Quantum Geometry Instantons & Elliptic Algebras

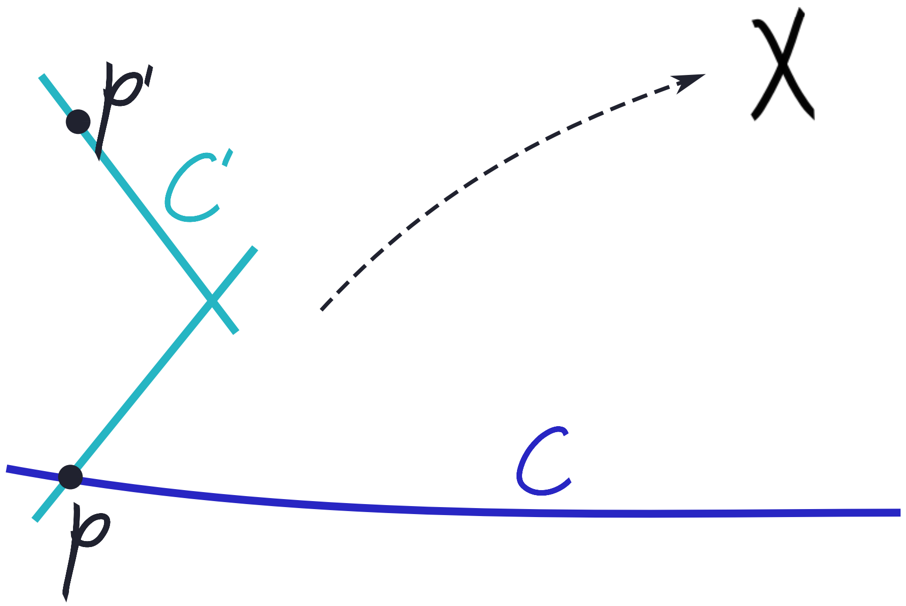
Peter Koroteev

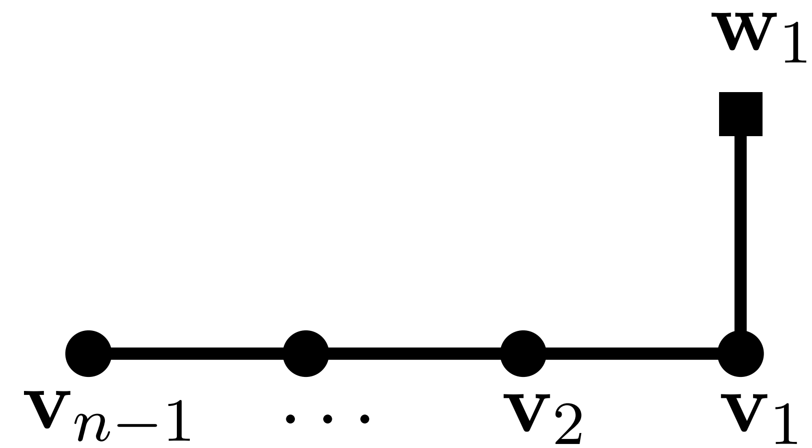
PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

THE UNIVERSITY OF CALIFORNIA DAVIS

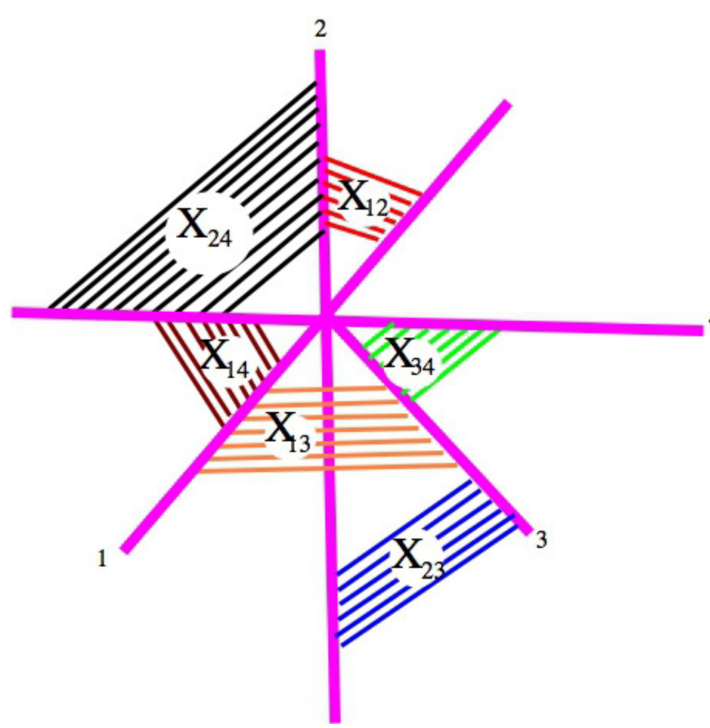
Talk at workshop `SCFTs in 6 and Lower Dimensions'
TSIMF, Sanya, China January 18th 2018

Based on new ideas, collaborations and discussions with





Aganagic
Okounkov
Zeitlin
Smirnov
Pushkar
Givental



Nekrasov
Pestun
Kimura
Sciarappa

Nekrasov

Costello, Gaiotto, Soibelman, Gukov, Nawata

Algebras from String/M-theory

In this talk we shall discuss **algebraic structures** which arise from CFTs. Often such algebras arise as quantizations of some moduli spaces.

Example — moduli spaces of SUSY vacua of gauge theories with 8 supercharges. Their quantization leads to vertex operator algebras which appear in 2d CFT (Virasoro, W-algebras, etc). This is a modern way to formulate the **BPS/CFT** correspondence:

Algebras from String/M-theory

In this talk we shall discuss **algebraic structures** which arise from CFTs. Often such algebras arise as quantizations of some moduli spaces.

Example — moduli spaces of SUSY vacua of gauge theories with 8 supercharges. Their quantization leads to vertex operator algebras which appear in 2d CFT (Virasoro, W-algebras, etc). This is a modern way to formulate the **BPS/CFT** correspondence:

- Connects BPS observables of $\mathcal{N}=2$ supersymmetric gauge theories with CFT correlators (*Mathematically*: Relates structures arising on moduli spaces of sheaves (instantons) with vertex operator algebras)

- Canonical example: [**A**lday **G**aiotto **T**achikawa]
  Partition functions vs. CFT conformal blocks
  Symmetries of the instanton moduli spaces vs.
  Vertex operator algebras

AGT

Class-*S* theories are constructed in M-theory with M5 branes wrapping $\mathcal{M}_4 \times \mathcal{C}$ [Gaiotto]

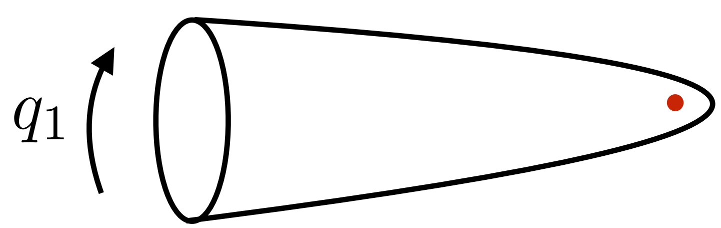
Twisted compactification of the theory on M5 branes — (2,0) 6d theory on $\mathcal{C}$ leads to $\mathcal{N}=2$ theory on $\mathcal{M}_4$

AGT

Class-*S* theories are constructed in M-theory with M5 branes wrapping $\mathcal{M}_4 \times \mathcal{C}$ [Gaiotto]

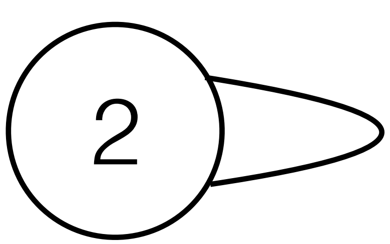
Twisted compactification of the theory on M5 branes — (2,0) 6d theory on $\mathcal{C}$ leads to $\mathcal{N}{=}2$ theory on $\mathcal{M}_4$

$\mathcal{N}{=}2^*$ SU(2) 4d gauge theory on $\mathbb{R}^2_{q_1} \times \mathbb{R}^2_{q_2}$



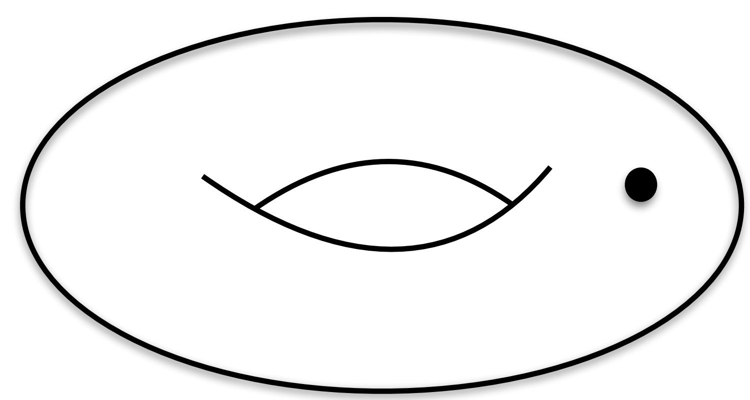
with adj hyper of mass ***m***

gauge coupling τ



Liouville CFT on a torus with one puncture

thin neck with sewing parameter $q = e^{2\pi i \tau}$



AGT: $\mathcal{Z}_{\mathrm{Nek}} = \mathcal{F}_{\mathrm{CFT}}$

Algebraic-geometric approach

Mathematicians have now several ***proofs*** of AGT in limiting cases (no fundamental matter), but those proofs do not use the original class-S construction [Schiffmann Vaserot] [Negut]

Physics **proof*** by Kimura and Pestun uses direct localization computations

Algebraic-geometric approach

Mathematicians have now several ***proofs*** of AGT in limiting cases (no fundamental matter), but those proofs do not use the original class-S construction [Schiffmann Vaserot] [Negut]

Physics **proof*** by Kimura and Pestun uses direct localization computations

One of our goals is to understand BPS/CFT **geometrically**

Namely we want describe instanton counting and vertex operator algebras in terms of **quantum geometry** (quantum cohomology or quantum K-theory) of some family of spaces

In other words we want VOAs to **emerge** from quantum geometry

Recent Developments

Vertex Algebras at the Corner [Gaiotto Rapcak]

VOAs at junctions of supersymmetric intersections in N=4 SYM

Quiver W-algebras [Kimura Pestun]

4,5,6d quiver gauge theories on R^4 x S in Omega background

The Magnificent Four [Nekrasov]

D8 brane probed by D0 branes in B field

$U(1)^4 \subset \mathrm{Spin}(8)$ + additional *nongeometric* U(1) symmetry

q_1, q_2, q_3, q_4

Large-n Limit

String theory enjoys **large-n** dualities

AdS/CFT, Gopakumar-Vafa

Gauge theories are known to have effective description when the rank of the gauge group becomes large $U(n)$ $n \to \infty$

