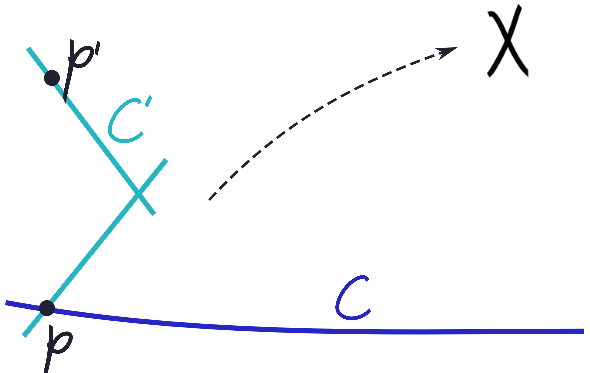
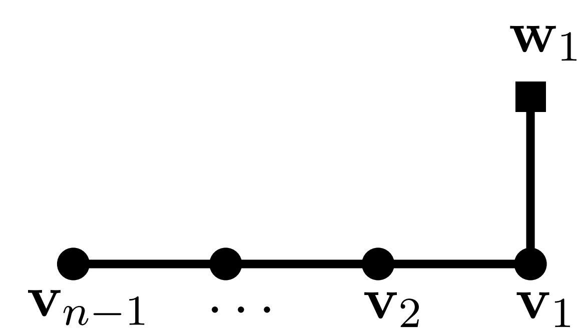
Quantum K-theory of Quiver Varieties and Integrability

Peter Koroteev





Talk at Algebra Seminar
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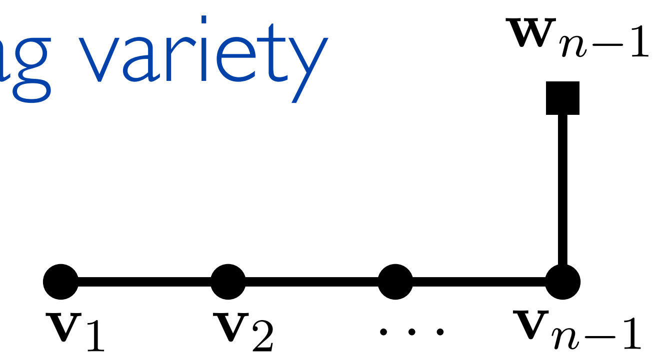
Main Theorem I

[PK Pushkar Smirnov Zeitlin]

Consider the cotangent bundle to the complete **n**-flag variety



Then its quantum equivariant K-theory is given by

$$QK_T(T^*\mathbb{F}l_n) = \frac{\mathbb{C}[\zeta_1^{\pm 1}, \dots, \zeta_n^{\pm 1}; a_1^{\pm 1}, \dots, a_n^{\pm 1}, \hbar^{\pm 1}; p_1^{\pm 1}, \dots, p_n^{\pm 1}]}{(H_r(\zeta_i, p_i, \hbar) - e_r(a_1, \dots, a_n))}$$

relations — integrals of motion of **trigonometric Ruijsenaars-Schneider** model

$$H_r = \sum_{\substack{\mathcal{I}\subset\{1,\dots,n\} \\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I} \\ j\notin\mathcal{I}}} \frac{\zeta_i\,\hbar^{-1/2} - \zeta_j\,\hbar^{1/2}}{\zeta_i - \zeta_j} \prod_{k\in\mathcal{I}} p_k$$

Maximal torus $\quad T = \mathbb{C}^\times_\hbar \times \mathbb{T}(U(n))$

Q: Have we seen something like this earlier?

Motivation

String theory have been suggesting for a long time that there is a strong connection between **geometry** and **integrability**

Study of ***Gromov-Witten*** invariants was influenced by progress in string theory. For a symplectic manifold $\mathcal{X}$ GW invariants appear in the expansion of quantum multiplication in ***quantum cohomology*** of $\mathcal{X}$.

A particular attention is given to genus zero GW invariants.

In this talk we shall study **equivariant quantum K-theory** of large family of symplectic varieties and its connection to **integrable systems**

Physics Motivation

To see why integrability is relevant one considers supersymmetric sigma model from an algebraic curve (P1 in our case) into $\mathcal{X}$

Witten demonstrated that relevant class of supersymmetric sigma models can be rewritten as supersymmetric gauge theories (**(2,2) GLSMs**) in two dimensions whose field content is related to geometry of $\mathcal{X}$. Sigma models thus describe infrared dynamics of GLSMs.

Nekrasov and Shatashvili showed how to obtain integrable systems from such GLSMs. It was conjectured that SUSY vacua of **2d** theories compute **quantum cohomology** ring of $\mathcal{X}$, while **3d** theories on $\mathbb{R}^2 \times S^1$ describe **quantum K-theory.**

Quantum groups

Let $\mathfrak{g}$ Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g}(t)$ loop algebra (Laurent poly valued in g)

Evaluation modules form a tensor category of $\hat{\mathfrak{g}}$

$$V_1(a_1) \otimes \cdots \otimes V_n(a_n)$$

V_i are representations of $\mathfrak{g}$

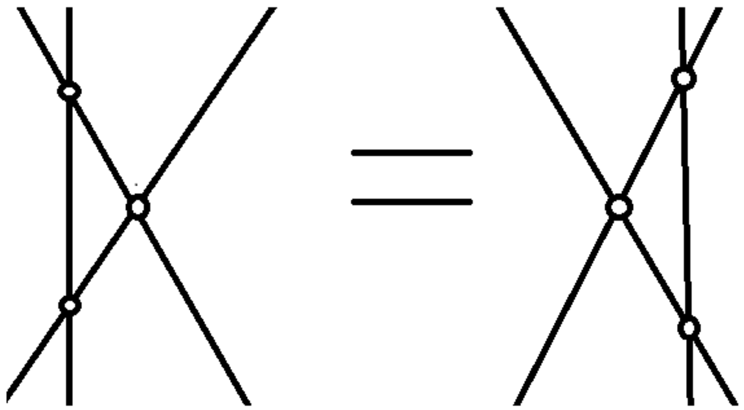
a_i are special values of spectral parameter t

