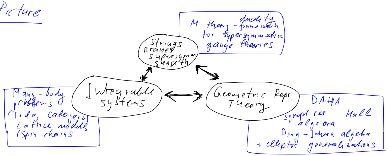
Virginia talk (Math colloquium 09/08)

Geometric Representation theory, Integrable Systems and String theory

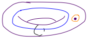
⓪ The Big Picture



We shall illustrate each corner of the triality by looking at one particular example

① Moduli space of flat $GL(n,\mathbb{C})$ connections of the punctured torus $\Sigma = T^2 - pt$

$\mathcal{M} = \mathcal{M}^{flat}_{gl(n)}$



character variety of Σ

Principal $GL(n,\mathbb{C})$-bundle over Σ

Allow only semi-simple monodromy around the puncture. Can find $\mathcal{M}_n$ by studying holonomies around A, B, C cycles which satisfy the fundamental group relation

$$\mathcal{M}_n = \frac{\{A, B, T\} \text{ st } ABA^{-1}B^{-1} = T, \quad T = \mathrm{diag}(t, t, \ldots t, t^{1-N})}{GL(n,\mathbb{C}) \text{ gauge transformations}}$$

Equivalently one can look at manifestly gauge invariant expressions. Ex $sl_2(\mathbb{C})$

$x = \operatorname{Tr} A$
$y = \operatorname{Tr} B$
$z = \operatorname{Tr} AB$

$$\mathbb{C}^3 \supset \mathcal{M}_{sl_2} = \{ \underbrace{xyz + x^2 + y^2 + z^2 + 2 + t + t^{-1}}_{= f(x,y,z)} \}$$

Side note when $t \to 1$ $\quad \mathcal{M}_{t} = \dfrac{\mathbb{C}^* \times \mathbb{C}^*}{\mathbb{Z}_2}$

$\mathcal{M}$ is a hyperkahler manifold and it has $\mathbb{CP}^1$ worth of complex structures usually called (I, J, K). In J the holomorphic symplectic form reads

$$\Omega = \frac{dx \wedge dy}{\partial f/\partial z} = \frac{dx \wedge dy}{xy + 2z}$$

<u>Exersize</u> Show that it vanishes on $f(x,y,z) = 0$

② So what can we do with $\mathcal{M}$?

First, let us notice that it coincides with the phase space of an integrable system (in fact there are 2 different systems depending on the complex structure in which we are looking at)

Write Ω and f in slightly different coordinates. Let us now take generic case gl_n

The flatness constraints can be solved via the following ansatz

$$\operatorname{Tr}_{\wedge^r}(A) = \sum \prod \alpha_i = S_r(\alpha_1 \ldots \alpha_n) \qquad \text{Classical phase space } (\mathbb{C}^\times)^{2n} \; (\alpha_i, p_\alpha^i)$$

$$\operatorname{Tr}_{\wedge^r}(B) = \sum \prod_{j \neq i} \frac{\alpha_i t - \alpha_j}{\alpha_i - \alpha_j} \prod p_\alpha^2$$

$H_r = \operatorname{Tr}_{\wedge^r}(B)$ - classical Hamiltonian functions of trigonometric Ruijsenaars Schneider model (relativistic Calogero system)

Symplectic form $\Omega = \sum_i \dfrac{dp_i}{p_i} \wedge \dfrac{d\alpha_i}{\alpha_i}$

Classical integrals of motion $H_r(p, \alpha, t) = S_r(\alpha)$

%add general discussion of classical int systems%

(3) Classical/quantum duality & K-theory of flag varieties [BKK 2014?]

Theorem (Proposition) The T-equivariant quantum K-theory of $T^*\mathbb{F}_N$ is the following ring

$$\mathcal{R} = K_T(T^*\mathbb{F}_n) = \frac{\mathbb{C}[p_1^{\pm1}, p_n^{\pm1}, \alpha_{\beta_1}^{\pm1}, \alpha_{\beta_2}^{\pm1}, t^{\pm1}]}{\{H_r(\alpha, p) - S_r\}}$$

$T = \mathbb{T}(U(N)\times U(1))$

This is a generalization of the classical result by Givental and Kim which relates cohomology (K-theory) of $\mathbb{F}_n$ and Toda lattice

Proposition [Nek Shat, Gaiotto PK] $\mathcal{R}$ is a space of solutions of Bethe Ansatz equations for anisotropic XXZ spin chain on n sites in the sector with n excitations transforming in fundamental representation of $U(n)$. Therefore there is a duality between quantum (XXZ) and classical (tRS) models when the Planck constant is related to t - parameter of the puncture

Ex for GL_2-flat connection $T^*\mathbb{P}^1$

$$\begin{cases} \dfrac{\beta_1 t - \beta_2}{\beta_1 - \beta_2} p_1 + \dfrac{\beta_2 t - \beta_1}{\beta_2 - \beta_1} p_2 = \alpha_1 + \alpha_2 \\ p_1 p_2 = \alpha_1\alpha_2 \end{cases}$$

introduce σ: $p_1 = \sigma$, $p_2 = \dfrac{\alpha_1\alpha_2}{\sigma}$

$$\frac{\sigma t - \alpha_1}{\sigma - t\alpha_1}\,\frac{\sigma t - \alpha_2}{\sigma - t\alpha_2}\,\frac{\beta_2}{\beta_1} = 1$$