Similar ideas work in mathematics — stable limits

Large-n Limit

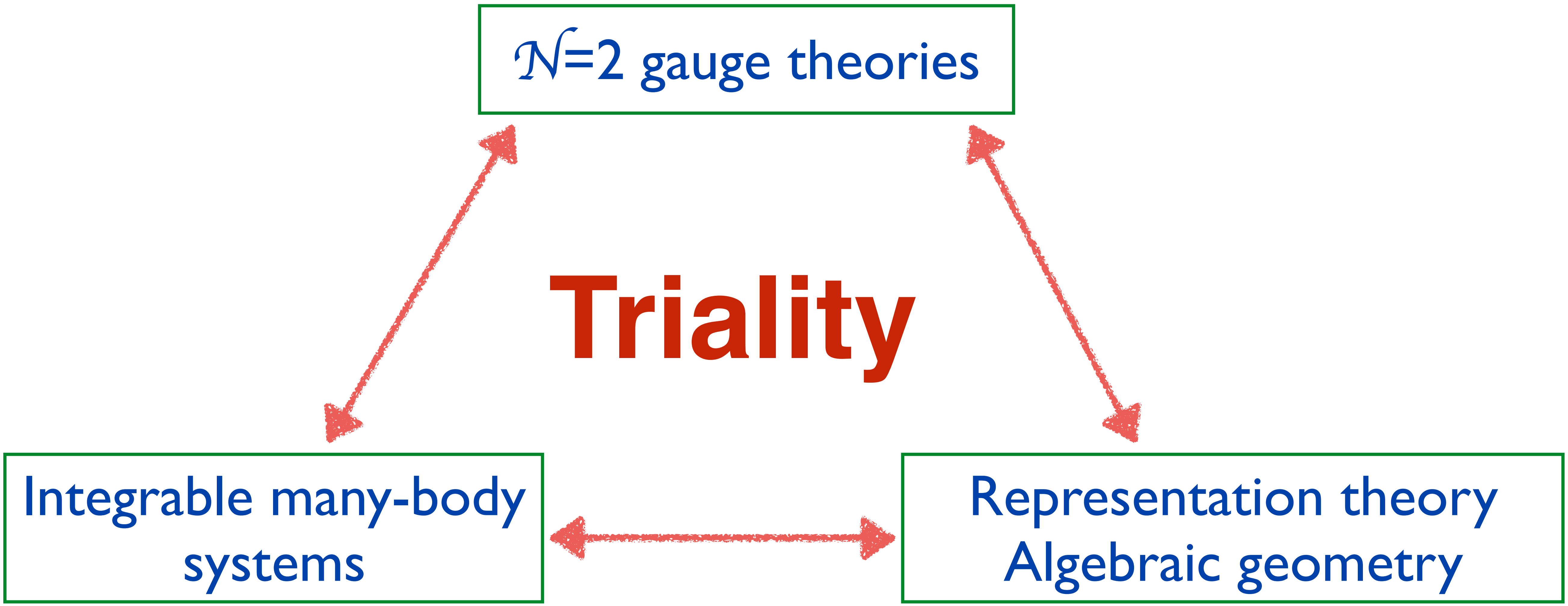
String theory enjoys **large-n** dualities

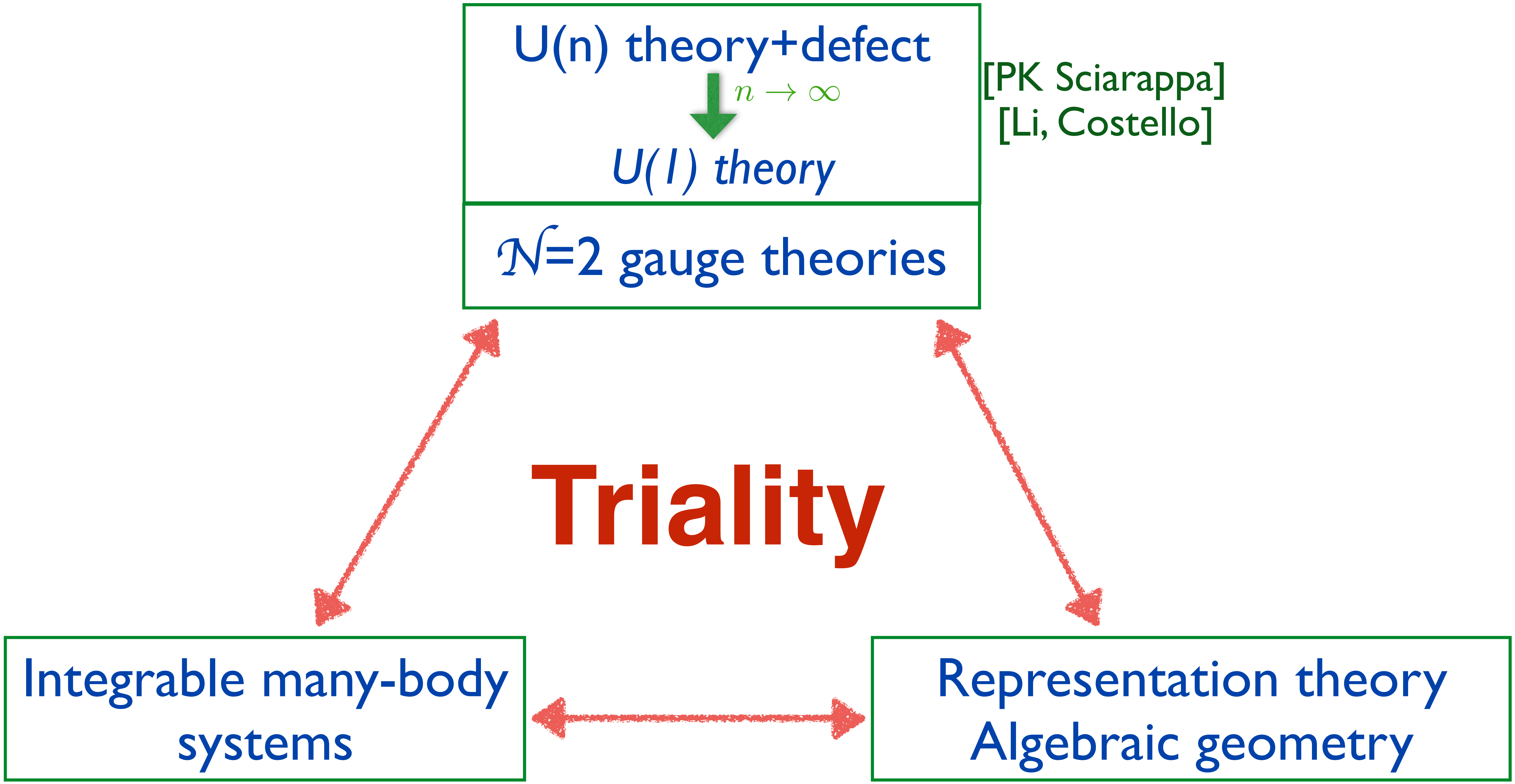
AdS/CFT, Gopakumar-Vafa

Gauge theories are known to have effective description when the rank of the gauge group becomes large $U(n)$ $n \to \infty$

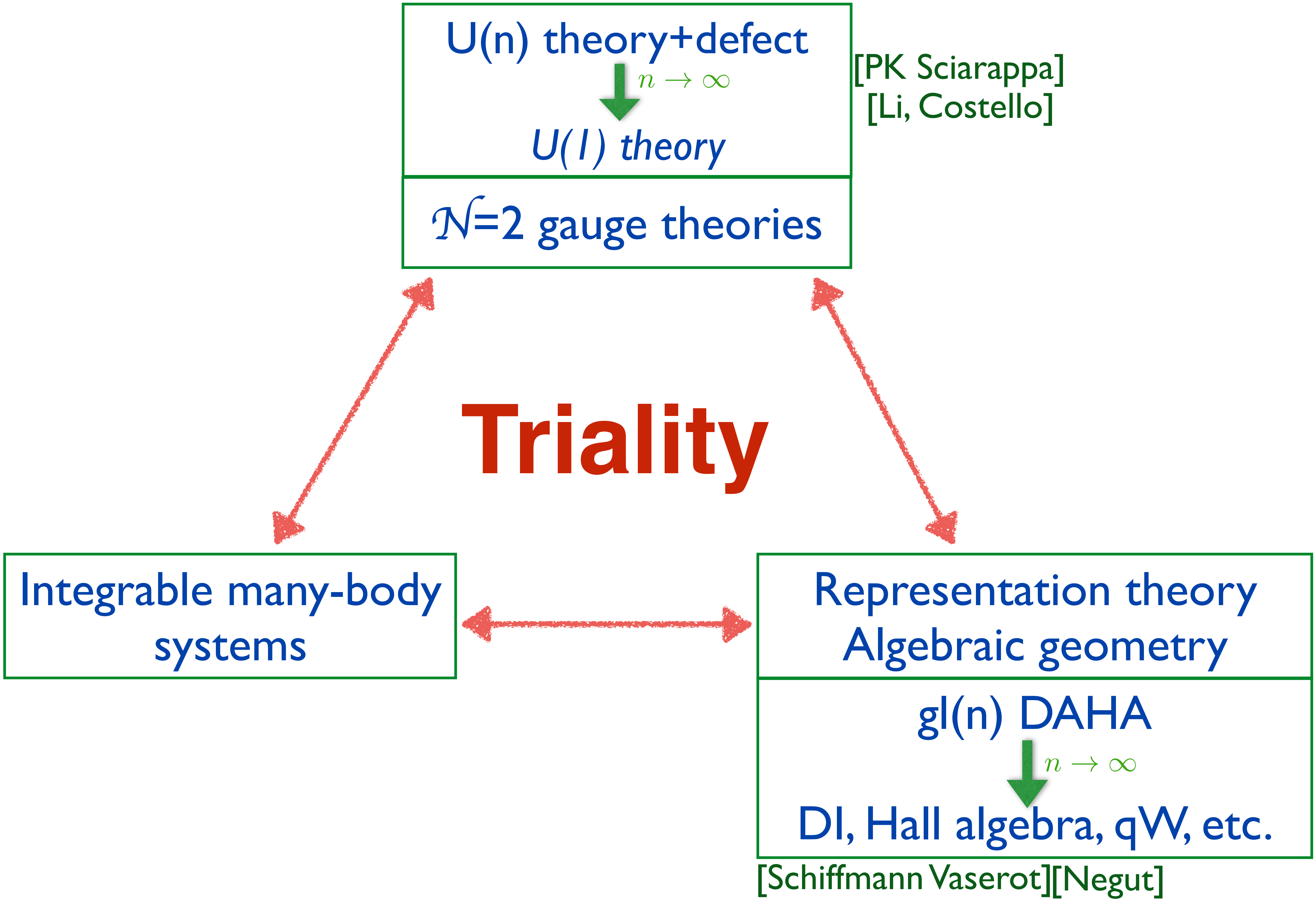
Similar ideas work in mathematics — stable limits

