Elliptic Algebras from Instanton Counting

Peter Koroteev

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In collaboration with

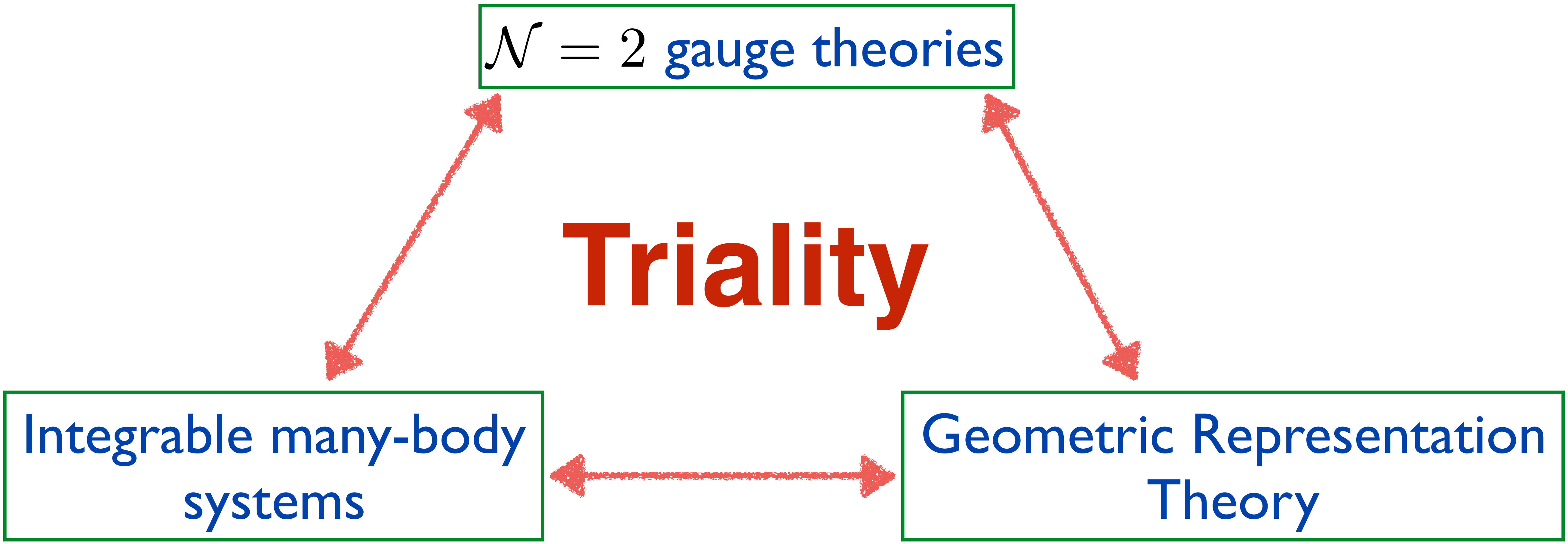
Davide Gaiotto
Matthew Bullimore
Hee-Cheol Kim
Antonio Sciarappa
Sergei Gukov
Satoshi Nawata
Shamil Shakirov
Amihay Hanany
Noppadol Mekareeya
Anindya Dey