Quantum group is a noncommutative deformation $U_\hbar(\hat{\mathfrak{g}})$

with a nontrivial intertwiner — R-matrix

$$R_{V_1,V_2}(a_1/a_2) : V_1(a_1) \otimes V_2(a_2) \to V_2(a_2) \otimes V_1(a_1)$$

satisfying Yang-Baxter equation

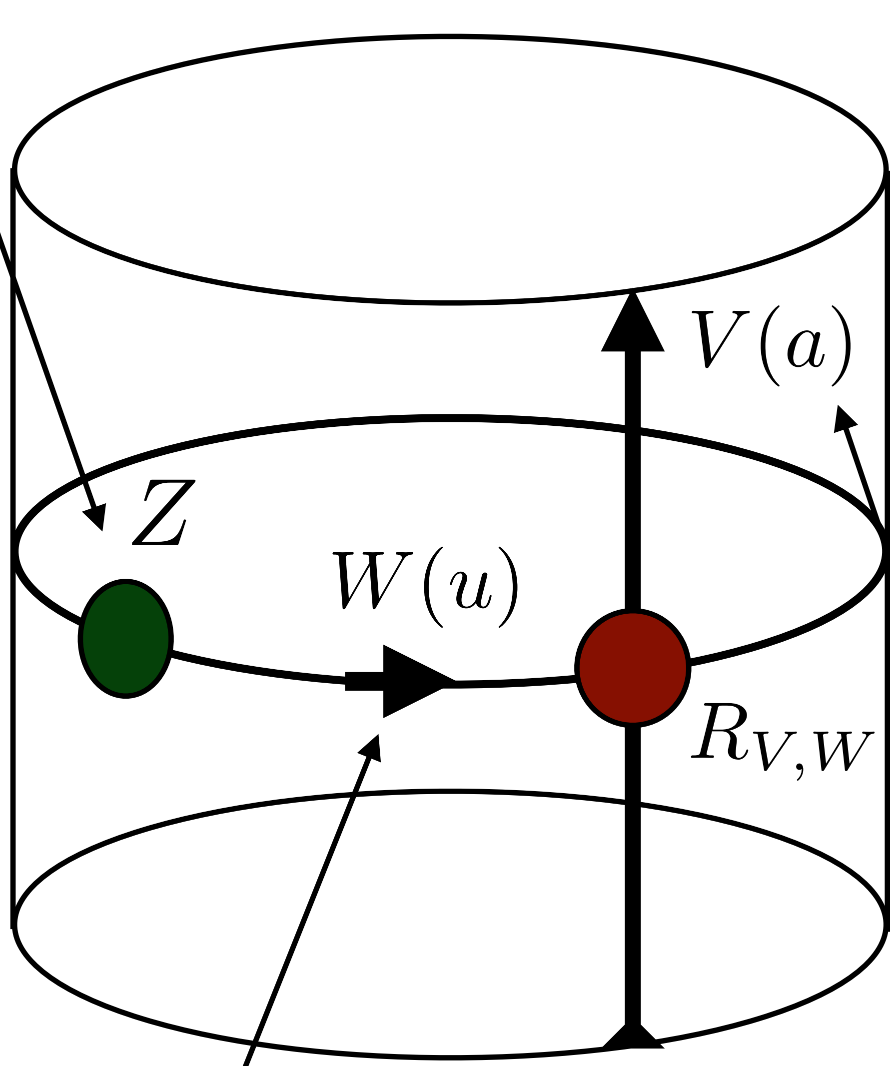


Quantum Integrability

[Faddeev Reshetikhin Tachtajan]

The intertwiner represents an interaction vertex in integrable models. The quantum group is generated by matrix elements of R

twist $Z \in e^{\mathfrak{h}}$



auxiliary space

physical space

Integrability comes from transfer matrix

$$T_W(u) = \mathrm{Tr}_{W(u)}\left((Z \otimes 1) T_{V,W}\right)$$

$$[T_W(u), T_W(u')] = 0$$

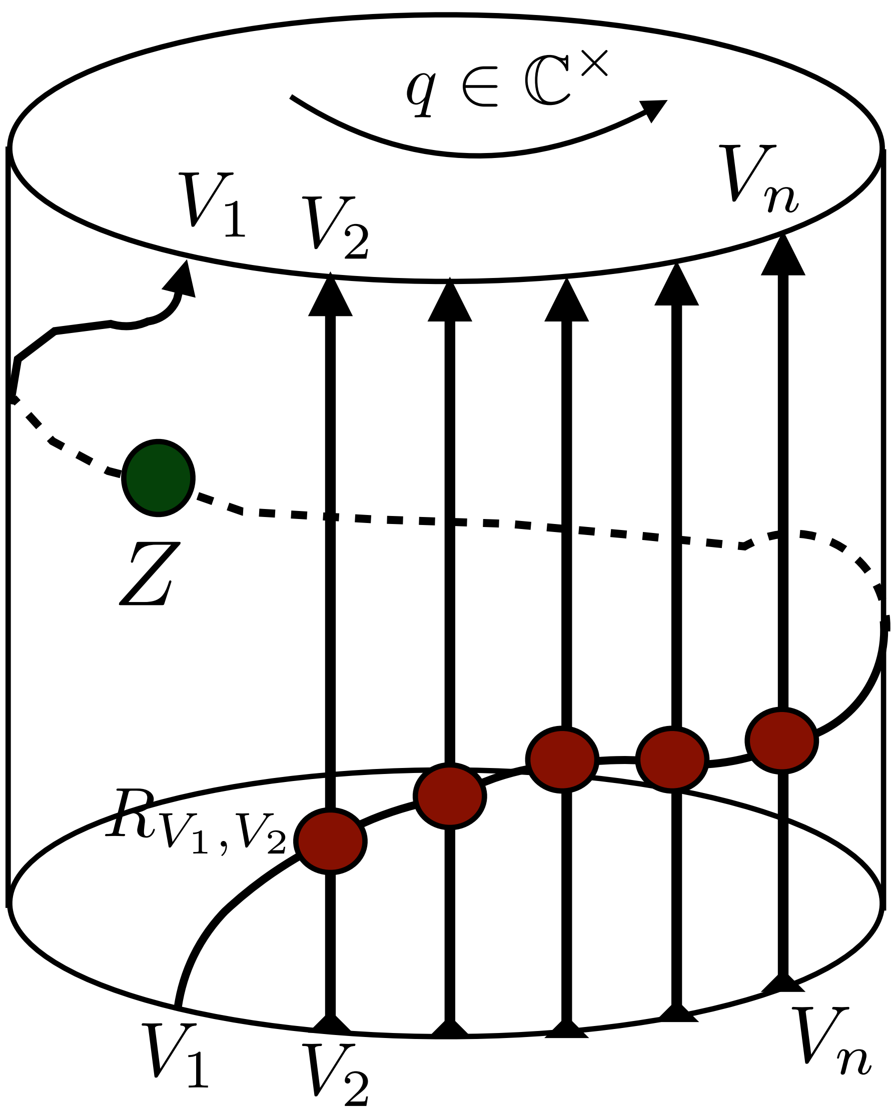
Transfer matrices are usually polynomials in u whose coefficients are the **integrals of motion**