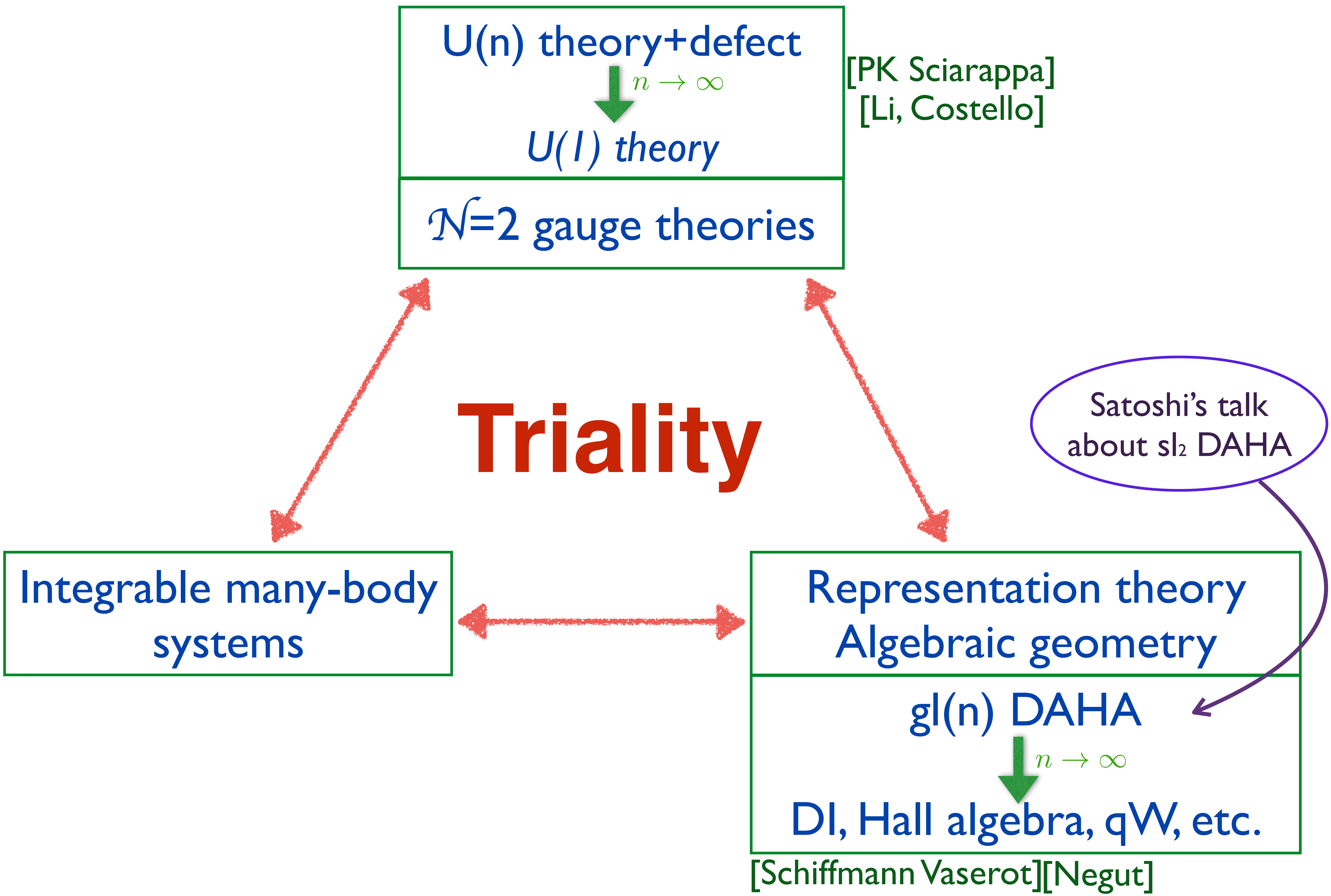
We shall see that BPS/CFT can be viewed as a large-n duality!


$\mathcal{N}$=2 gauge theories
Triality
Integrable many-body systems
Representation theory
Algebraic geometry

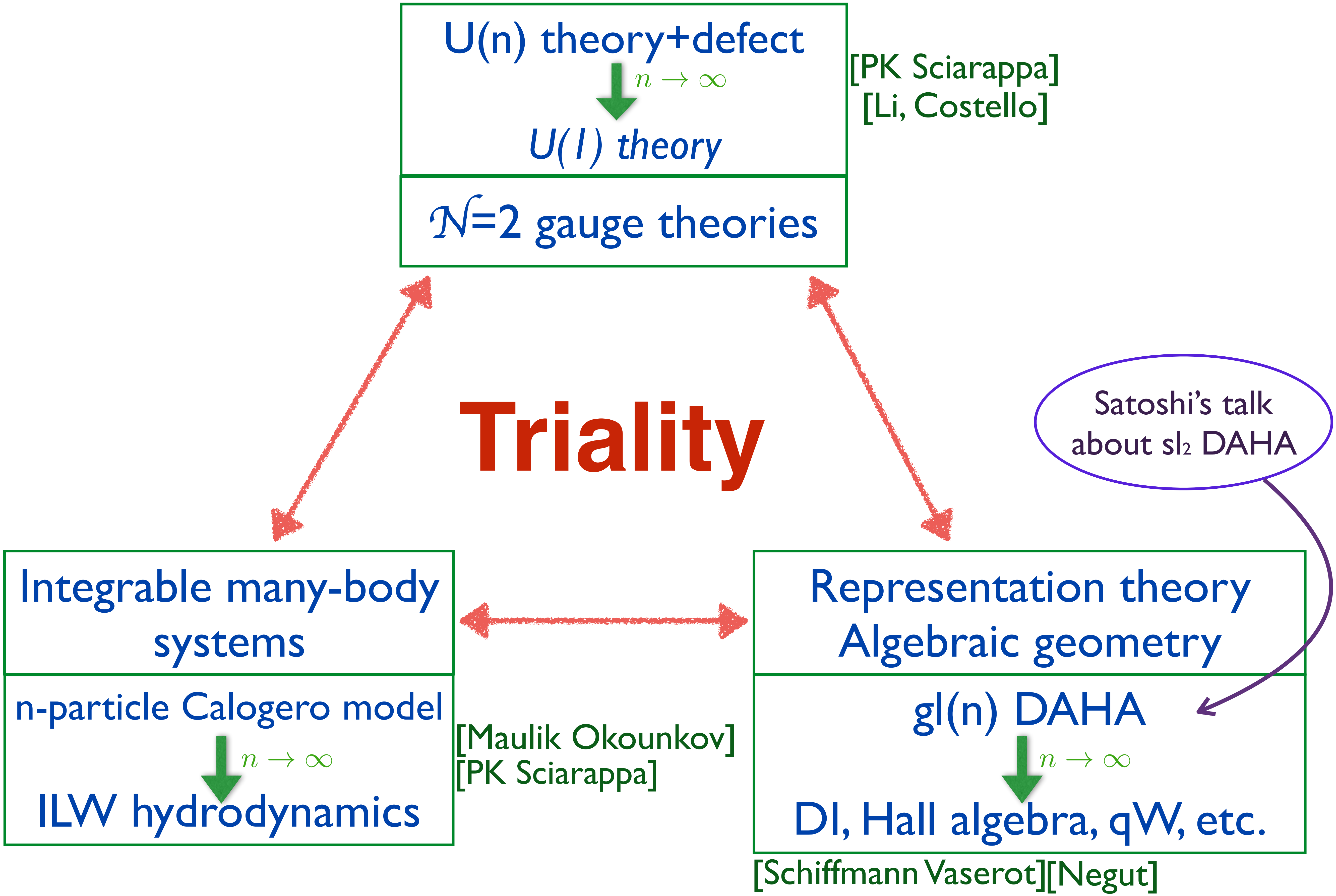


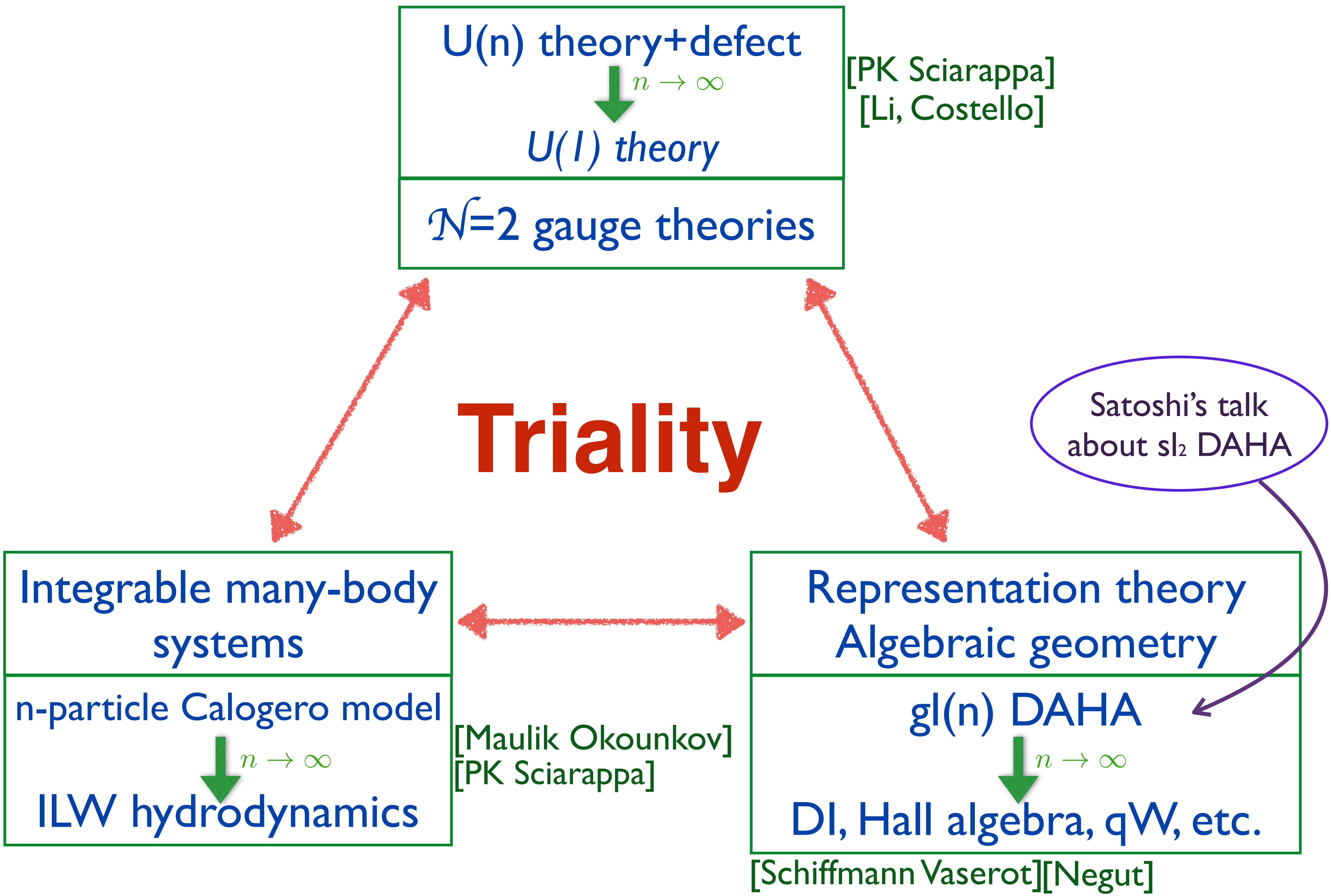
U(n) theory+defect
$n \to \infty$
U(1) theory
[PK Sciarappa]
[Li, Costello]
𝒩=2 gauge theories
Triality
Integrable many-body systems
Representation theory
Algebraic geometry


U(n) theory+defect
$n \to \infty$
U(1) theory
$\mathcal{N}$=2 gauge theories
[PK Sciarappa]
[Li, Costello]
Triality
Integrable many-body systems
Representation theory
Algebraic geometry
gl(n) DAHA
$n \to \infty$
DI, Hall algebra, qW, etc.
[Schiffmann Vaserot][Negut]



U(n) theory+defect
$n \to \infty$
U(1) theory
$\mathcal{N}$=2 gauge theories
[PK Sciarappa]
[Li, Costello]
Triality
Satoshi's talk about sl_2 DAHA
Integrable many-body systems
Representation theory
Algebraic geometry
gl(n) DAHA
$n \to \infty$
DI, Hall algebra, qW, etc.
[Schiffmann Vaserot][Negut]

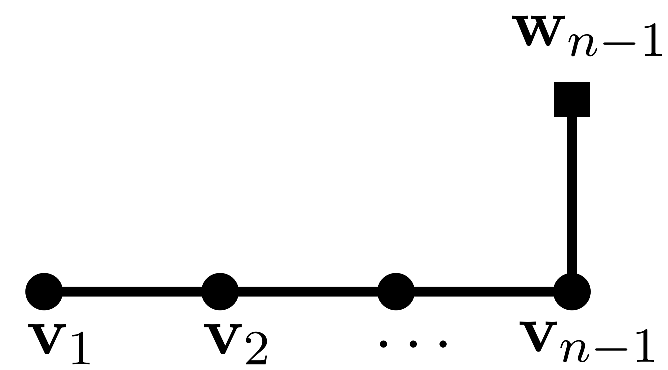

U(n) theory+defect
$n \to \infty$
U(1) theory
$\mathcal{N}$=2 gauge theories
[PK Sciarappa]
[Li, Costello]
Triality
Satoshi's talk about sl_2 DAHA
Integrable many-body systems
n-particle Calogero model
$n \to \infty$
ILW hydrodynamics
[Maulik Okounkov]
[PK Sciarappa]
Representation theory
Algebraic geometry
gl(n) DAHA
$n \to \infty$
DI, Hall algebra, qW, etc.
[Schiffmann Vaserot][Negut]



Large-n limits are manifest in each description!