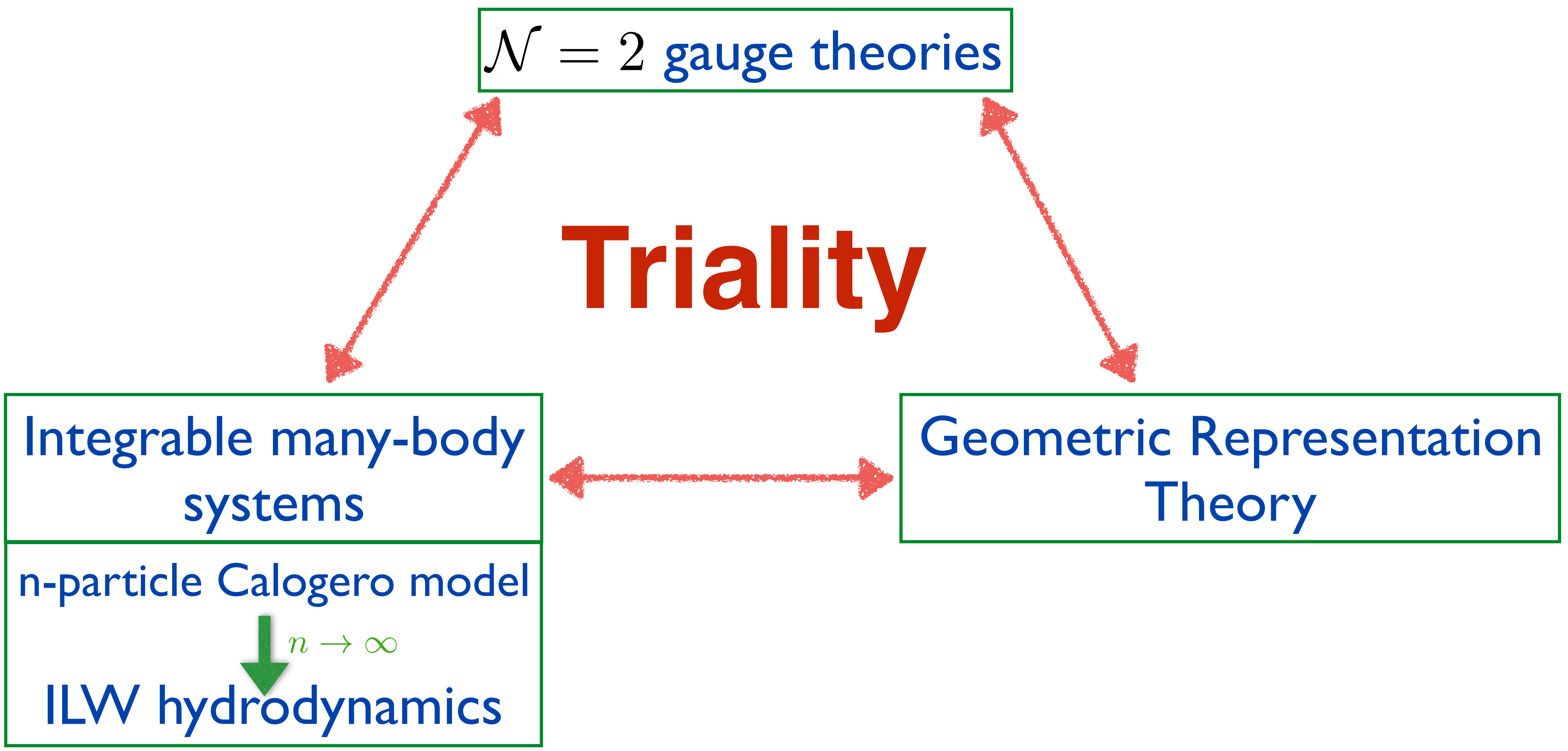
Large-N Approach

Gauge theories are known to have effective descriptions when the rank of the gauge group becomes large $U(N) \quad N \to \infty$

For supersymmetric gauge theories we expect to compute the effective large-N theory exactly

There are similar ideas which work in mathematics


$\mathcal{N} = 2$ gauge theories
Triality
Integrable many-body systems
Geometric Representation Theory