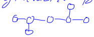
$$\Updownarrow$$

$$\frac{\sinh(s - a_1 + \hbar)}{\sinh(-s - a_1 + \hbar)}\,\frac{\sinh(s - a_2 + \hbar)}{\sinh(-s - a_2 + \hbar)}\, e^{\tau} = 1$$

$$XY = \Phi^2$$

s - spectral parameter
a_1, a_2 - shifts of spectral parameter
τ - twist of bc
$\hbar$ - Planck cnst

Can generalize to other Nakajima quiver varieties

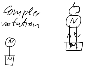


Many (simpler) integrable systems are contained as rational limits of this construction

(4) Gauge theory interpretation

Proposition [Nekrasov-Shatashvili, Gaiotto - PK] Space of massive supersymmetric vacua of 3d $\mathcal{N}=2^*$ quiver gauge theories on $\mathbb{R}^2\times S^1$ (a.k.a. twisted chiral ring) are isomorphic to equivariant quantum K-ring of the corresponding Nakajima quiver variety

complex notation



A_1

$T^*Gr_{N,M}$

Twisted superpotential $W(\sigma, t, \alpha_i, \beta_i)$ ($W = 2^*$ mass; masses; FI parameters; complex Coulomb branch parameters) $\quad \dfrac{\partial W}{\partial\sigma} = 0$

$Gr \quad Q(x)Q^{\perp}(x) = \prod_{i=1}^{M}(x - m_i)$

$T^*Gr \quad Q(x)Q^{\perp}(x) = \prod_{i=1}^{M}(x - m_i - \frac{\epsilon}{2})$

$\Rightarrow P(s_i + \frac{\epsilon}{2})Q(s_i - \epsilon) - \tau P(s_i + \frac{\epsilon}{2})Q(-s_i - \epsilon) = 0$

$P(x) = \prod_{i=1}^{M}(x - m_i)$
$Q(x) = \prod_{i=1}^{N}(x - s_i)$
Chern Polys for flavor and gauge symmetry

$$P(x - \tfrac{\epsilon}{2})Q(x+\epsilon) = t(x)\,Q(x)$$
$$t(x) = Q^{\perp}Q(x+\epsilon)$$

(5) Quantization

Deformation quantization [Kontsevich] $\quad \{p, x\}_{PB} \to \frac{1}{\hbar}[p, x]$

Proposition [Oblomkov] [Gukov-Koroteev-PK] Deformation quantization of $\mathcal{M}_{gl(n)}$ is spherical DAHA for $gl(n)$ in complex structure J]

$[x, y]_q = q^{1/2}xy - q^{-1/2}yx$

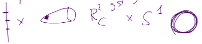
$$[x, y]_q = (q - q^{-1})z$$
$$[y, z]_q = (q - q^{-1})x$$
$$(z, x]_q = (q - q^{-1})y$$

Casimir: $f \to \hat{f}$

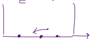
$$\Omega = qx^2 + qy^2 + q^{-1}z^2 - q^{1/2}yzx$$

$$\Omega = (q^{1/2}t^{-1} - q^{-1/2}t)^2 + (q^{1/2} + q^{-1/2})^2$$

Proposition Physically Sp DAHA is the algebra of line operators in 4d $\mathcal{N}=2^*$ theory on $\mathbb{R}^3\times S^1$. $\mathbb{R}^3 \to \mathbb{R}^2_\epsilon \times \mathbb{R}$ Omega background [Nekrasov]

$\mathbb{R} \times$  $\mathbb{R}^2_\epsilon$ S^1 $\times S^1$

After reduction on $S^1\times S^1_\epsilon$ we have



$\mathbb{R}$

Claim tRS operators H_1 $p_1 \alpha^{\pm 1} = q^{\pm 1} \alpha^{\pm 1} p_1$ form a maximal commuting subalgebra inside sperical DAHA
(aka Macdonald operators)
y above $y = Y + Y^{-1}$

DAHA reps "weigts" $\alpha_1, \ldots, \alpha_n$ (or Hamiltonian energy levels) are complex and at generic values of those "weights" reps are ∞ - dimensional

(no actual highest weight reps
most of the time people study rational and trig degenerations)

To illustrate this fact let us look at spectrum problem of Macdonald operators

$$\left(\frac{t\beta - \beta^{-1}}{\beta - \beta^{-1}} p + \frac{\beta - t\beta^{-1}}{\beta^{-1} - \beta} p^{-1}\right) Z = (\alpha + \alpha^{-1}) Z$$

$$Z = {}_2\phi_1\left(t, t^2\alpha^2, q\alpha^2, q; \frac{q}{t}\beta\right) = \sum_{k=0}^{\infty} a_k \beta^k$$

For some values of α the series truncates
$\alpha^2 = q^{\lambda} t^{-1}$
$Z \to P_{\lambda}(\beta, \beta^{-1} | q, t)$ ← Macdonald Poly of corresponding to Tableau λ

for sl_2 DAHA there are 5 types of such finite-dimensional reps

$\mathrm{Rep}(\mathrm{DAHA}^{S}_{\mathrm{gen}})$ has 5 spherical objects

We suggest a (derived) equivalence with $\mathrm{Fuk}(\mathcal{M}_n)$