Classical K-theory

Rep(**v**,**w**) — linear space of quiver reps



$\mu : T^*\mathrm{Rep}(\mathbf{v}, \mathbf{w}) \to \mathrm{Lie}(G)^*$ moment map

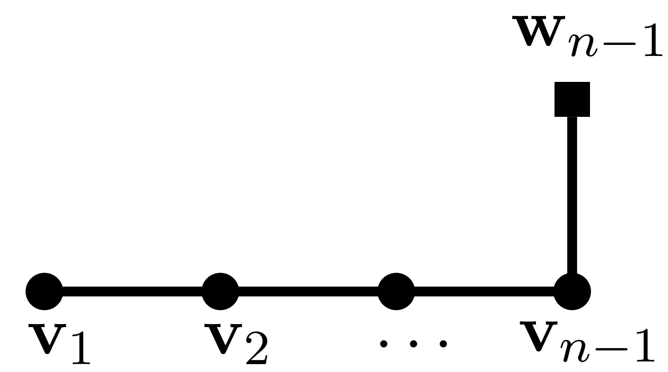
Nakajima quiver variety $X = \mu^{-1}(0) /\!\!/ G$ $G = \prod GL(V_i)$

Automorphism group $\mathrm{Aut}(X) = \prod GL(Q_{ij}) \times \prod GL(W_i) \times \mathbb{C}^\times_\hbar$

Maximal torus $T = \mathbb{T}(\mathrm{Aut}(X))$

Classical K-theory

Rep(**v**,**w**) — linear space of quiver reps



$\mu : T^*\mathrm{Rep}(\mathbf{v},\mathbf{w}) \to \mathrm{Lie}(G)^*$ moment map

Nakajima quiver variety $X = \mu^{-1}(0)/\!\!/G$ $G = \prod GL(V_i)$

Automorphism group $\mathrm{Aut}(X) = \prod GL(Q_{ij}) \times \prod GL(W_i) \times \mathbb{C}^\times_\hbar$

Maximal torus $T = \mathbb{T}(\mathrm{Aut}(X))$

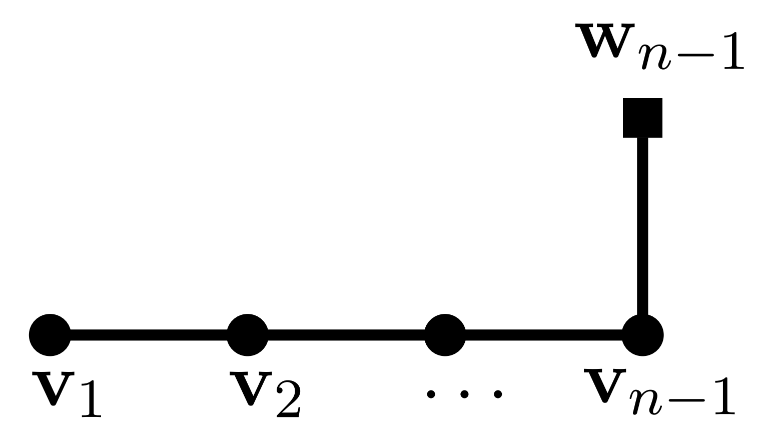
Tensorial polynomials of tautological bundles V_i, W_i and their duals generate *classical T-equivariant K-theory* ring of X

Quasimaps

Quasimap $f : \mathcal{C} - - \to X$ is described by collection of vector bundles $\mathscr{V}_i$ on $\mathcal{C}$ of ranks $\mathbf{v}_i$ with section $f \in H^0(\mathcal{C}, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar)$ satisfying $\mu = 0$

where $\mathscr{M} = \sum_{i \in I} Hom(\mathscr{W}_i, \mathscr{V}_i) \oplus \sum_{i,j \in I} Q_{ij} \otimes Hom(\mathscr{V}_i, \mathscr{V}_j)$

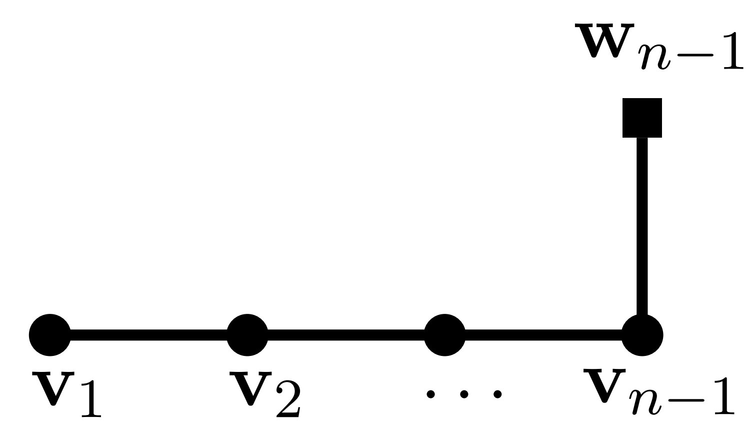
Degree $(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$



Quasimaps

Quasimap $f : \mathcal{C} - - \to X$ is described by collection of vector bundles $\mathscr{V}_i$ on $\mathcal{C}$ of ranks $\mathbf{v}_i$ with section $f \in H^0(\mathcal{C}, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar)$ satisfying $\mu = 0$

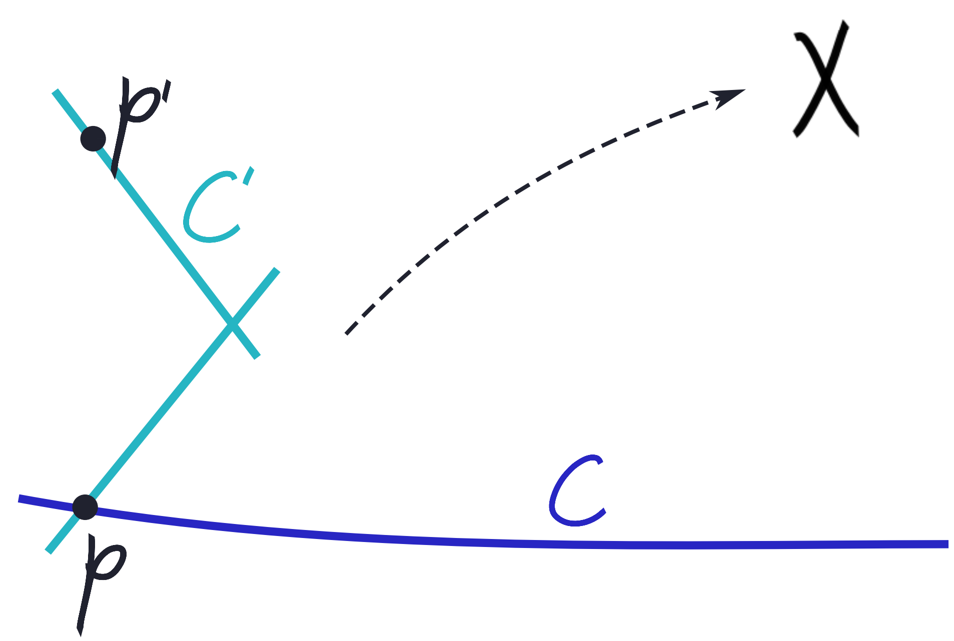
where $\mathscr{M} = \sum_{i \in I} Hom(\mathscr{W}_i, \mathscr{V}_i) \oplus \sum_{i,j \in I} Q_{ij} \otimes Hom(\mathscr{V}_i, \mathscr{V}_j)$



Degree $(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$

Evaluation map

$$\mathrm{ev}_p(f) = f(p) \in [\mu^{-1}(0)/G] \supset X$$



Stable if $f(p) \in X$

for all but finitely many singular points

Resolve to make *proper* ev map

Vertex Function (g=0)

Spaces of quasimaps admit an action of an extra torus $\mathbb{C}_q$ which scales the base $\mathbb{P}^1$ keeping two fixed points

Define **vertex function** with quantum (Novikov) parameters $z^{\mathbf{d}} = \prod_{i \in I} z_i^{d_i}$

$$V^{(\tau)}(z) = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_2,*}\Big(QM^{\mathbf{d}}_{\text{nonsing}\, p_2}, \widehat{\mathcal{O}}_{\text{vir}} \tau(\mathscr{V}_i|_{p_1})\Big) \in K_{\mathsf{T}_q}(X)_{loc}[[z]]$$

[Okounkov]
[PK Pushkar Smirnov Zeitlin]

Vertex Function (g=0)

Spaces of quasimaps admit an action of an extra torus $\mathbb{C}_q$ which scales the base $\mathbb{P}^1$ keeping two fixed points

Define **vertex function** with quantum (Novikov) parameters $z^{\mathbf{d}} = \prod_{i\in I} z_i^{d_i}$

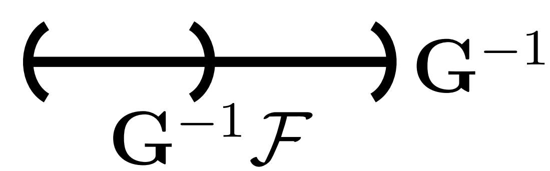
$$V^{(\tau)}(z) = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_2,*}\Big(QM^{\mathbf{d}}_{\mathrm{nonsing}\, p_2}, \widehat{\mathcal{O}}_{\mathrm{vir}} \tau(\mathscr{V}_i|_{p_1})\Big) \in K_{\mathsf{T}_q}(X)_{loc}[[z]]$$

[Okounkov]
[PK Pushkar Smirnov Zeitlin]