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$n \to \infty$
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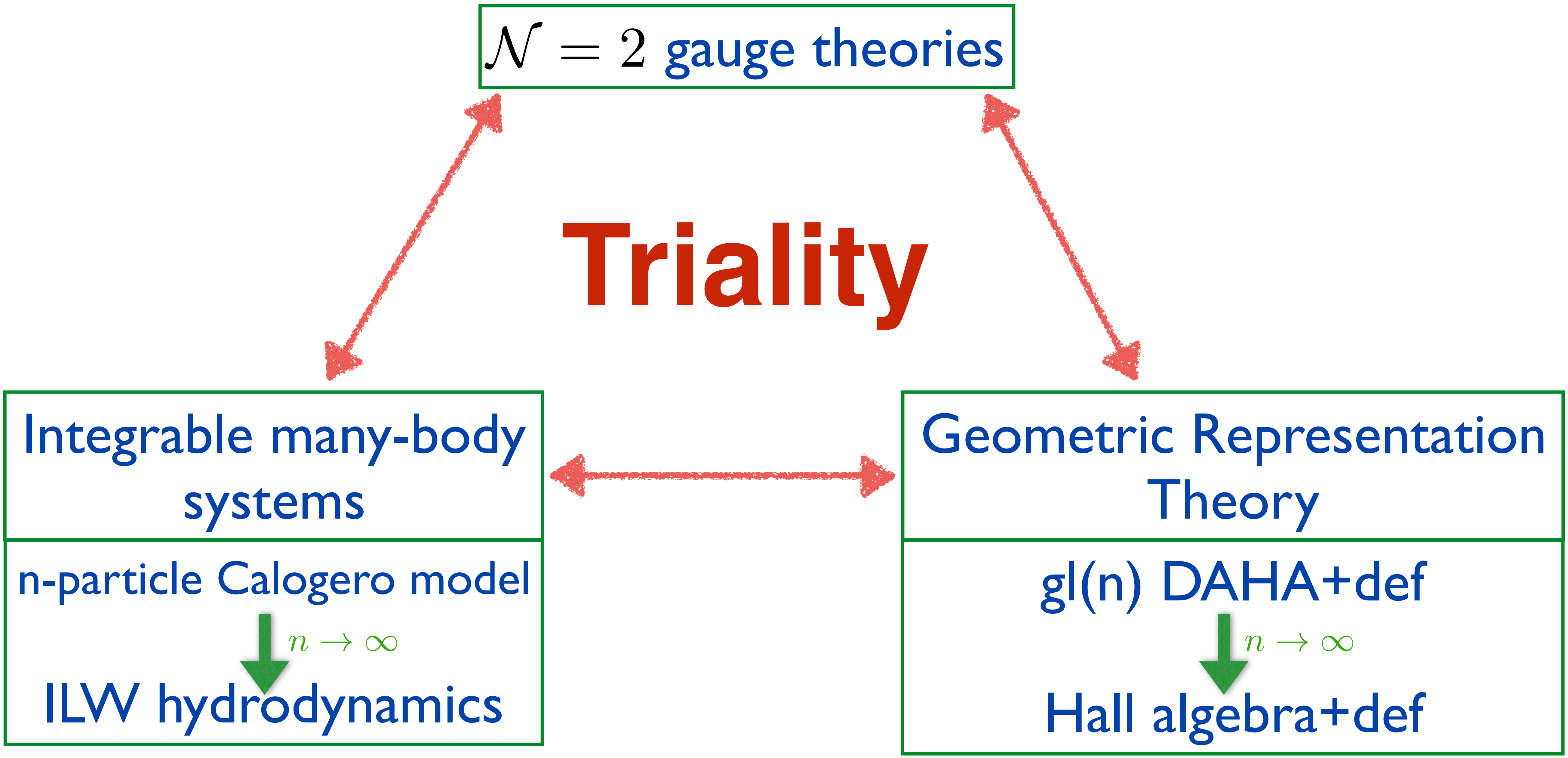
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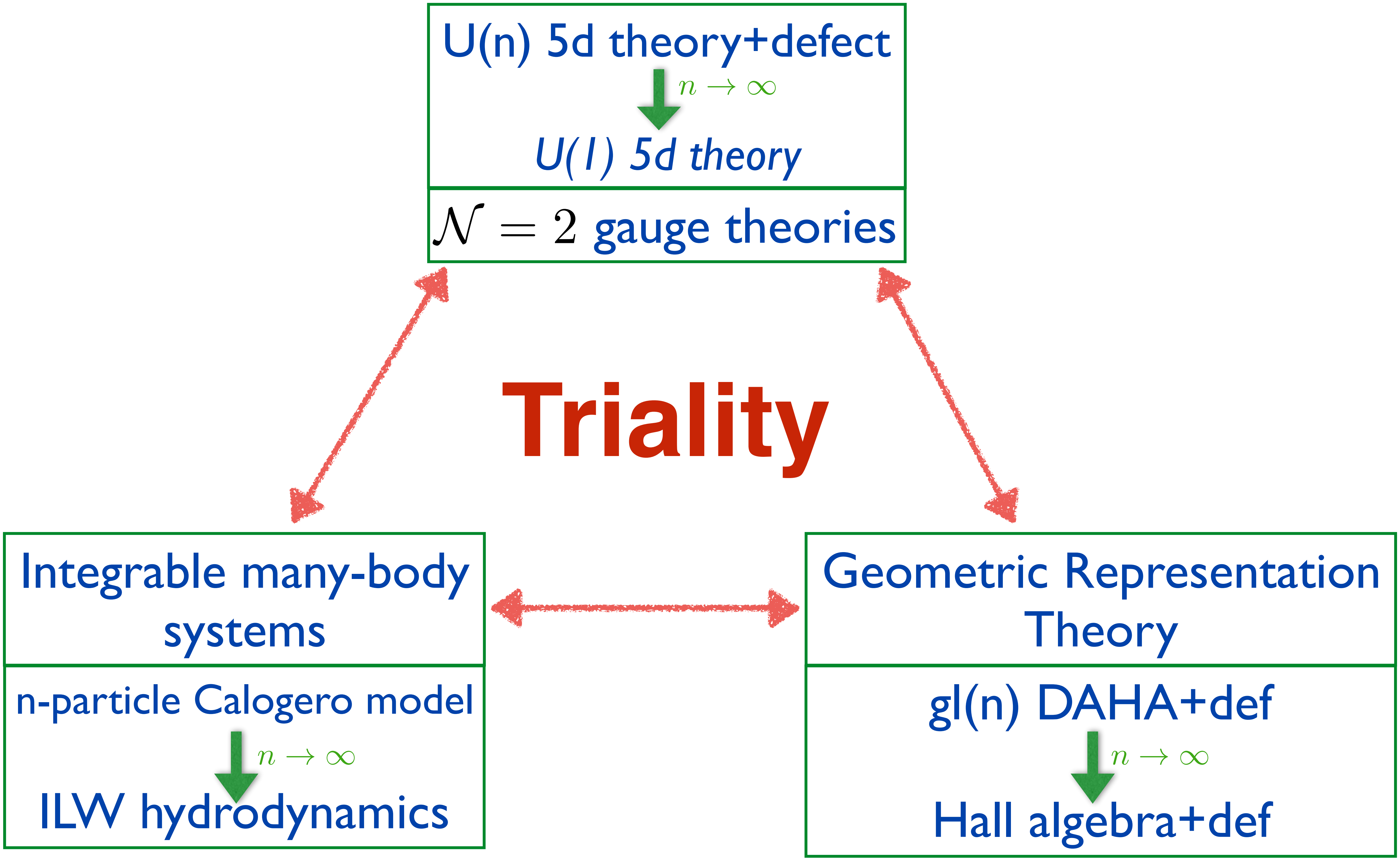
U(n) 5d theory+defect
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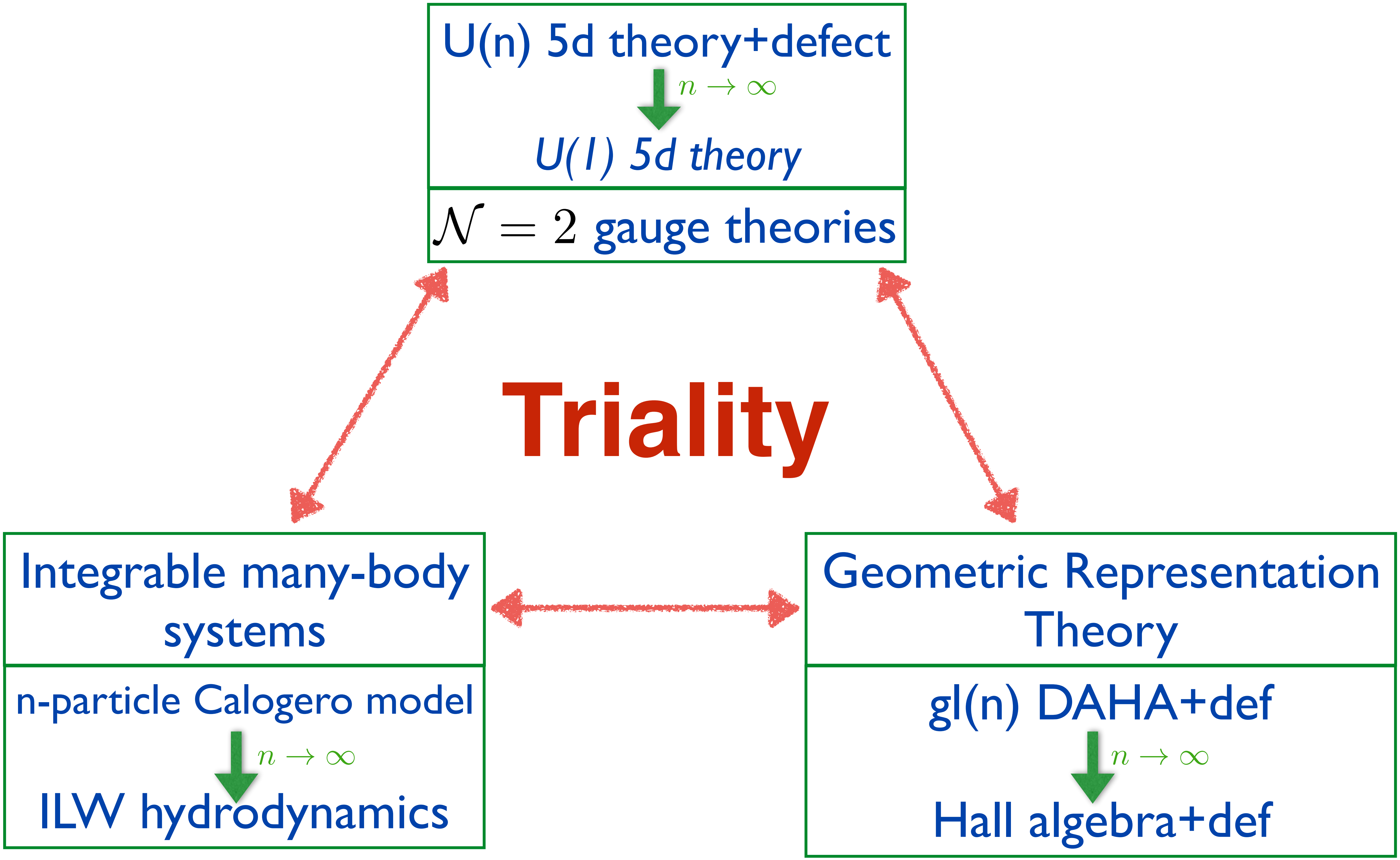
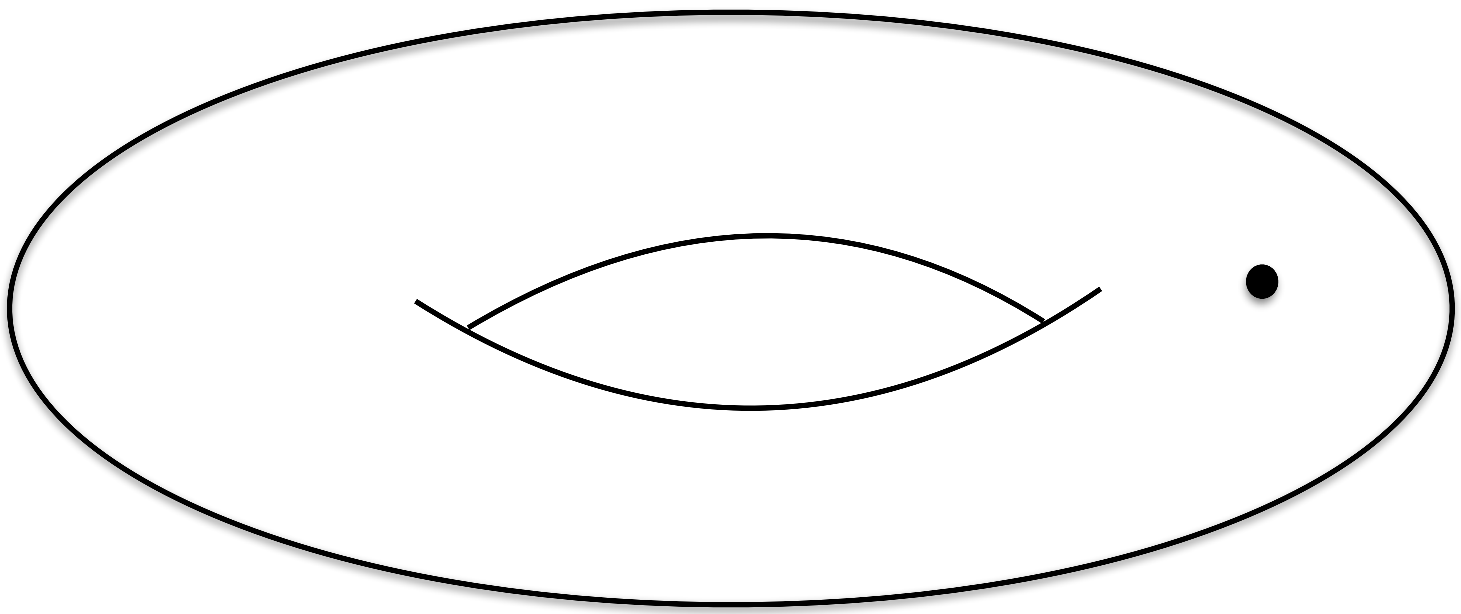