XXZ Spin Chain

$\mathfrak{g} = \mathfrak{sl}_2$ spin-1/2 chain on n sites $V = \mathbb{C}^2(a_1) \otimes \cdots \otimes \mathbb{C}^2(a_n)$

Spectrum can be found using Bethe Ansatz techniques. However, if we want to understand the problem for more general algebras we need to think of the Knizhnik-Zamolodchikov difference equation (qKZ)

$$\Psi(qa_1, \ldots a_n) = (Z \otimes 1 \otimes \cdots \otimes 1) R_{V_1,V_n} \cdots R_{V_1,V_2} \Psi(a_1, \ldots a_n)$$



where

$$\Psi(a_1, \ldots, a_n) \in V_1(a_1) \otimes \cdots \otimes V_n(a_n)$$

[I. Frenkel Reshetikhin]

In the limit $q \to 1$
qKZ becomes an eigenvalue problem

Solutions of qKZ

[Aganagic Okounkov]

Schematic solution

$$\Psi_\alpha = \int \frac{d\mathbf{x}}{\mathbf{x}} f_\alpha(\mathbf{x}, a)\, \mathcal{K}(\mathbf{x}, z, a, q)$$

indexed by physical space

representation

universal kernel

$$\log \mathcal{K}(\mathbf{x}, z, a, q) \underset{q \to 1}{\sim} \frac{S(\mathbf{x}, z, a)}{\log q}$$

$$\frac{\partial S}{\partial x_i} = 0$$ Bethe equations for Bethe roots **x**

$$a_i \frac{\partial S}{\partial a_i} = \Lambda_i$$ Eigenvalues of qKZ operators

The map $\alpha \mapsto f_\alpha(\mathbf{x}^*)$ provides diagonalization

So we need to find `off shell' Bethe eigenfunctions $f_\alpha(\mathbf{x}, a)$

Nekrasov-Shatashvili correspondence

The answer will come from enumerative AG inspired by physics

Hilbert space of states of quantum integrable system $\longleftrightarrow$ Equivariant K-theory of Nakajima quiver varitey (line operators in 3d SUSY gauge theory)

gauge group $G = \prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}} U(v_i)$ (v_1,v_2,...) encode weight of rep α

Bethe roots **x** live in maximal torus of G, by integrating over **x** we project on Weyl invariant functions of Bethe roots

Flavor group $G_F = \prod_i U(w_i)$ whose maximal torus gives parameters **a**

Bifundamental matter $\mathrm{Hom}(V_i, V_j)$

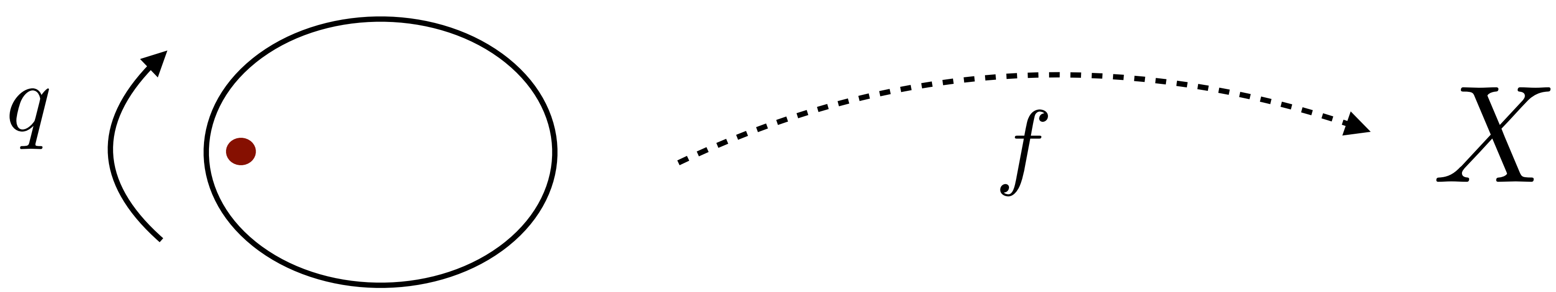
Nekrasov-Shatashvili correspondence

The quiver variety   X = {Matter fields}/gauge group

X is a module of some quantum group in Nakajima correspondence construction

We will be computing integrals in K-theory of the space of quasimaps $f : \mathcal{C} - -- > X$ weighted by degree $\mathbf{z}^{\deg f}$ subject to equivariant action on the base nodal curve $\mathbb{C}_q^\times$

(cf Gromov-Witten invariants)

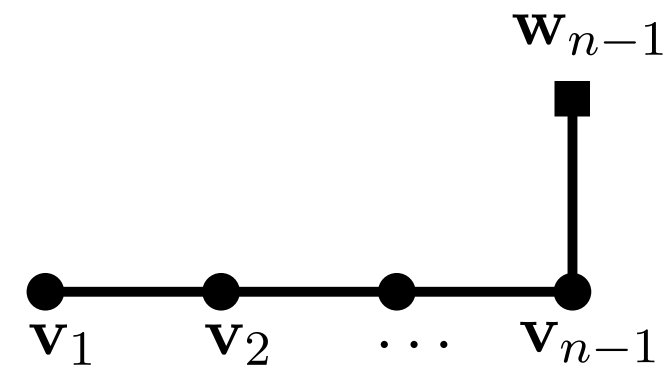


In particular we shall study quantum K-theory ring with quantum parameters **z** whose structure constants arise from 3 point correlators

Nakajima Quiver Varieties

Rep(**v**,**w**) — linear space of quiver reps

$\mu : T^*\mathrm{Rep}(\mathbf{v}, \mathbf{w}) \to \mathrm{Lie}(G)^*$ moment map



Nakajima quiver variety $X = \mu^{-1}(0)//_\theta G = \mu^{-1}(0)_{ss}/G$ $G = \prod GL(V_i)$

Automorphism group $\mathrm{Aut}(X) = \prod GL(Q_{ij}) \times \prod GL(W_i) \times \mathbb{C}_\hbar^\times$