Define **quantum K-theory** as a ring with multiplication

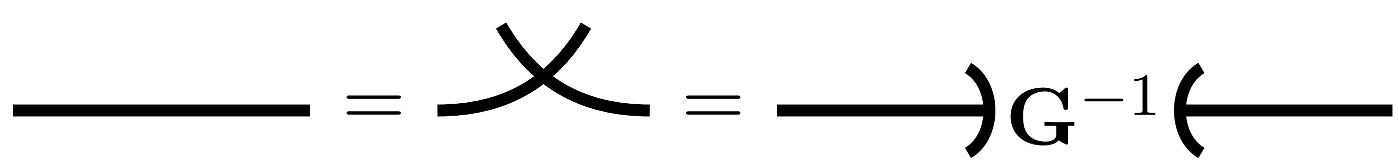
$$A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$$

$$\mathcal{F}\circledast = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_1,p_3 *}\Big(\mathrm{QM}^{\mathbf{d}}_{p_1,p_2,p_3}, \mathrm{ev}^*_{p_2}(\mathbf{G}^{-1}\mathcal{F})\widehat{\mathcal{O}}_{\mathrm{vir}}\Big)\, \mathbf{G}^{-1}$$



gluing

$$\mathcal{C}_0 = \mathcal{C}_{0,1} \cup_p \mathcal{C}_{0,2}$$

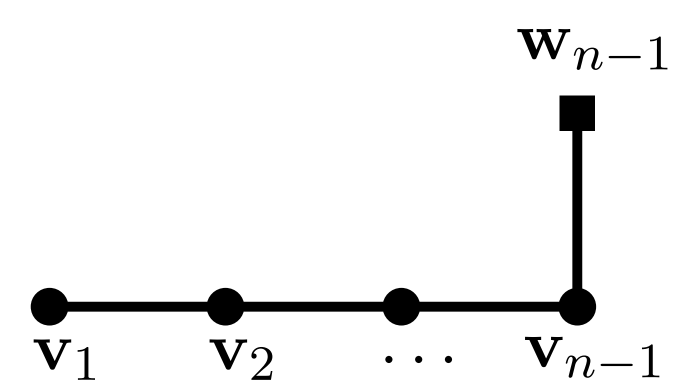


Vertex Functions

After classifying fixed points of space of nonsingular quasimaps we can compute the vertex

$$V_{\boldsymbol{p}}^{(\tau)}(z) = \sum_{d_{i,j}\in C} z^{\mathbf{d}} q^{N(\mathbf{d})/2}\, EHG \quad \tau(x_{i,j} q^{-d_{i,j}})$$

$$E = \prod_{i=1}^{n-1} \prod_{j,k=1}^{\mathbf{v}_i} \{x_{i,j}/x_{i,k}\}^{-1}_{d_{i,j}-d_{i,k}} \qquad x_{i,j} \in \{a_1, \dots a_{\mathbf{w}_n}\}$$

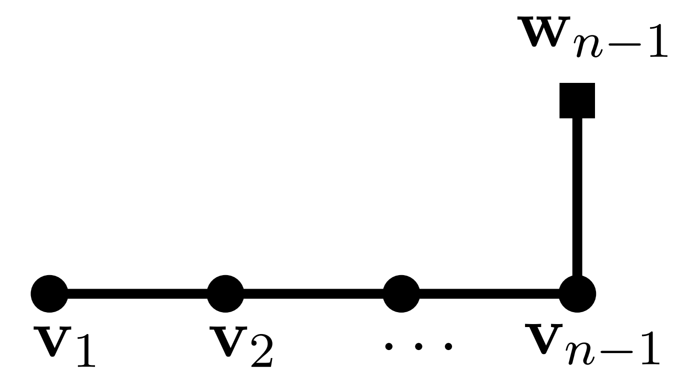


Vertex Functions

After classifying fixed points of space of nonsingular quasimaps we can compute the vertex

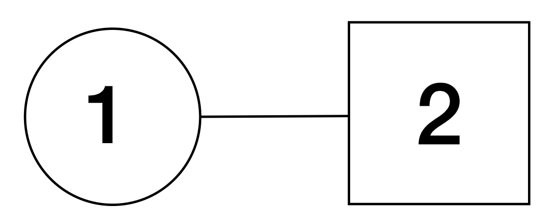
$$V_{\boldsymbol{p}}^{(\tau)}(z) = \sum_{d_{i,j}\in C} z^{\mathbf{d}} q^{N(\mathbf{d})/2}\, EHG \quad \tau(x_{i,j} q^{-d_{i,j}})$$

$$E = \prod_{i=1}^{n-1} \prod_{j,k=1}^{\mathbf{v}_i} \{x_{i,j}/x_{i,k}\}^{-1}_{d_{i,j}-d_{i,k}} \qquad x_{i,j} \in \{a_1, \dots a_{\mathbf{w}_n}\}$$



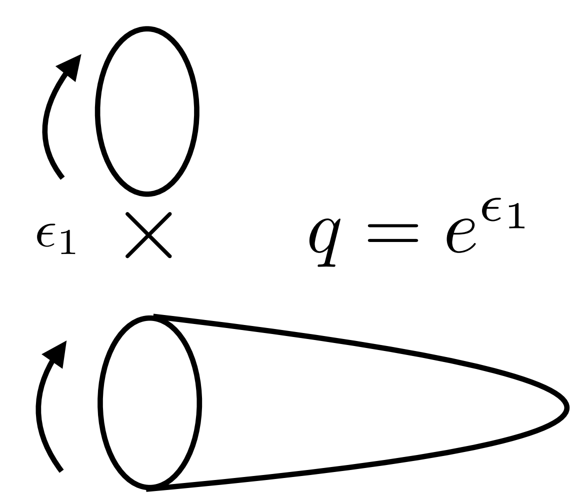
Vertex

$$V = {}_2\phi_1\left(\hbar, \hbar\frac{a_1}{a_2}, q\frac{a_1}{a_2}; q; z\right)$$



Vortex

$\mathcal{N} = 2^*$ quiver gauge theory on $X_3 = \mathbb{C}_{\epsilon_1} \times S^1_\gamma$



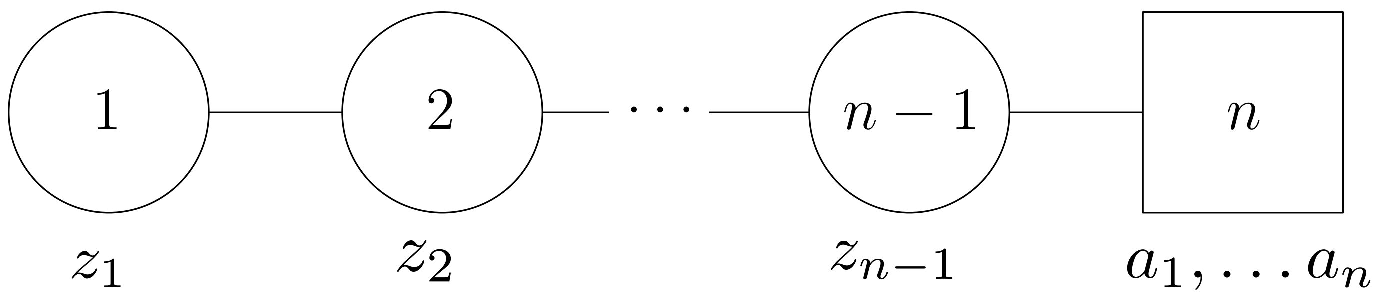
$q = e^{\epsilon_1}$

Lagrangian depends on twisted masses a_1, a_2

FI parameter z and U(1) R-symmetry fugacity $\log \hbar$

Difference Equations

[PK Pushkar Smirnov Zeitlin]



Ring relations

$$QK_T(T^*\mathbb{F}l_n) = \frac{\mathbb{C}[z_i^{\pm 1}, a_i^{\pm 1}, \hbar, q]}{\mathcal{I}_{\text{tRS}}}$$

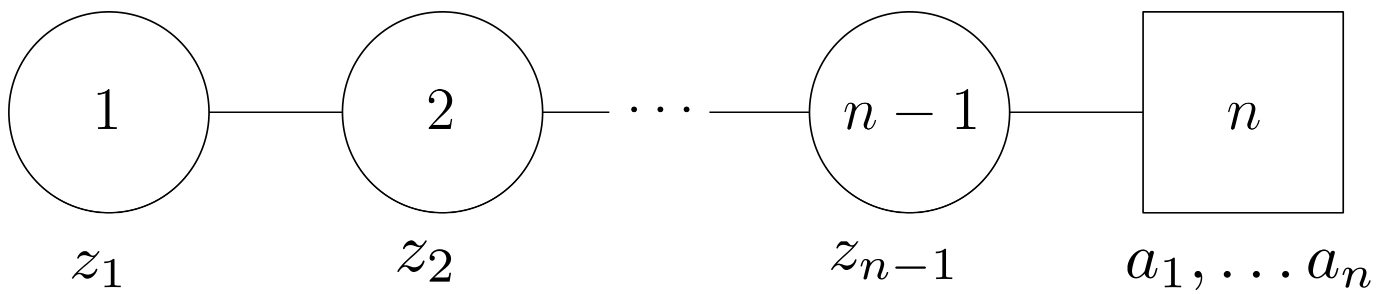
Let $X = T^*\mathbb{F}l_n$ Then K-theory vertex function satisfies equation of motion of trigonometric Ruijsenaars-Schneider model

$$\hat{H}_d V = e_d(z_1, \ldots, z_{n-1}) V$$

$$\hat{H}_d = \sum_{I \subset \{1,\ldots,n\}, |I|=d} \Big(\prod_{i\in I, j \notin I} \frac{a_i \hbar^{\frac{1}{2}} - a_j \hbar^{-\frac{1}{2}}}{a_i - a_j} \Big) \prod_{i \in I} T_i^q$$

Difference Equations

[PK Pushkar Smirnov Zeitlin]



Ring relations

$$QK_T(T^*\mathbb{F}l_n) = \frac{\mathbb{C}[z_i^{\pm 1}, a_i^{\pm 1}, \hbar, q]}{\mathcal{I}_{\mathrm{tRS}}}$$

Let $X = T^*\mathbb{F}l_n$ Then K-theory vertex function satisfies equation of motion of trigonometric Ruijsenaars-Schneider model

$$\hat{H}_d V = e_d(z_1, \dots, z_{n-1}) V$$

$$\hat{H}_d = \sum_{I \subset \{1,\dots,n\}, |I|=d} \Big(\prod_{i \in I, j \notin I} \frac{a_i \hbar^{\frac{1}{2}} - a_j \hbar^{-\frac{1}{2}}}{a_i - a_j} \Big) \prod_{i \in I} T_i^q$$

3d Mirror version (a.k.a. bispectral dual)

$$\hat{H}_d^! V = e_d(a_1, \dots, a_{n-1}) V$$

$$\hat{H}_d^!(a_i, \hbar, T_a^q) = \hat{H}_d(z_i/z_{i+1}, \hbar^{-1}, T_z^q)$$

Spherical DAHA

[Satoshi's talk]

Trigonometric Ruijsenaars-Schneider Hamiltonians form a maximal commuting subalgebra inside **spherical double affine Hecke algebra for gl(n)**

$$\{\hat{H}_1, \ldots, \hat{H}_n\} \subset \mathrm{DAHA}^{\mathfrak{S}_n}_{q,\hbar}(\mathfrak{gl}_n) =: \mathcal{A}_n$$

$\hat{H}_d$ are also known as *Macdonald* operators

Spherical DAHA

[Satoshi's talk]