Large-n limits are manifest in each description!

Flat connections

Moduli space of flat GL(n;C) connections on punctured torus



$$\mathcal{M}_n = \{A, B, C\}/GL(n;\mathbb{C})$$

$$ABA^{-1}B^{-1} = C$$

$$C = \mathrm{diag}(t, t, \ldots t, t^{1-n})$$

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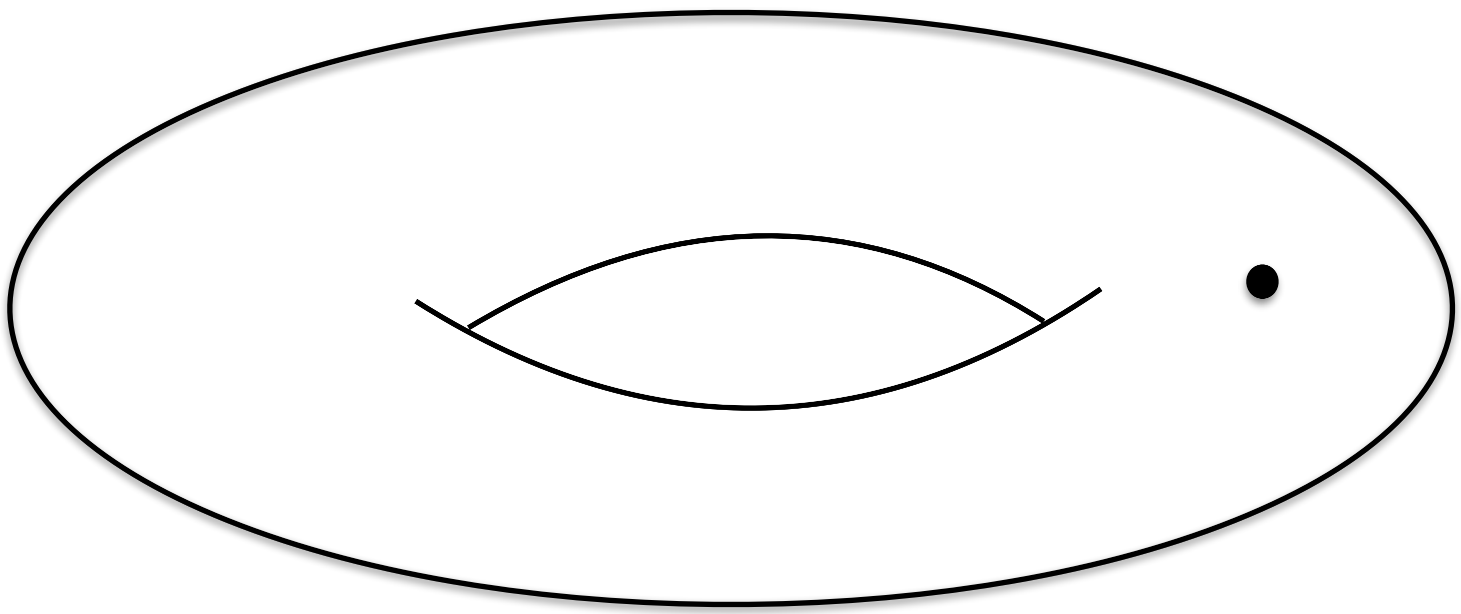