Maximal torus $T = \mathbb{T}(\mathrm{Aut}(X))$

Tensorial polynomials of tautological bundles V_i, W_i and their duals generate *classical T-equivariant K-theory* ring of X

Ex: T*Gr(k,n)

$$\tau(V) = V^{\otimes 2} - \Lambda^3 V^*$$

$$\mathbf{v}_1 = k, \ \mathbf{w}_1 = n$$

$$\tau(s_1, \cdots, s_k) = (s_1 + \cdots + s_k)^2 - \sum_{1 \le i_1 < i_2 < i_3 \le k} s_{i_1}^{-1} s_{i_2}^{-1} s_{i_3}^{-1}$$

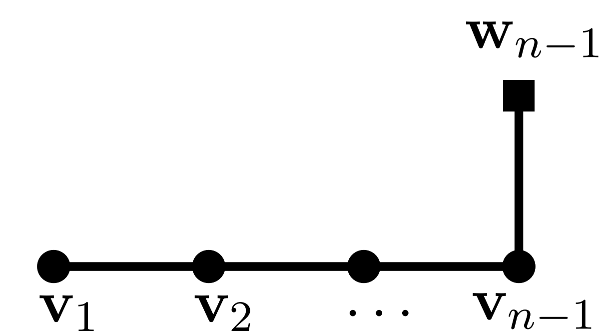
value of a quasimap defines a map to a quotient stack which contains stable locus as an open subset

Quasimaps

[Ciocan-Fontanine, Kim, Maulik]
[Okounkov]

Quasimap $f : \mathcal{C} - - \to X$ is described by collection of vector bundles $\mathscr{V}_i$ on $\mathcal{C}$ of ranks $\mathbf{v}_i$ with section $f \in H^0(\mathcal{C}, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar)$ satisfying $\mu = 0$

where $\mathscr{M} = \sum_{i \in I} Hom(\mathscr{W}_i, \mathscr{V}_i) \oplus \sum_{i,j \in I} Q_{ij} \otimes Hom(\mathscr{V}_i, \mathscr{V}_j)$



d_i degrees of $\mathscr{V}_i$.

Evaluation map to quotient stack

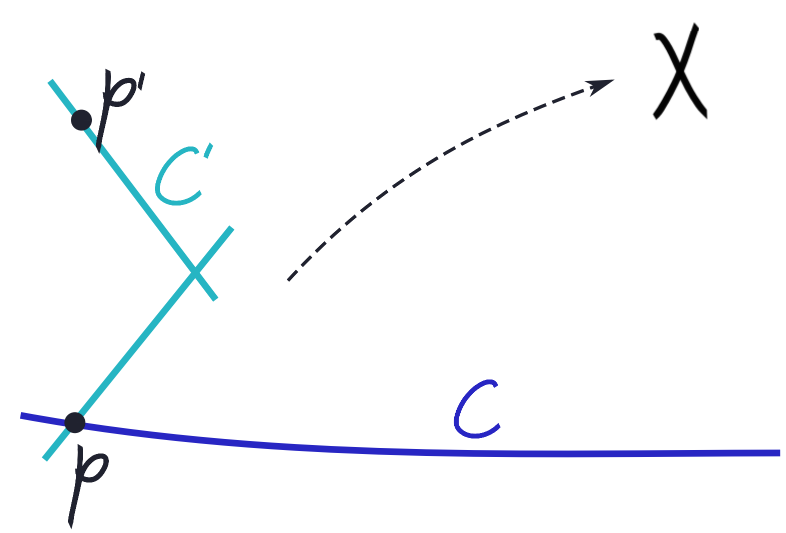
$$\mathrm{ev}_p : \mathbf{QM}^{\mathbf{d}} \to \mu^{-1}(0)/G$$
$$p \mapsto f(p)$$

Resolve to make ***proper*** ev map

$$\begin{tikzcd} & \mathrm{QM}^{\mathbf{d}}{}_{\text{relative } p} \arrow[dr, "\widetilde{\mathrm{ev}}_p"] & \\ \mathrm{QM}^{\mathbf{d}}{}_{\text{nonsing } p} \arrow[ur, hook] \arrow[rr, "\mathrm{ev}_p"] & & X \end{tikzcd}$$

QM is nonsingular if $f(p) \in X$

for all but finitely many singular points



Virtual Sheaves

Deformation-obstruction theory allows one to construct virtual tangent bundle and virtual structure sheaf

[Ciocan-Fontanine, Kim, Maulik]

Fiber of the reduced virtual tangent bundle to $\mathsf{QM}^{\mathbf{d}}{}_{\mathrm{nonsing}\,p}$

$$T^{\mathrm{vir}}_{(\{\mathscr{V}_i\},\{\mathscr{W}_i\})}\mathsf{QM}^{\mathbf{d}}_{\text{nonsing p}} = H^\bullet(\mathscr{M} \oplus \hbar\,\mathscr{M}^*) - (1+\hbar)\bigoplus_i Ext^\bullet(\mathscr{V}_i, \mathscr{V}_i).$$

moment map, deformations
C* factorizations in GIT

Symmetrized virtual structure sheaf

(possible to do for quiver varieties)

$$\hat{\mathcal{O}}_{\mathrm{vir}} = \mathcal{O}_{\mathrm{vir}} \otimes \mathscr{K}_{\mathrm{vir}}^{1/2} q^{\deg(\mathscr{P})/2}$$

virtual canonical bundle

polarization bundle

Standard bilinear form on K-theory

(twisting by root of K will be important)

$$(\mathcal{F}, \mathcal{G}) = \chi(\mathcal{F} \otimes \mathcal{G} \otimes K^{-1/2})$$

canonical class

Vertex Function

[Okounkov]
[Pushkar Smirnov Zeitlin]

Spaces of quasimaps admit an action of an extra torus $\mathbb{C}_q^\times$ which scales the base $\mathbb{P}^1$ keeping two fixed points (0, infinity)

Define **vertex function** with quantum (Novikov) parameters $z^{\mathbf{d}} = \prod_{i\in I} z_i^{d_i}$