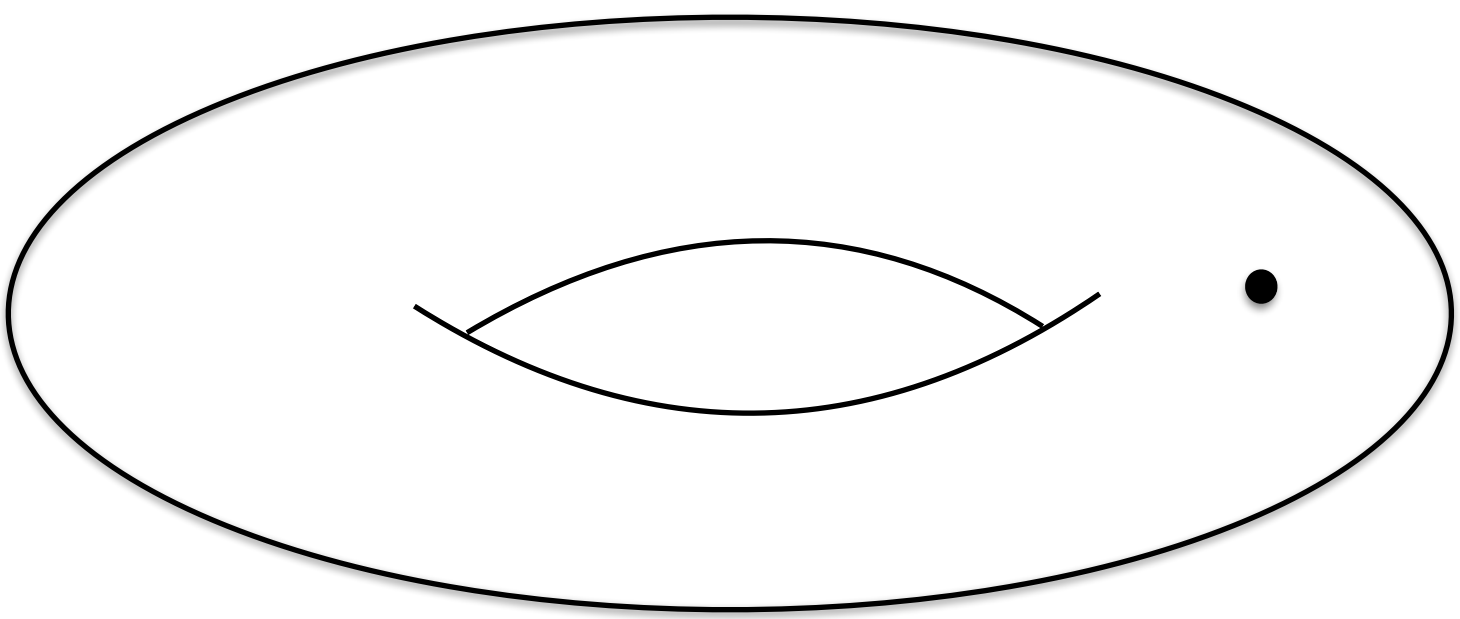
Trigonometric Ruijsenaars-Schneider Hamiltonians form a maximal commuting subalgebra inside **spherical double affine Hecke algebra for gl(n)**

$$\{\hat{H}_1, \ldots, \hat{H}_n\} \subset \mathrm{DAHA}_{q,\hbar}^{\mathfrak{S}_n}(\mathfrak{gl}_n) =: \mathcal{A}_n$$

$\hat{H}_d$ are also known as Macdonald operators

[Oblomkov]

Spherical gl(n) DAHA is a **deformation quantization** of the moduli space of flat GL(n;C) connections on a torus with one simple puncture



$$\mathcal{M}_n = \{A, B, C\}/GL(n;\mathbb{C})$$

$$ABA^{-1}B^{-1} = C$$

$$C = \mathrm{diag}(\hbar, \ldots, \hbar, \hbar^{1-n})$$

$$\mathcal{A}_n = \widehat{\mathbb{C}_J[\mathcal{M}_n]}$$

Line Operators and Branes

$\mathcal{M}_n$ is the moduli space of vacua in $\mathcal{N}$=2* gauge theory on $\mathbb{R}^3 \times S^1$ with gauge group U(n) and is described by VEVs of line operators wrapping the circle.

A and *B* are holonomies of *electric* and *magnetic* line operators

Line Operators and Branes

$\mathcal{M}_n$ is the moduli space of vacua in $\mathcal{N}$=2* gauge theory on $\mathbb{R}^3 \times S^1$ with gauge group U(n) and is described by VEVs of line operators wrapping the circle.

A and *B* are holonomies of *electric* and *magnetic* line operators

Omega background along real 2-plane $\mathbb{R}^2_q \times \mathbb{R} \times S^1$

Line operators are forced to stay at the tip of the cigar and slide along the remaining line, hence **non-commutativity**

Line Operators and Branes

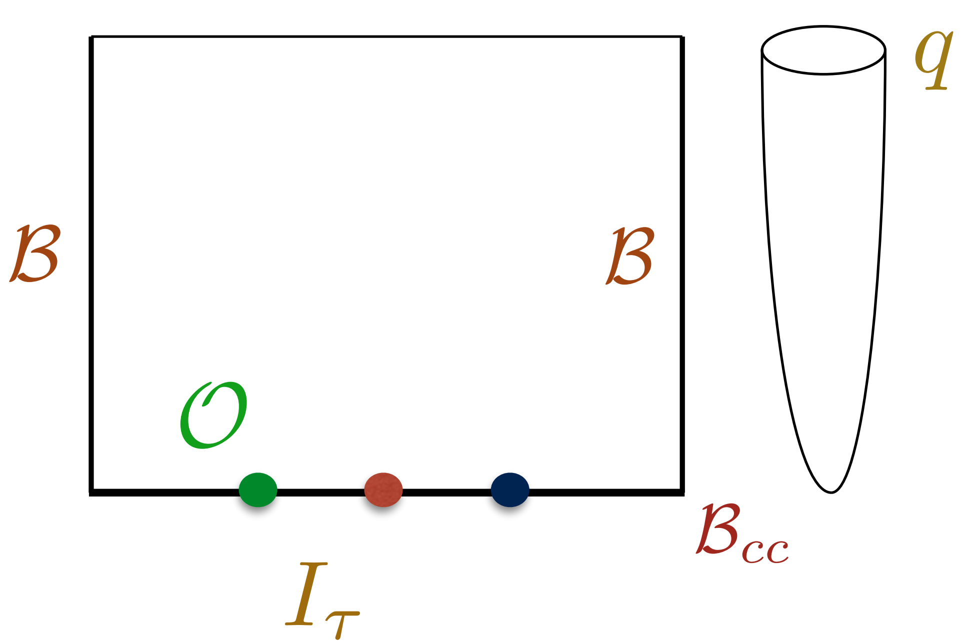
$\mathcal{M}_n$ is the moduli space of vacua in $\mathcal{N}$=2* gauge theory on $\mathbb{R}^3 \times S^1$ with gauge group U(n) and is described by VEVs of line operators wrapping the circle.

A and *B* are holonomies of *electric* and *magnetic* line operators

Omega background along real 2-plane $\mathbb{R}^2_q \times \mathbb{R} \times S^1$

Line operators are forced to stay at the tip of the cigar and slide along the remaining line, hence **non-commutativity**

[Gukov-Witten]
[Nekrasov-Witten]



algebra — open strings

$$\mathcal{A}_n = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc})$$

representations

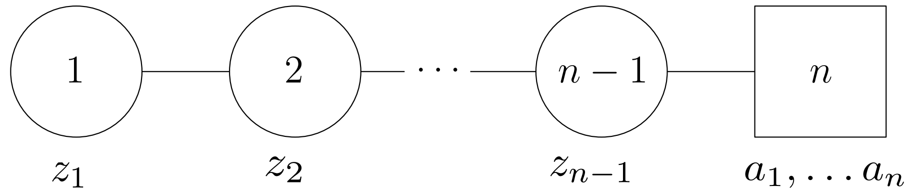
(Hilbert space of SUSY QM)

$$\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B})$$

[Satoshi's talk on Tuesday]

DAHA Reps

Start with a vertex function for T^*F_n

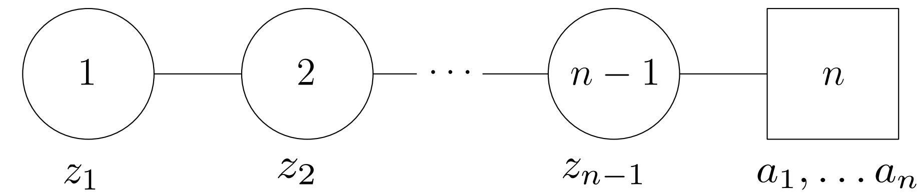


Specify equivariant parameters $a_k = q^{\lambda_k} \hbar^{n-k}$

q-hypergeometric series → Macdonald polynomials with $\hbar = t$

DAHA Reps

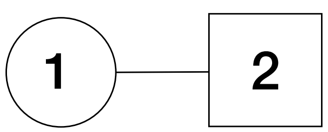
Start with a vertex function for T*Fn



Specify equivariant parameters $a_k = q^{\lambda_k} \hbar^{n-k}$

q-hypergeometric series ⟶ Macdonald polynomials with $\hbar = t$

E.g. k=2, n=2

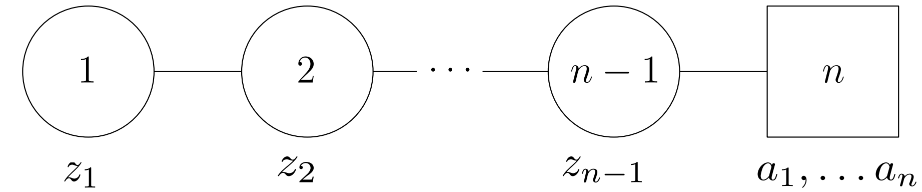


$$V(z; tq, q) = P_{(1,1)}(z|q,t)$$

$$V(z; tq^2, 1) = P_{(2,0)}(z|q,t)$$

DAHA Reps

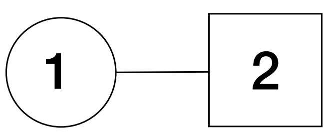
Start with a vertex function for T*Fn



Specify equivariant parameters $a_k = q^{\lambda_k} \hbar^{n-k}$

q-hypergeometric series → Macdonald polynomials with $\hbar = t$

E.g. k=2, n=2



$$V(z; tq, q) = P_{(1,1)}(z|q,t)$$

$$V(z; tq^2, 1) = P_{(2,0)}(z|q,t)$$

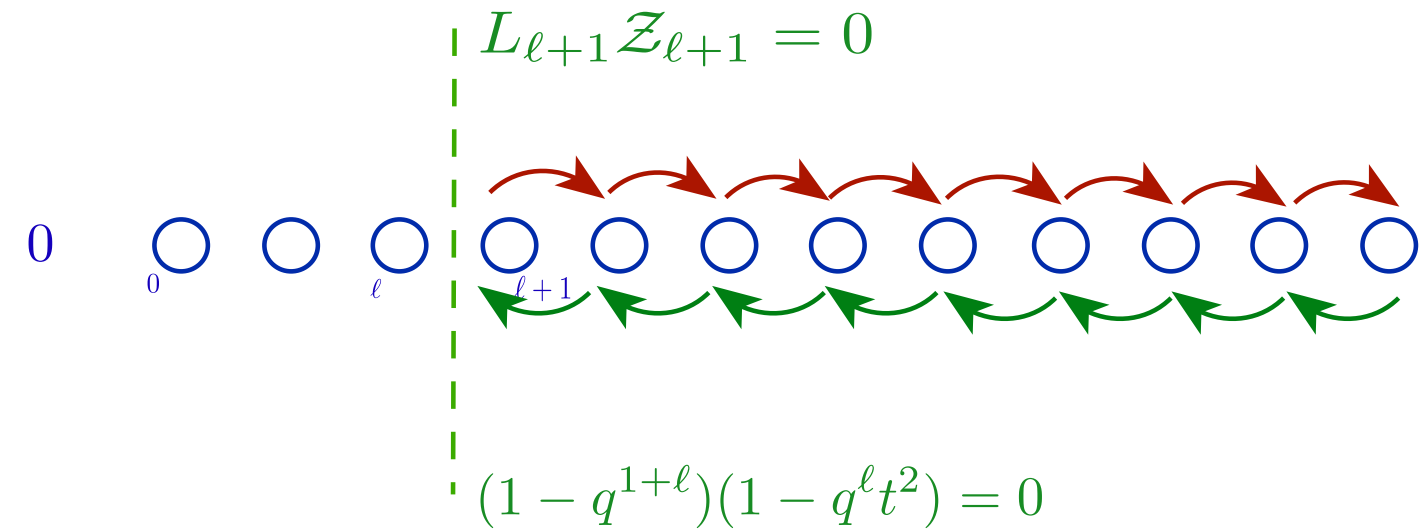
Raising and lowering operators of sl(2) DAHA