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Lemma: $\dim \mathcal{M}_n = 2n$

Can be interpreted as a phase space of some Hamiltonian system

locally $\mathcal{M}_n = T^*M$

Pick holomorphic Lagrangian $\mathcal{L} \subset \mathcal{M}_n$
by fixing eigenvalues of A or B

Modular duality

$ABA^{-1}B^{-1} = C$ is invariant under $A \leftrightarrow B$ $C \leftrightarrow C^{-1}$

We can naturally double the phase space $\mathcal{M} = \mathcal{M}_n^A \times \mathcal{M}_n^B$

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$A = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ B derived from the relation

equivalently

$B = \mathrm{diag}(\beta_1, \ldots, \beta_n)$ A derived from the relation

Pick a corresponding Lagrangian $\mathcal{L} \subset \mathcal{M}$

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In terms of different variables $(\alpha_1, \ldots, \alpha_n; p_\alpha^1, \ldots, p_\alpha^n)$

A can be written as a Lax matrix of an integrable system

$$\det(u - A(\alpha_i, p_\alpha^i)) = \prod_j (u - \beta_j)$$

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$$\det(u - A(\alpha_i, p_\alpha^i)) = \prod_j (u - \beta_j)$$ + dual

Quantization

Quantization

D***eformation quantization*** of $\mathcal{M}_n$ in complex structure **J** gives is *spherical* DAHA for gl(n) [Oblomkov]

$$\alpha_i p^j_\alpha = q^{\delta_{ij}} p^j_\alpha \alpha_i \qquad q = e^\hbar \qquad \omega = \sum_i \frac{dp^i_\alpha}{p^i_\alpha} \wedge \frac{d\alpha_i}{\alpha_i}$$

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$\mathcal{M}_n$ is the moduli space of vacua in N=2* gauge theory on $\mathbb{R}^3 \times S^1$ with gauge group U(n) and is described by VEVs of line operators wrapping the circle.

A and B matrices are holonomies of electric and magnetic Wilson lines

sl(2) spherical DAHA

For sl(2) x,y,z are VEVs of Wilson, t'Hooft and dyonic loops

$$x = TrA \qquad y = TrB \qquad z = TrAB$$

y is symmetric Macdonald operator

$$\mathcal{M}_n \qquad x^2 + y^2 + z^2 + xyz = \mathrm{tr}C + 2 \qquad C = \mathrm{diag}(t, t^{-1})$$

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quantization gives $[x, y]_q = (q - q^{-1})z$ +cyclic

Casimir $\quad \Omega = qx^2 + qy^2 + q^{-1}z^2 - q^{1/2}yzx$

satisfying relation

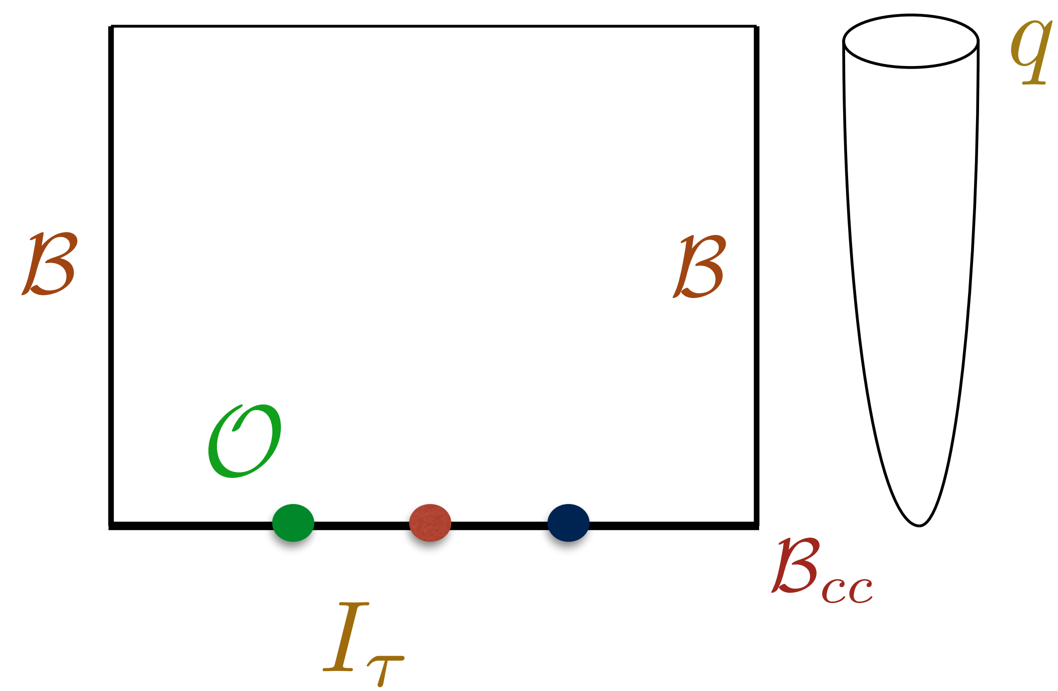
$$\Omega = (q^{1/2}t^{-1} - q^{-1/2}t)^2 + (q^{1/2} + q^{-1/2})^2$$

DAHA cont'd

[Gukov PK Nawata]

Physically we describe quantization by introducing Omega background to one of the spacetime 2-planes $\mathbb{R}^2_{\epsilon_1} \times \mathbb{R} \times S^1$