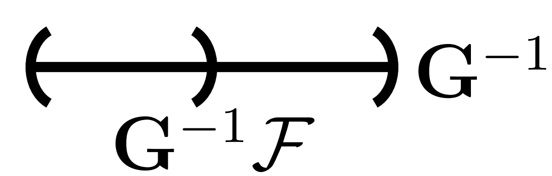
$$V^{(\tau)}(z) = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_2,*}\Big(\mathsf{QM}^{\mathbf{d}}_{\mathrm{nonsing}\,p_2}, \widehat{\mathcal{O}}_{\mathrm{vir}}\tau(\mathscr{V}_i|_{p_1})\Big) \in K_{\mathsf{T}_q}(X)_{loc}[[z]]$$

descendent

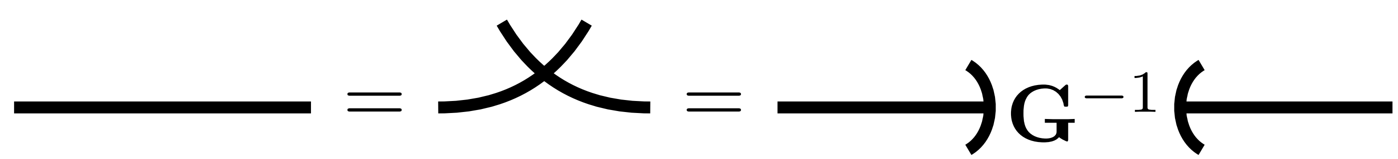
Define **quantum K-theory** as a ring with multiplication

$$A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$$

$$\mathcal{F}\circledast = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_1,p_3*}\Big(\mathsf{QM}^{\mathbf{d}}_{p_1,p_2,p_3}, \mathrm{ev}^*_{p_2}(\mathbf{G}^{-1}\mathcal{F})\widehat{\mathcal{O}}_{\mathrm{vir}}\Big)\,\mathbf{G}^{-1}$$



gluing $\quad \mathcal{C}_0 = \mathcal{C}_{0,1} \cup_p \mathcal{C}_{0,2}$



Theorem: QK(X) is a commutative associative unital algebra

Vertex computation for T*Fln

At a given fixed point of extended maximal torus tangent space has

$$\mathcal{M} = \left(\mathcal{O}(d) \otimes q^{-d}\right) \oplus \left(\mathcal{O}(d) \otimes q^{-d} \otimes \frac{a_i}{a_j}\right)$$

character $\quad H^\bullet\left(\mathcal{O}(d) \otimes q^{-d} \otimes \frac{a_i}{a_j}\right) = \frac{a_i}{a_j}\left(1 + q^{-1} + \ldots q^{-d}\right)$ similar to rest

Overall the contribution of $xq^{-d}\mathcal{O}(d)$ to the character is

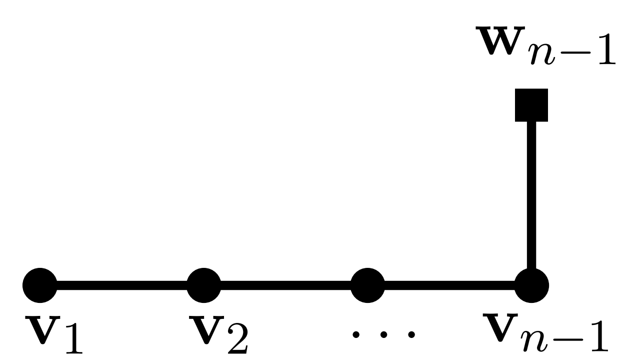
$$\{x\}_d = \frac{(\hbar/x, q)_d}{(q/x, q)_d}\,(-q^{1/2}\hbar^{-1/2})^d, \quad \text{where} \quad (x,q)_d = \frac{\varphi(x)}{\varphi(q^d x)} \qquad \varphi(x) = \prod_{i=0}^{\infty}(1 - q^i x)$$

Vertex $\quad V_{\mathbf{p}}^{(\tau)}(z) = \sum_{\mathbf{d}\in\mathbb{Z}^n_{\geq 0}} \sum_{(\mathscr{V},\mathscr{W})\in(\mathsf{QM}^{\mathbf{d}}_{\text{nonsing}\, p_2})^{\mathsf{T}}} \hat{s}(\chi(\mathbf{d}))\, z^{\mathbf{d}} q^{\deg(\mathscr{P})/2} \tau(\mathscr{V}|_{p_1}).$

fixed point **p** contributes

$$V_{\boldsymbol{p}}^{(\tau)}(z) = \sum_{d_{i,j}\in C} z^{\mathbf{d}} q^{N(\mathbf{d})/2}\, EHG \quad \tau(x_{i,j} q^{-d_{i,j}})$$

$$E = \prod_{i=1}^{n-1} \prod_{j,k=1}^{\mathbf{v}_i} \{x_{i,j}/x_{i,k}\}^{-1}_{d_{i,j}-d_{i,k}}$$



Example for T*P$_1$

$$\mathbf{v}_1 = 1,\, \mathbf{w}_1 = 2$$

Vertex with trivial insertion

$$V_{\mathbf{p}}^{(1)} = \sum_{d>0} z^d \prod_{i=1}^{2} \frac{\left(\frac{q}{\hbar}\frac{a_{\mathbf{p}}}{a_i};q\right)_d}{\left(\frac{a_{\mathbf{p}}}{a_i};q\right)_d} = {}_2\phi_1\left(\hbar, \hbar\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}, q\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}; q; \frac{q}{\hbar}z\right) .$$

two fixed points

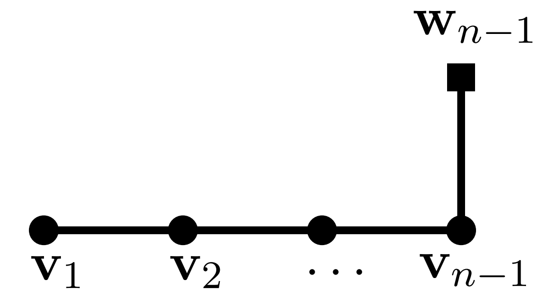
$\mathbf{p} = \{a_1\}$ and $\mathbf{p} = \{a_2\}$.