$$R_a = x + a_k^{-1} z$$

$$L_a = x + a_k z$$

$$R_a \mathcal{Z}_a = r_a \mathcal{Z}_{a+1}$$

$$L_a \mathcal{Z}_a = l_a \mathcal{Z}_{a-1}$$



Fock Space

Change of variables $$p_m = \sum_{l=1}^{n} z_l^m$$

Macdonald polynomials depend only on k and the partition

$$P_{\square\square} = \frac{1}{2}(p_1^2 - p_2)\,, \qquad P_{\substack{\square\\\square}} = \frac{1}{2}(p_1^2 - p_2) + \frac{1-qt}{(1+q)(1-t)}p_2$$

Fock Space

Change of variables $$p_m = \sum_{l=1}^{n} z_l^m$$

Macdonald polynomials depend only on k and the partition

$$P_{\square\square} = \frac{1}{2}(p_1^2 - p_2)\,, \qquad P_{\substack{\square \\ \square}} = \frac{1}{2}(p_1^2 - p_2) + \frac{1-qt}{(1+q)(1-t)} p_2$$

Starting with Fock vacuum $$|0\rangle$$

Construct Hilbert space $$a_{-\lambda}|0\rangle \longleftrightarrow p_\lambda$$

for each partition $$a_{-\lambda}|0\rangle = a_{-\lambda_1} \cdots a_{-\lambda_l}|0\rangle$$

Commutators $$[a_m, a_n] = m \frac{1-q^{|m|}}{1-t^{|m|}} \delta_{m+n,0}$$

DAHA Action

[PK to appear]

Vertex functions or quantum classes for X are elements of quantum K-theory of X. Equivalently we can view them as elements of equivariant K-theory of the space of quasimaps from P1 to X

$V \in K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)$ with maximal torus $T = \mathbb{T}(U(n) \times U(1)_\hbar \times U(1)_q)$.

Specification $a_k = q^{\lambda_k} t^{n-k}$ restricts us to the Fock space representation of (q,t)-Heisenberg algebra which is DAHA module

DAHA Action

[PK to appear]

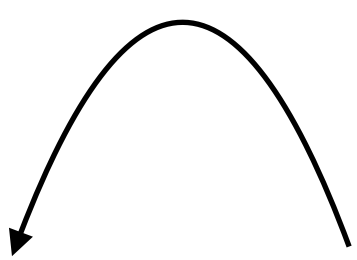
Vertex functions or quantum classes for X are elements of quantum K-theory of X. Equivalently we can view them as elements of equivariant K-theory of the space of quasimaps from P1 to X

$V \in K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)$ with maximal torus $T = \mathbb{T}(U(n) \times U(1)_\hbar \times U(1)_q).$

Specification $a_k = q^{\lambda_k} t^{n-k}$ restricts us to the Fock space representation of (q,t)-Heisenberg algebra which is DAHA module

In other words, we can define the following action

gl(n) DAHA



$$K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)\Big|_{a_k = q^{\lambda_k} t^{n-k}}$$

λ not more than n columns

DAHA Action

[PK to appear]

Vertex functions or quantum classes for X are elements of quantum K-theory of X. Equivalently we can view them as elements of equivariant K-theory of the space of quasimaps from P1 to X

$V \in K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)$ with maximal torus $T = \mathbb{T}(U(n) \times U(1)_\hbar \times U(1)_q)$.

Specification $a_k = q^{\lambda_k} t^{n-k}$ restricts us to the Fock space representation of (q,t)-Heisenberg algebra which is DAHA module

In other words, we can define the following action

gl(n) DAHA

$$K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)\Big|_{a_k = q^{\lambda_k} t^{n-k}} \xrightarrow{n \to \infty} K_{q,t}\left(\oplus_i \mathcal{M}^{\text{inst}}_{i,1}\right)$$

λ not more than n columns

$\mathbb{C}[p_1, p_2, \dots] \otimes \mathbb{C}[q,t]$

DAHA Action

[PK to appear]

Vertex functions or quantum classes for X are elements of quantum K-theory of X. Equivalently we can view them as elements of equivariant K-theory of the space of quasimaps from P1 to X

$V \in K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)$ with maximal torus $T = \mathbb{T}(U(n) \times U(1)_\hbar \times U(1)_q)$.

Specification $a_k = q^{\lambda_k} t^{n-k}$ restricts us to the Fock space representation of (q,t)-Heisenberg algebra which is DAHA module

In other words, we can define the following action

gl(n) DAHA

$$K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)\Big|_{a_k = q^{\lambda_k} t^{n-k}} \xrightarrow{n \to \infty} K_{q,t}\left(\oplus_i \mathcal{M}^{\text{inst}}_{i,1}\right)$$

λ not more than n columns

$\mathbb{C}[p_1, p_2, \dots] \otimes \mathbb{C}[q,t]$

DAHA Action

[PK to appear]

Vertex functions or quantum classes for X are elements of quantum K-theory of X. Equivalently we can view them as elements of equivariant K-theory of the space of quasimaps from P1 to X

$V \in K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)$ with maximal torus $T = \mathbb{T}(U(n) \times U(1)_\hbar \times U(1)_q)$.

Specification $a_k = q^{\lambda_k} t^{n-k}$ restricts us to the Fock space representation of (q,t)-Heisenberg algebra which is DAHA module

In other words, we can define the following action

gl(n) DAHA $\xrightarrow[\text{[Schiffmann Vaserot]}]{n \to \infty}$ Elliptic Hall alg

$K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)\Big|_{a_k = q^{\lambda_k} t^{n-k}} \xrightarrow{n \to \infty} K_{q,t}\left(\oplus_i \mathcal{M}^{\text{inst}}_{i,1}\right)$

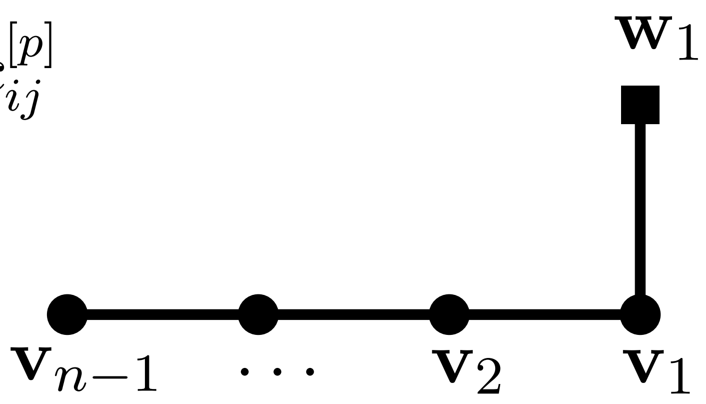
λ not more than n columns

$\mathbb{C}[p_1, p_2, \dots] \otimes \mathbb{C}[q, t]$

Quiver qW-algebra

Construction of qW algebra from free-boson representation of *extended* Nekrasov partition function [Kimura Pestun]

$$\mathcal{Z}_{\text{Nek}} = \widehat{\mathcal{Z}}_{\text{Nek}}|0\rangle \qquad [s_{i,p}, s_{j,p'}] = -\delta_{p+p',0}\frac{1}{p}\frac{1-q_1^p}{1-q_2^{-p}}c_{ij}^{[p]}$$



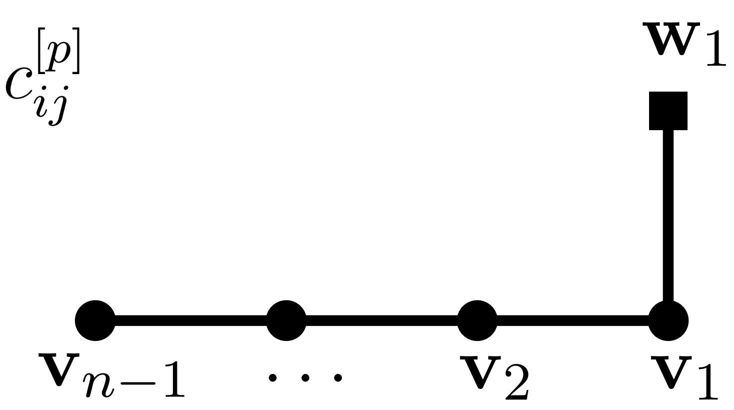
Start with quiver gauge theory on $\mathbb{C}_{q_1} \times \mathbb{C}_{q_2} \times S^1$

Quiver qW-algebra

Construction of qW algebra from free-boson representation of *extended* Nekrasov partition function [Kimura Pestun]

$$\mathcal{Z}_{\text{Nek}} = \widehat{\mathcal{Z}}_{\text{Nek}}|0\rangle \qquad [s_{i,p}, s_{j,p'}] = -\delta_{p+p',0}\frac{1}{p}\frac{1-q_1^p}{1-q_2^{-p}}c_{ij}^{[p]}$$

Start with quiver gauge theory on $\mathbb{C}_{q_1} \times \mathbb{C}_{q_2} \times S^1$



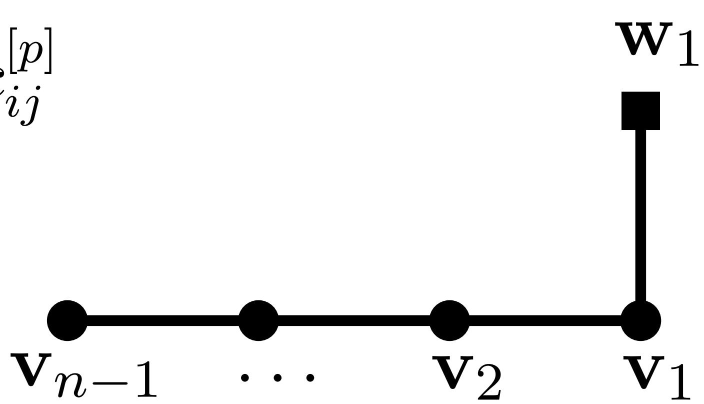
Moduli space of vacua is the space of A_{n-1} periodic monopoles with $\mathbf{w}_1$ Dirac singularities whose charges are given by the numbers of colors [Nekrasov Pestun Shatashvili]

Quiver qW-algebra

Construction of qW algebra from free-boson representation of *extended* Nekrasov partition function [Kimura Pestun]

$$\mathcal{Z}_{\text{Nek}} = \widehat{\mathcal{Z}}_{\text{Nek}}|0\rangle \qquad [s_{i,p}, s_{j,p'}] = -\delta_{p+p',0}\frac{1}{p}\frac{1-q_1^p}{1-q_2^{-p}}c_{ij}^{[p]}$$