We can now reduce along the circle action which acts on this 2-plane



Line operators are forced to stay at the tip of the cigar and slide along the remaining line thus they do not commute anymore

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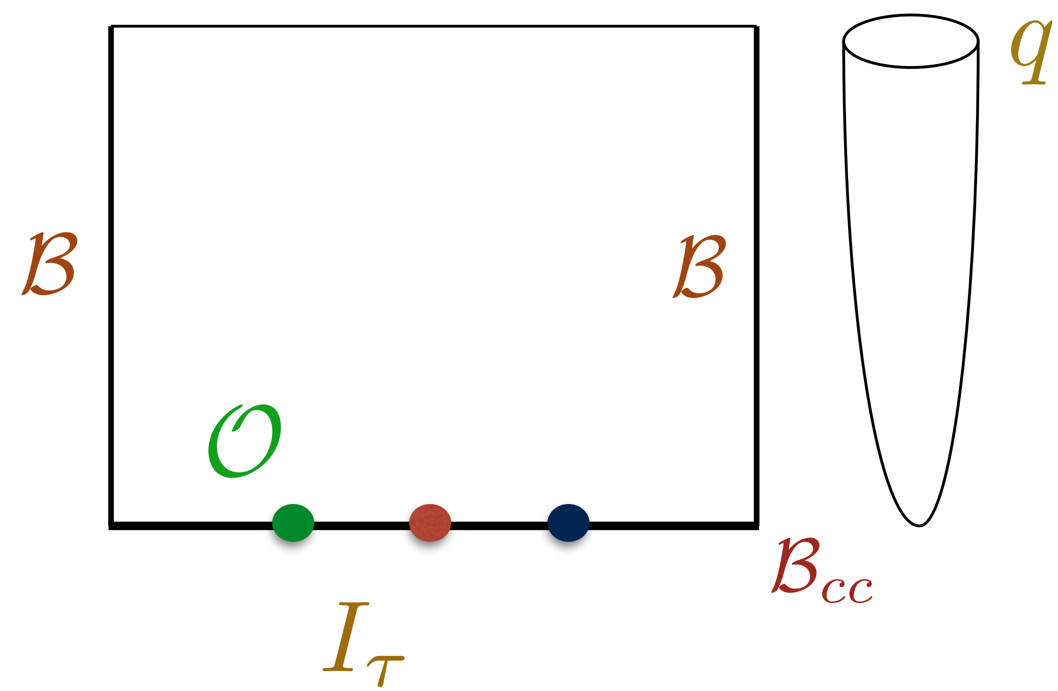

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Representations of DAHA can be understood by introducing boundaries

Equivariant K-theory

Eigenfunctions of quantum Hamiltonians gives Givental J-function for equvariant quantum K-theory of the cotangent bundle of the n-flag

$$D^{(k)}(\alpha_i, p^i_\alpha)\mathcal{B} = h_k(\beta_i)\mathcal{B}$$

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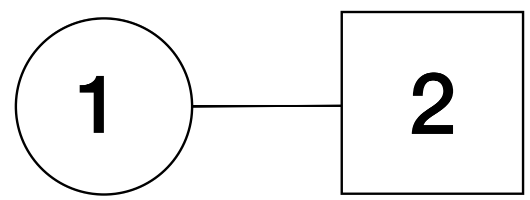
J-functions for T*P1



$$\mathcal{B} \sim {}_2\phi_1\left(t, t\frac{\beta_1}{\beta_2}, q\frac{\beta_1}{\beta_2}; q; \frac{\alpha_1}{\alpha_2}\right)$$

is the eigenstate of the trigonometric Ruijsenaars-Schneider system!

$$D^{(1)}\mathcal{B} = (\beta_1 + \beta_2)\mathcal{B}$$

$$D^{(1)} \sim \sum_{i \neq j} \frac{t\alpha_i - \alpha_j}{\alpha_i - \alpha_j} p^i_\alpha$$

Quantum K-theory

Theorem [Givental Lee]

Quantum equivariant K-theory of complete n-flag variety is

$$K_T \simeq \mathbb{C}[\mu_1, \ldots, \mu_n, p_1, \ldots, p_n]/\mathcal{I}$$

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Claim [Kim Bullimore PK]

Quantum equivariant K-theory of the cotangent bundle to the complete n-flag variety is

$$K_T(T^*\mathbb{F}_n) \simeq \mathbb{C}[\alpha_i^{\pm 1}, p_\alpha^{\pm 1}, t, \beta_i^{\pm 1}]/\widetilde{\mathcal{I}}$$

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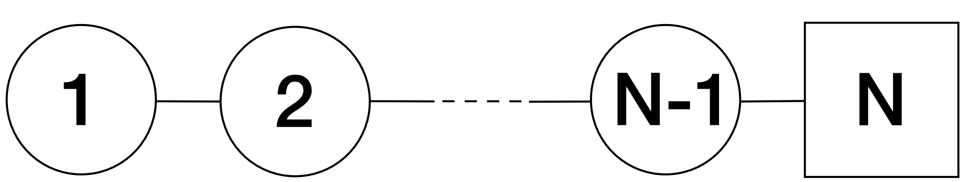
Claim [Nekrasov-Shatashvili, Gaiotto PK]

$K_T(T^*\mathbb{F}_N)$ is isomorphic to the twisted chiral ring of



on $\mathbb{R}^2 \times S^1$

Elliptic RS model

$$D^{(1)}_{p,q,t} \sim \sum \frac{\theta(t\frac{\tau_i}{\tau_j}|p)}{\theta(q\frac{\tau_i}{\tau_j}|p)} e^{\hbar\partial_{\log\tau_i}}$$

$$D^{(k)}_{p,q,t}\mathcal{Z}^{5\mathrm{d}/3\mathrm{d}} = \left\langle W^{U(n)}_{\Lambda^k} \right\rangle \mathcal{Z}^{5\mathrm{d}/3\mathrm{d}}$$

K-theoretic equivariant holomorphic Euler characteristic on the moduli space of ramified U(n) instantons

$$\mathcal{Z}^{5d/3d} = \sum_{k,l} R_{k,l}(t,q,\mu)\,\tau^k \left(\frac{p}{\tau}\right)^l$$