As a contour integral

$$V = \frac{e^{-\frac{\log z \cdot \log a_1 a_2}{\log q}}}{2\pi i} \int_C \frac{ds}{s}\, e^{\frac{\log z \cdot \log s}{\log q}} \frac{\varphi\left(\hbar \frac{s}{a_1}\right)}{\varphi\left(\frac{s}{a_1}\right)} \frac{\varphi\left(\hbar \frac{s}{a_2}\right)}{\varphi\left(\frac{s}{a_2}\right)}$$

Bethe Equations

[PK Pushkar Smirnov Zeitlin]



Saddle point approximation provides the operator of quantum multiplication

$$\tau_{\boldsymbol{p}}(z) = \lim_{q\to 1} \frac{V_{\boldsymbol{p}}^{(\tau)}(z)}{V_{\boldsymbol{p}}^{(1)}(z)}$$

For the cotangent bundle to partial flag variety we get

Theorem 3.4. *The eigenvalues of $\hat{\tau}(z)\circledast$ is given by $\tau(s_{i,k})$, where $s_{i,k}$ satify Bethe equations:*

$$\prod_{j=1}^{\mathbf{v}_2} \frac{s_{1,k}-s_{2,j}}{s_{1,k}-\hbar s_{2,j}} = z_1\left(-\hbar^{1/2}\right)^{-\mathbf{v}_1'} \prod_{\substack{j=1\\ j\neq k}}^{\mathbf{v}_1} \frac{s_{1,j}-s_{1,k}\hbar}{s_{1,j}\hbar-s_{1,k}}\,,$$

$$\prod_{j=1}^{\mathbf{v}_{i+1}} \frac{s_{i,k}-s_{i+1,j}}{s_{i,k}-\hbar s_{i+1,j}} \prod_{j=1}^{\mathbf{v}_{i-1}} \frac{s_{i-1,j}-\hbar s_{i,k}}{s_{i-1,j}-s_{i,k}} = z_i\left(-\hbar^{1/2}\right)^{-\mathbf{v}_i'} \prod_{\substack{j=1\\ j\neq k}}^{\mathbf{v}_i} \frac{s_{i,j}-s_{i,k}\hbar}{s_{i,j}\hbar-s_{i,k}}\,, \tag{23}$$

$$\prod_{j=1}^{\mathbf{w}_{n-1}} \frac{s_{n-1,k}-a_j}{s_{n-1,k}-\hbar a_j} \prod_{j=1}^{\mathbf{v}_{n-2}} \frac{s_{n-2,j}-\hbar s_{n-1,k}}{s_{n-2,j}-s_{n-1,k}} = z_{n-1}\left(-\hbar^{1/2}\right)^{-\mathbf{v}_{n-1}'} \prod_{\substack{j=1\\ j\neq k}}^{\mathbf{v}_{n-1}} \frac{s_{n-1,j}-s_{n-1,k}\hbar}{s_{n-1,j}\hbar-s_{n-1,k}}\,,$$

where $k=1,\dots,\boldsymbol{v}_i$ for $i=1,\dots,\boldsymbol{v}_{n-1}$.

which are Bethe Ansatz Equations for gl(n) XXZ spin chain

Bethe Equations for T*Gr(k,n)

[Pushkar Smirnov Zeitlin]

T*Gr(k,n) $\quad \mathbf{v}_1 = k,\ \mathbf{w}_1 = n$

Theorem 2. *The eigenvalues of operators of quantum multiplication by $\hat{\tau}(z)$ are given by the values of the corresponding Laurent polynomials $\tau(s_1, \cdots, s_k)$ evaluated at the solutions of the following equations:*

$$\prod_{j=1}^{n} \frac{s_i - a_j}{\hbar a_j - s_i} = z\, \hbar^{-n/2} \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{s_i \hbar - s_j}{s_i - s_j \hbar}, \quad i = 1 \cdots k.$$

twisting by K^1/2

In the limit z —> 0 obtain classical relations

$$\prod_{j=1}^{n} (s_i - a_j) = 0, \quad i = 1 \cdots k$$

K-theory and Many-Body systems

Now we would like to connect quantum K-theory of X with *integrable many-body systems*

Consider vertex for T*Fln with trivial insertion V(**z**,**a**,h,q)

Theorem 1 [PK]:

Given integrals of motion of trigonometric Ruijsenaars-Schneider model

$$T_r(\vec{\zeta}) = \sum_{\substack{\mathcal{I}\subset\{1,\ldots,n\}\\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I}\\ j\notin\mathcal{I}}} \frac{\hbar\,\zeta_i - \zeta_j}{\zeta_i - \zeta_j} \prod_{i\in\mathcal{I}} p_k \qquad p_k f(\zeta_k) = f(q\zeta_k) \qquad z_i = \zeta_{i+1}/\zeta_i$$

then vertex is their mutual eigenfunction $T_r(\vec{\zeta})\mathrm{V}^{(1)}_{\boldsymbol{p}} = e_r(\mathbf{a})\mathrm{V}^{(1)}_{\boldsymbol{p}}\,, \qquad r = 1,\ldots,n$

Theorem 2 [PK Zeitlin]:

Given integrals of motion of dual trigonometric Ruijsenaars-Schneider model

$$T_r(\mathbf{a}) = \sum_{\substack{\mathcal{I}\subset\{1,\ldots,n\}\\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I}\\ j\notin\mathcal{I}}} \frac{t\,a_i - a_j}{a_i - a_j} \prod_{i\in\mathcal{I}} p_k \qquad p_k f(a_k) = f(qa_k) \qquad t = \frac{q}{\hbar}$$

then vertex is their mutual eigenfunction

$$T_r(\boldsymbol{a})\mathrm{V}(\boldsymbol{a},\vec{\zeta}) = S_r(\vec{\zeta},t)\mathrm{V}(\boldsymbol{a},\vec{\zeta})\,, \qquad r = 1,\ldots,\mathbf{w_{n-1}}$$

qKZ/tRS and Stable Basis

[PK Zeitlin]

[Zabrodin Zotov]

Solution $\Phi = \sum_J \Phi_J E_J$ of qKZ equation $p_i \Phi = K_i^{(q)} \Phi$,

$$K_i^{(q)} = R_{i\,i-1}\left(\frac{a_i q}{a_{i-1}}\right) \dots R_{i\,1}\left(\frac{a_i q}{a_1}\right) Z^{(i)} R_{i\,\mathbf{w_{n-1}}}\left(\frac{a_i}{a_{\mathbf{w_{n-1}}}}\right) \dots R_{i\,i+1}\left(\frac{a_i}{a_{i+1}}\right)$$

Claim: $V = \sum_J \hbar^{\frac{\ell(J)}{2}} \Phi_J$

Theorem:

Vertex function is given by linear combination of vertices with insertions

$$\mathrm{V}(\boldsymbol{a}, \vec{\zeta}) = c \cdot \Theta \cdot \sum_{\mathbf{q}} \hbar^{\frac{\ell(\mathbf{q})}{2}} V^{((Stab_+^{-1}\ f)_{\mathbf{q}})}$$

