (s_1 + \dots + s_k)^2 - \sum_{1 \le i < j < k} \frac{1}{s_i} \frac{1}{s_j} \frac{1}{s_k}$$

Vertex w/ descendent $\tau \in K_T(X)$

$$V^{(\tau)}(X) = \sum_{\vec{d} = \vec{0}}^{\infty} z^{\vec{d}} \, ev_{p_2,*}\left(QM^d_{\text{nonsing } p_2}, \hat{\mathcal{O}}_{vir} \, \tau(\mathcal{V}_i|_{p_1})\right) \in K_{T_q}(X)[[z]]$$

$$\tau(*) = \lim_{q \to 1} \frac{V^{(\tau)}}{V^{(1)}}$$

3) Take $X_n = T^*\mathcal{F}\ell_n$ V_i – tautological bundles on X_n, $i=1,\dots,n$

$$\begin{tikzcd} \mathbb{C} \arrow[r, dash] & \mathbb{C}^2 \arrow[r, dash] & \mathbb{C}^3 \arrow[r, dash] & \bullet \arrow[r, dash] & \mathbb{C}^{n-1} \arrow[d, dash] \\ &&&& \boxed{\mathbb{C}^n} \end{tikzcd}$$

$\hat{Y}_i$ – operator of quantum multiplication by $\Lambda^i V_i(z) \otimes \Lambda^{i-1} V^*_{i-1}(z)$

$T_r = e_r(\hat{Y}_1 \dots \hat{Y}_n)$ – Macdonald operators

Theorem 1 $V^{(1)} = \sum_p V_p^{(1)} \alpha_p$ ← basis of fixed points of the action of maximal torus

Can be computed using localization (log formula) (see paper)

write an example for $T^*\mathbb{P}^1$

Theorem 2 $T_r V_p^{(1)} = e_r(a) V_p^{(1)}$

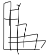
vertex function is an eigenfunction of Macdonald (trig Ruijsenaars Schneider) hamilt.

T_r can be realized as difference operator on quantum parameters

$$T_1 = \sum_{j \neq i} \frac{\hbar z_i - z_j}{z_i - z_j} p_i, \qquad \hat{p}_i z_j = q^{\delta_{ij}} z_j p_i$$

Thus K-theoretic vertex functions are Macdonald series in z

They are of some hypergeom type and can truncate

Theorem 3 λ 

$\lambda_1 \geq \dots \geq \lambda_n$

$a_i = a q^{\lambda_i} \hbar^{i-n}$, $i=1,\dots,n$

then $\exists$ p – fixed point (and only one)

s/t $V_p^{(1)} = P_\lambda(z_1 \dots z_n | q, \hbar)$ ← symmetric Macdonald polynomial for λ

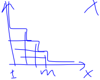
Ex for $T^*\mathbb{P}^1$

(from the paper)

4) ADHM P_λ – classes of fixed points of equivariant K theory of $Hilb^k(\mathbb{C}^2)$

$J \subset \mathbb{C}[x,y]$ – ideal sheaves of colength k

$\dim_{\mathbb{C}} \mathbb{C}[x,y]/J = k \longleftarrow$



λ – partition of k

generated by monomials $x^{\lambda_1}, x^{\lambda_2}y \; x^{\lambda_m}y^{m} \; y^{m+1}$

$\mathbb{C}^\times_q \times \mathbb{C}^\times_\hbar \quad (x,y) \mapsto (qx, \hbar y)$

$K_{q,\hbar}(Hilb^k(\mathbb{C}^2)) = \mathbb{C}[x_1 \; x_n] \otimes \mathbb{C}[q,\hbar]$

$[J] \leftrightarrow P_\lambda$

Main theorem $n > k$ there is an embedding

1) $\bigoplus_{\ell=0}^{k} K_{q,\hbar}(Hilb^\ell(\mathbb{C}^2)) \hookrightarrow K_T(QM(\mathbb{P}^1, X_n))$

$[\lambda] \longmapsto V_q \Big|_{a_i = a q^{\lambda_i} \hbar^{1-i}}$

2) $\bigoplus_{\ell=0}^{\infty} K_{q,\hbar}(Hilb^\ell(\mathbb{C}^2)) \hookrightarrow K_T(QM(\mathbb{P}^1, X_\infty))$

stable limit

Spectra (Lemma 3.4)

Theorem $\lim_{n\to\infty} e_r(\vec{a})\Big| = \mathcal{E}_1(\lambda) = a\Big(1 - (1-\hbar)(1-q)\sum_{(i,j)\in\lambda} s_i\Big|$