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Wilson loop in fundamental representation

$$\langle W_{(1)}\rangle = \frac{\sum_{\vec{\lambda}} p^{|\vec{\lambda}|}\chi^{(\mathcal{E})}_{\vec{\lambda}} \prod_\alpha \left(2\sinh\left(\frac{w_\alpha}{2}\right)\right)^{-n_\alpha}}{\sum_{\vec{\lambda}} p^{|\vec{\lambda}|} \prod_\alpha \left(2\sinh\left(\frac{w_\alpha}{2}\right)\right)^{-n_\alpha}}$$

$$E^{U(2)}_{(1)} = (\mu_1+\mu_2)\left(1+\sum_n F_n(q,t,\mu)p^n\right)$$

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Now let's discuss large-n limit

Mapping States

Consider partition λ of $k < n$ (assume p=0)

Specify $\mu_a = q^{\lambda_a} t^{n-a} \quad , \quad a = 1, \ldots, n$ for T[U(n)] theory

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J-function series truncates to Macdonald polynomials!

$$D^{(1)}_{n,\vec{\tau}}(q,t) P_\lambda(\vec{\tau}; q, t) = E^{(\lambda;n)}_{tRS} P_\lambda(\vec{\tau}; q, t)$$

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E.g. k=2

$$\mathcal{B}(\tau_1, \tau_2; t^{-1/2} q, t^{1/2} q) = P_{\square\square}(\tau_1, \tau_2; q, t)$$

$$\mathcal{B}(\tau_1, \tau_2; t^{-1/2}, t^{-1/2} q^2) = P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(\tau_1, \tau_2 | q, t)\,.$$

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Their exact form depends on n

$$P_{(2,0)}(\tau_1, \tau_2; q, t) = \tau_1 \tau_2 + \frac{1 - qt}{(1+q)(1-t)}(\tau_1^2 + \tau_2^2)$$

Change of Variables

However, after change of variables $p_m = \sum_{l=1}^{n} \tau_l^m$

Macdonald polynomials depend only on k and the partition

$$P_{\square\square} = \frac{1}{2}(p_1^2 - p_2)\,, \qquad P_{\substack{\square\\ \square}} = \frac{1}{2}(p_1^2 - p_2) + \frac{1-qt}{(1+q)(1-t)}p_2$$

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Starting with Fock vacuum $|0\rangle$

Construct Hilbert space $a_{-\lambda}|0\rangle \longleftrightarrow p_\lambda$

for each partition $a_{-\lambda}|0\rangle = a_{-\lambda_1}\cdots a_{-\lambda_l}|0\rangle$

Free boson realization

$$[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}$$

(more involved with p)

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Now need to describe eigenvalues

Free Boson Realization

Introduce vertex operators [Ding Iohara]

$$\eta(z) =: \exp\left(-\sum_{k\neq 0}\frac{1-t^k}{k}a_k z^{-k}\right):$$

$$\phi(z) = \exp\left(\sum_{n>0}\frac{1-t^n}{1-q^n}a_{-n}\frac{z^n}{n}\right)$$

Define $\phi_n(\tau) = \prod_{i=1}^n \phi(\tau_i)$

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Define $\phi_n(\tau) = \prod_{i=1}^n \phi(\tau_i)$ then

$$[\eta(z)]_1\phi_n(\tau)|0\rangle = \left[t^{-n} + t^{-n+1}(1-t^{-1})D^{(1)}_{n,\vec{\tau}}(q,t)\right]\phi_n(\tau)|0\rangle$$

Assuming |t|<1

$$\mathcal{E}_1^{(\lambda)} = \lim_{n\to\infty}\left[t^{-n+1}(1-t^{-1})E^{(\lambda;n)}_{tRS}\right]$$

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For elliptic model replace [Feigin Hashizume Hoshino Shiraishi Yanagida]

$$\eta(z;pq^{-1}t) = \exp\left(\sum_{n>0}\frac{1-t^{-n}}{n}\frac{1-(pq^{-1}t)^n}{1-p^n}a_{-n}z^n\right)\exp\left(-\sum_{n>0}\frac{1-t^n}{n}a_n z^{-n}\right)$$

Free Boson Realization

Assuming |t|<1

$$\mathcal{E}_1^{(\lambda)}(p) = \lim_{n\to\infty}\left[t^{-n+1}(1-t^{-1})\frac{(pt^{-1};p)_\infty(ptq^{-1};p)_\infty}{(p;p)_\infty(pq^{-1};p)_\infty}E_{eRS}^{(\lambda;n)}(p)\right]$$

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[Feigin Hashizume Hoshino Shiraishi Yanagida]

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Free Boson Realization

From gauge theory we can compute

$$\frac{(pt^{-1};p)_\infty (ptq^{-1};p)_\infty}{(p;p)_\infty (pq^{-1};p)_\infty} E^{(\lambda;n)}_{eRS}(p) = \left\langle W^{U(1)}_{\square} \right\rangle E^{(\lambda;n)}_{eRS}(p) = \left\langle W^{U(n)}_{\square} \right\rangle \Big|_{\lambda}$$

Assuming |t|<1

$$\mathcal{E}^{(\lambda)}_1(p) = \lim_{n\to\infty} \left[t^{-n+1}(1-t^{-1}) \frac{(pt^{-1};p)_\infty (ptq^{-1};p)_\infty}{(p;p)_\infty (pq^{-1};p)_\infty} E^{(\lambda;n)}_{eRS}(p) \right]$$

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[Feigin Hashizume Hoshino Shiraishi Yanagida]

$$\eta(z;pq^{-1}t) = \exp\left(\sum_{n>0} \frac{1-t^{-n}}{n} \frac{1-(pq^{-1}t)^n}{1-p^n} a_{-n} z^n \right) \exp\left(-\sum_{n>0} \frac{1-t^n}{n} a_n z^{-n} \right)$$

U(1) Instantons

Heisenberg algebra (and ***elliptic Hall algebra***) which we have seen earlier appears in the study of moduli space of U(1) (non-commutative) instantons