$f_\alpha \in \mathbb{Q}(T \times T_G / W_G)$

R matrix $R(a) = \mathrm{Stab}_-^{-1} \mathrm{Stab}_+$

Example (T*P1)

$$T_1 \mathrm{V} = \zeta_2 \frac{e^{\frac{\log \zeta_2 \log a_1 a_2}{\log q}}}{2\pi i} \int_C \frac{ds}{s} \left[\frac{t a_1 - a_2}{a_1 - a_2} \frac{\hbar a_1 - s}{a_1 - s} + \frac{t a_2 - a_1}{a_2 - a_1} \frac{\hbar a_2 - s}{a_2 - s} \right] \left(\frac{\zeta_1}{\zeta_2}\right)^{-\frac{\log s}{\log q}} \prod_{i=1}^{2} \frac{\varphi\left(\frac{q}{\hbar} \frac{s}{a_i}\right)}{\varphi\left(\frac{s}{a_i}\right)}$$

Stabs

$$= (\zeta_1 + \zeta_2)\mathrm{V}$$

Baxter Operator

Consider quantum tautological bundles $\widehat{\Lambda^k V_i}(z),\ k=1,\dots,\mathbf{v}_i$

and their generating function —

Baxter Q-operator

$$\mathbf{Q}_i(u)=\sum_{k=0}^{\mathbf{v}_i}(-1)^k u^{\mathbf{v}_i-k}\hbar^{\frac{ik}{2}}\widehat{\Lambda^k V_i}(z)$$

Proposition: The eigenvalue of quantum multiplication by **Qi** is

$$Q_i(u)=\prod_{k=1}^{\mathbf{v_i}}(u-\hbar^{\frac{i}{2}}s_{i,k})$$

Using results from integrability we can write XXZ Bethe equations in term of polynomials

$$Q_i(u)=\prod_{\alpha=1}^{\mathbf{v}_i}(u-\sigma_{i,\alpha})\ , \qquad P(u)=Q_n(u)=\prod_{a=1}^{\mathbf{w}_{n-1}}(u-\alpha_a)$$

Theorem [PK]: Given Lax matrix of tRS model

$$L_{ij}=\frac{\prod\limits_{k\neq j}^{n}\left(\hbar^{-1/2}\zeta_i-\hbar^{1/2}\zeta_k\right)}{\prod\limits_{k\neq i}^{n}(\zeta_i-\zeta_k)}p_j \qquad p_j=-\frac{\mathbf{Q}_j(0)}{\mathbf{Q}_{j-1}(0)}=\hbar^{j-\frac{1}{2}}\widehat{\Lambda^j V_j}(z)\circledast\widehat{\Lambda^{j-1}V^*}_{j-1}(z)$$

$$p_i=\frac{s_{i+1,1}\cdot\dots\cdot s_{i+1,i+1}}{s_{i,1}\cdot\dots\cdot s_{i,i}},\qquad i=1,\dots,n-1$$

we get $P(u)=det\Big(u-L\Big)$

Example for T*P1

Vertex

$$V = \frac{e^{\frac{\log \zeta_2 \cdot \log a_1 \cdots a_n}{\log q}}}{2\pi i} \int_C \frac{ds}{s}\, e^{\frac{\log \zeta_1/\zeta_2 \cdot \log s}{\log q}} \frac{\varphi\left(\hbar \frac{s}{a_1}\right)}{\varphi\left(\frac{s}{a_1}\right)} \frac{\varphi\left(\hbar \frac{s}{a_2}\right)}{\varphi\left(\frac{s}{a_2}\right)}$$

tRS Hamiltonians

$$T_1(\vec{\zeta}) = \frac{\hbar\zeta_1 - \zeta_2}{\zeta_1 - \zeta_2} p_1 + \frac{\hbar\zeta_2 - \zeta_1}{\zeta_2 - \zeta_1} p_2$$

$$T_2(\vec{\zeta}) = p_1 p_2 \,.$$

tRS class

Energy equations

$$T_1(\vec{\zeta})\mathrm{V} = \mathrm{V}^{(T_1(s))} = (a_1 + a_2)\mathrm{V}$$

$$T_2(\vec{\zeta})\mathrm{V} = a_1 a_2 \mathrm{V}\,,$$

K theory via tRS

Classical limit q —>1 implies

$$QK_T(T^*\mathbb{F}l_n) = \frac{\mathbb{C}[\zeta_1^{\pm 1}, \dots, \zeta_n^{\pm 1}; a_1^{\pm 1}, \dots, a_n^{\pm 1}, \hbar^{\pm 1}; p_1^{\pm 1}, \dots, p_n^{\pm 1}]}{(H_r(\zeta_i, p_i, \hbar) - e_r(a_1, \dots, a_n))}$$

where the ideal is generated by energy equations of all Hamiltonians of **tRS model**

$\zeta_1, \dots, \zeta_n$ are coordinates $p_1, \dots, p_n$ are momenta

symplectic form $\Omega = \sum_{i=1}^{n} \frac{dp_i}{p_i} \wedge \frac{d\zeta_i}{\zeta_i}$

Momenta can be determined from derivatives of Yang-Yang function XXZ for Bethe equations. They define Lagrangian $\mathcal{L} \subset T^* \left(\mathbb{C}^\times\right)^n$ whose generating function is given by the Yang-Yang function.

[Gaiotto PK]
[Bullimore Kim PK]

tRS/XXZ duality

Compact limit

Equivariant push-forward $$V_{\mathbf{p}}^{(\tau)}(z) = \sum_{\mathbf{d}\in\mathbb{Z}^n_{\geq 0}} \sum_{(\mathscr{V},\mathscr{W})\in(\mathsf{QM}^{\mathbf{d}}_{\mathrm{nonsing}\, p_2})^{\mathsf{T}}} \hat{s}(\chi(\mathbf{d}))\, z^{\mathbf{d}} q^{\deg(\mathscr{P})/2} \tau(\mathscr{V}|_{p_1}).$$

s-roof class $$\hat{s}(x) = \frac{1}{x^{1/2} - x^{-1/2}} \qquad \hat{s}(x+y) = \hat{s}(x)\hat{s}(y)$$

Contributions from the base and the fiber in T*G/B split $(\omega, \omega^{-1}\hbar)$

$$\frac{1}{\omega^{1/2} - \omega^{-1/2}} \frac{1}{(\hbar\omega^{-1})^{1/2} - (\hbar\omega^{-1})^{-1/2}} = \frac{1}{1-\omega^{-1}} \frac{-\hbar^{1/2}}{1-\hbar^{-1}\omega^{-1}}$$