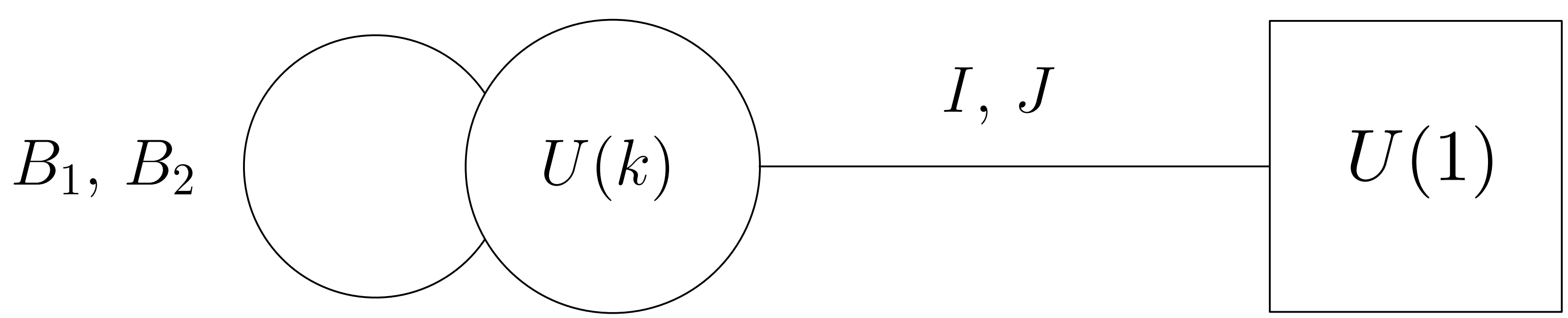
[Nakjima]
[Schiffmann Vaserot]

U(1) Instantons

Heisenberg algebra (and ***elliptic Hall algebra***) which we have seen earlier appears in the study of moduli space of U(1) (non-commutative) instantons

[Nakjima]
[Schiffmann Vaserot]

Moduli space $\mathcal{M}_{k,1}$ described by ADHM quiver



Quantum Cohomology

Using supersymmetry we can effectively describe quantum cohomology (K-theory) of the instanton moduli space $\mathcal{M}_{k,1}$

We need to find the twisted chiral ring of the ADHM gauge theory— Jacobian ring for effective twisted superpotential

$$H^{\bullet}_{T}(\mathcal{M}_{k,1}) \simeq \frac{\{\sigma_1, \ldots \sigma_s\}}{\{\partial \widetilde{\mathcal{W}} / \partial \sigma_s = 0\}}$$

[Nekrasov Shatashvili]

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$$(\sigma_s - 1)\prod_{\substack{t=1\\t\neq s}}^{k}\frac{(\sigma_s - q\sigma_t)(\sigma_s - t^{-1}\sigma_t)}{(\sigma_s - \sigma_t)(\sigma_s - qt^{-1}\sigma_t)} = \frac{\widetilde{p}}{\sqrt{qt^{-1}}}(1 - qt^{-1}\sigma_s)\prod_{\substack{t=1\\t\neq s}}^{k}\frac{(\sigma_s - q^{-1}\sigma_t)(\sigma_s - t\sigma_t)}{(\sigma_s - \sigma_t)(\sigma_s - q^{-1}t\sigma_t)}$$

where $\sigma_s = e^{i\gamma\Sigma_s},\ q = e^{i\gamma\epsilon_1},\ t = e^{-i\gamma\epsilon_2}$ $\widetilde{p} = e^{-2\pi\xi}$ FI coupling

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Calogero ***Hamiltonian*** *contains the operator of* ***quantum multiplication*** *in small* ***quantum cohomology*** *ring of the instanton moduli space*

The Duality

Eigenvalues at large-n

[PK Sciarappa]

$$\left\langle W_{\square}^{U(n)} \right\rangle \Big|_{\lambda} \sim \mathcal{E}_1^{(\lambda)} = 1 - (1-q)(1-t^{-1}) \sum_s \sigma_s \Big|_{\lambda}$$

Wilson line VEV becomes an equivariant Chern character for $\mathcal{M}_{k,1}$

In other words there exists a stable limit of the equivariant Chern character of the universal bundle over the U(n) instanton moduli space in terms of the same character only for U(1) instantons

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elliptic RS	**3d ADHM theory**	**3d/5d coupled theory,** $n \to \infty$
coupling t	twisted mass $e^{-i\gamma\epsilon_2}$	5d $\mathcal{N}=1^*$ mass deformation $e^{-i\gamma m}$
quantum shift q	twisted mass $e^{i\gamma\epsilon_1}$	Omega background $e^{i\gamma\widetilde{\epsilon}_1}$
elliptic parameter p	FI parameter $\widetilde{p} = -p/\sqrt{qt^{-1}}$	5d instanton parameter
eigenstates λ	ADHM Coulomb vacua	5d Coulomb branch parameters
eigenvalues	$\langle \mathrm{Tr}\,\sigma \rangle$	$\langle W_{\square}^{U(\infty)} \rangle$ in NS limit $\widetilde{\epsilon}_2 \to 0$

Stable limits

Moduli space $\widetilde{\mathcal{M}_1} = \bigoplus_{k=0}^{\infty} \mathcal{M}_{1,k}$

$$\begin{array}{ccc} \mathfrak{gl}_n \mathrm{DAHA}^{S_n} & \longrightarrow & \text{ell Hall algebra} \\ \curvearrowright & & \curvearrowright \\ K_T(T^*\mathbb{F}_n) & \longrightarrow & K_{q,t}^{\text{class}}\left(\widetilde{\mathcal{M}_1}\right) \end{array}$$

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Analogous object related to the spectrum of elliptic RS

$$\mathcal{E}_T^Q(T^*\mathbb{F}_n) := \mathbb{C}[p_i^{\pm 1}, \tau_i^{\pm 1}, Q, t, \mu_i^{\pm 1}]/\mathcal{I}_{\text{eRS}}$$

Large-n limit

$$\lim_{n\to\infty} \mathcal{E}_T^Q(T^*\mathbb{F}_n) \simeq K_{q,t}\left(\widetilde{\mathcal{M}_1}\right)$$