After rescaling we can take the limit $\hbar \to \infty$ $$\hat{s}(\omega, \omega^{-1}\hbar) \to \frac{1}{1-\omega^{-1}}$$

Vertex functions

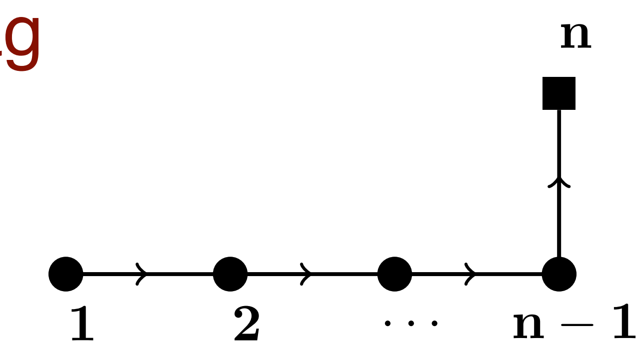
[cf Givental Lee]

$$V_{\mathbf{p}}^{(1)} \to {}_2\phi_1\left(0, 0, \frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}; q; z^\sharp\right) =: {}_1\phi_0\left(\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}; q; z^\sharp\right) = \sum_{k=0}^{\infty} \frac{(z^\sharp)^k}{\left(\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}, q\right)_k (q,q)_k}$$

satisfy q-Toda difference relations

Five-Vertex model and qToda

In the limit we can recover the K-theory ring for complete n-flag



$$QK_{T'}(\mathbb{F}l_n) = \frac{\mathbb{C}[\mathfrak{z}_1^{\pm 1}, \dots, \mathfrak{z}_n^{\pm 1}; \mathfrak{a}_1^{\pm 1}, \dots, \mathfrak{a}_n^{\pm 1}; \mathfrak{p}_1^{\pm 1}, \dots, \mathfrak{p}_n^{\pm 1}]}{\left(H_r^{q\text{-}Toda}(\mathfrak{z}_i, \mathfrak{p}_i) = e_r(\mathfrak{a}_1, \dots, \mathfrak{a}_n)\right)}$$

q-Toda Hamiltonians

$$H_r^{\text{q-Toda}} = \sum_{\substack{\mathcal{I}=\{i_1<\cdots<i_r\} \\ \mathcal{I}\subset\{1,\dots,n\}}} \prod_{\ell=1}^{r} \left(1 - \frac{\mathfrak{z}_{i_\ell - 1}}{\mathfrak{z}_{i_\ell}}\right)^{1-\delta_{i_\ell - i_{\ell-1},1}} \prod_{k\in\mathcal{I}} \mathfrak{p}_k$$

Analogously to **XXZ/tRS** duality we can formulate **5-vert/qToda** duality

Bethe equations
$$\prod_{j=1}^{n}(s_i - \mathfrak{a}_j) = z \prod_{j\neq i} \frac{s_i}{s_j}$$

Geometric q-Langlands

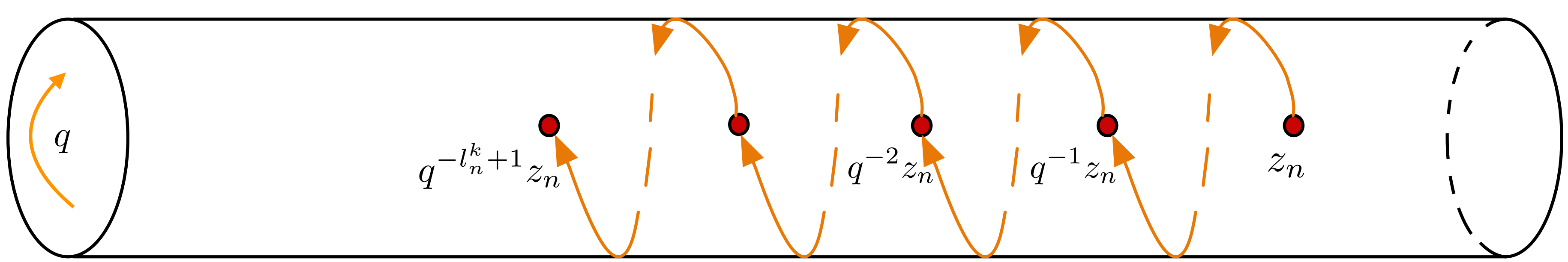
[PK Sage Zeitlin]
[Frenkel PK Sage Zeitlin]

tRS/XXZ duality can be upgraded to classical geometric q-Langlands

(G,q) opers on $\mathbb{P}^1$ ⟷ Solutions of ${}^L G$ -QQ system (XXZ)

Line subbundle $\mathcal{L}_i \longrightarrow \mathcal{W}_i \longrightarrow \mathcal{W}_i/\mathcal{L}_i$

qConnection $A(z) = g(qz) Z g^{-1}(z)$



Locally $$s(qz) \wedge A(z) s(z) = \prod_{m=1}^{L} \prod_{j=0}^{k_m-1} (z - q^{-j} z_m)$$

(SL(2),q)-Opers

qOper with regular singularities $s(z) = \begin{pmatrix} Q_-(z) \\ Q_+(z) \end{pmatrix}$ $Z = \mathrm{diag}(\zeta, \zeta^{-1})$

$$\zeta Q_-(z) Q_+(qz) - \zeta^{-1} Q_-(qz) Q_+(z) = \Lambda(z) := \prod_{m=1}^{L} \prod_{j=0}^{k_m-1} (z - q^{-j} z_m).$$

leads to

$$\prod_{m=1}^{L} \frac{w_i - q^{1-k_m} z_m}{w_i - q z_m} = -\zeta^{-2} q^{l_- - k} \prod_{j=1}^{l_-} \frac{q w_i - w_j}{w_i - q w_j}, \qquad i = 1, \ldots, l_-$$

compare with T*Gr. Let $Q_- = z - p_-$ and $Q_+ = c(z - p_+)$

$$z^2 - \frac{z}{q}\left[\frac{\zeta - q\zeta^{-1}}{\zeta - \zeta^{-1}} p_+ + \underbrace{\frac{q\zeta - \zeta^{-1}}{\zeta - \zeta^{-1}}}_{T_1} p_-\right] + \underbrace{\frac{p_+ p_-}{q}}_{T_2} = (z - z_+)(z - z_-)$$

qOper condition yields tRS Hamiltonians!

$$\det(z - L_{tRS}) = (z - z_+)(z - z_-)$$