Start with quiver gauge theory on $\mathbb{C}_{q_1} \times \mathbb{C}_{q_2} \times S^1$



Moduli space of vacua is the space of A_{n-1} periodic monopoles with $\mathbf{w}_1$ Dirac singularities whose charges are given by the numbers of colors [Nekrasov Pestun Shatashvili]

Quantization of this moduli space in carefully chosen complex structure gives qW(q1,q2) algebra modulo Virasoro constraints!

$$\widehat{\mathbb{C}}[\mathcal{M}_{\text{mon}}] = \frac{qW_{q_1,q_2}}{\text{Vir}(\mathbf{v}_1,\ldots,\mathbf{v}_{n-1})} \qquad T_{i,-k}|\psi\rangle = 0, \quad k > \mathbf{v}_i$$

Virasoro constrains can be removed by taking $\mathbf{v}_i \to \infty$

Gauge Origami

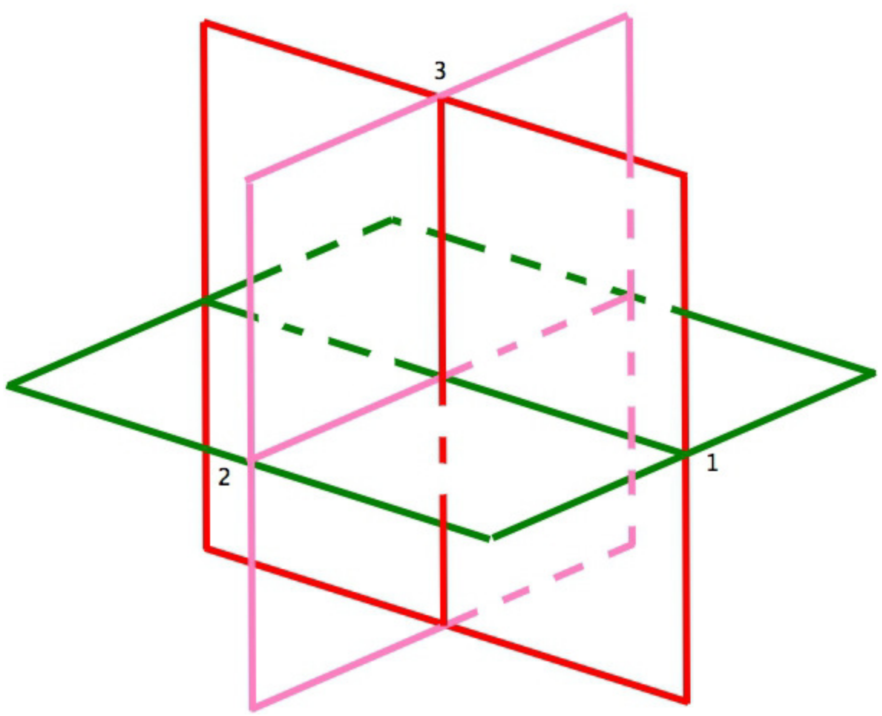
[Nekrasov]

Type IIB on Calabi-Yau 4 $\mathcal{X}_4 \times \Sigma$

singular hypersurface $Z_2 \subset \mathcal{X}_4$

Local model: $\cup_{a<b} \mathbb{C}^2_{ab} \subset \mathbb{C}^4$

For example, when $1 \leq a, b \leq 3$



$$\mathcal{X}_4 = \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times \mathbb{C}_{\epsilon_3} \times \mathbb{C}_{\epsilon_4} \qquad \sum_a \epsilon_a = 0$$

Gauge Origami

[Nekrasov]

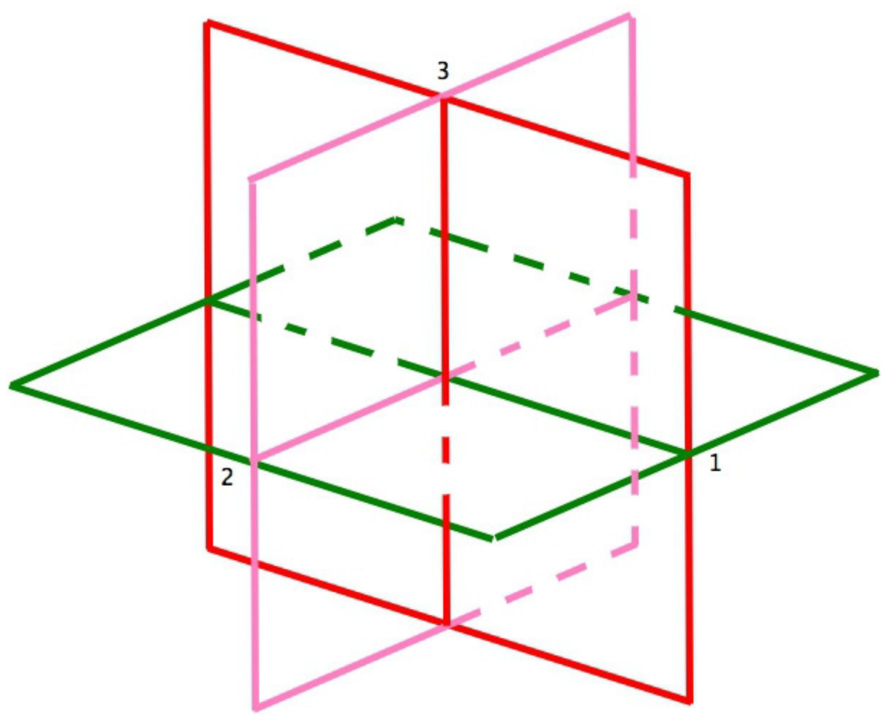
Type IIB on Calabi-Yau 4 $\mathcal{X}_4 \times \Sigma$

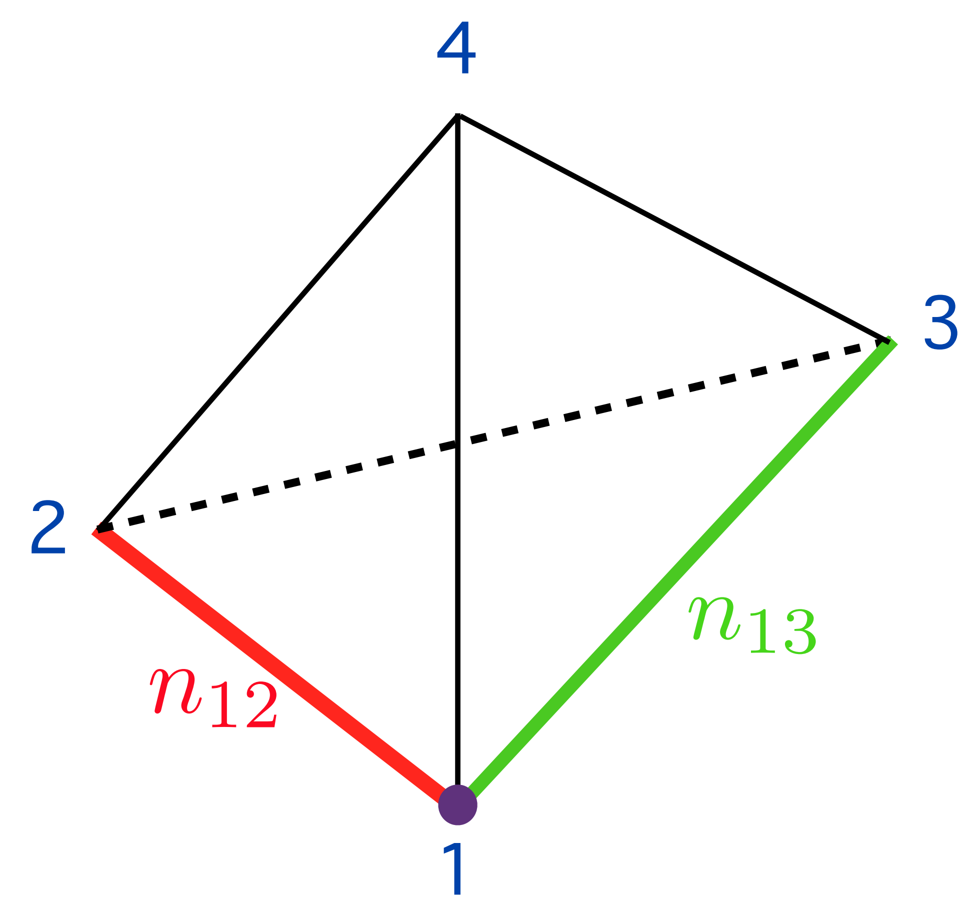
singular hypersurface $Z_2 \subset \mathcal{X}_4$

Wrap D3 branes on 2-planes in Z_2 pointlike on Σ

Local model: $\cup_{a<b} \mathbb{C}^2_{ab} \subset \mathbb{C}^4$

For example, when $1 \leq a, b \leq 3$





$$\mathcal{X}_4 = \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times \mathbb{C}_{\epsilon_3} \times \mathbb{C}_{\epsilon_4} \qquad \sum_a \epsilon_a = 0$$

Folded Instantons

Take $n_{12}=n$, $n_{13}=2$

In the presence of Abelian orbifold

$$\Gamma = \mathrm{diag}(1\ \omega\ 1\ \omega^{-1})$$

$$\epsilon_1\,\epsilon_2\,\epsilon_3\,\epsilon_4$$

$$\omega^n = 1$$

Folded Instantons

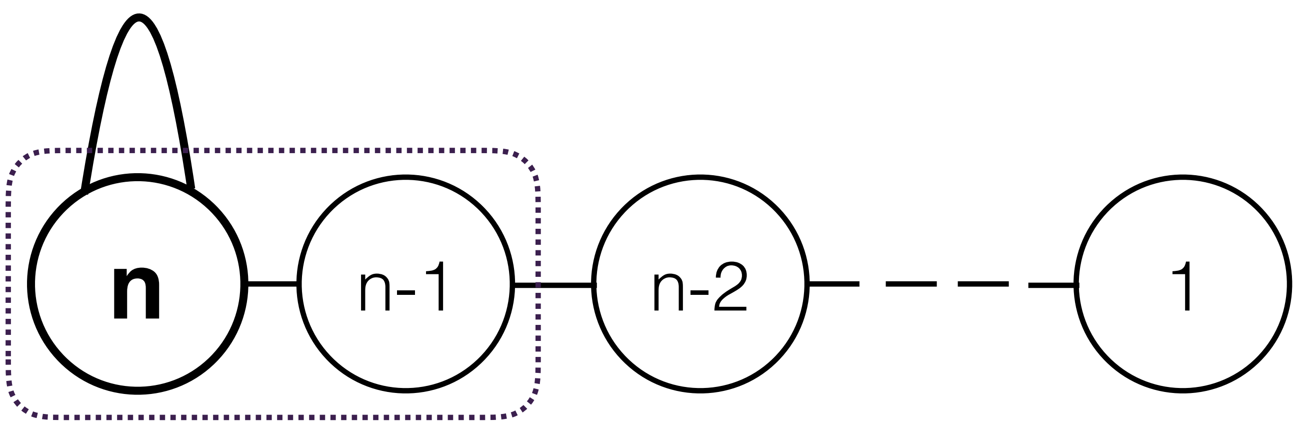
Take n_{12}=n, n_{13} = 2

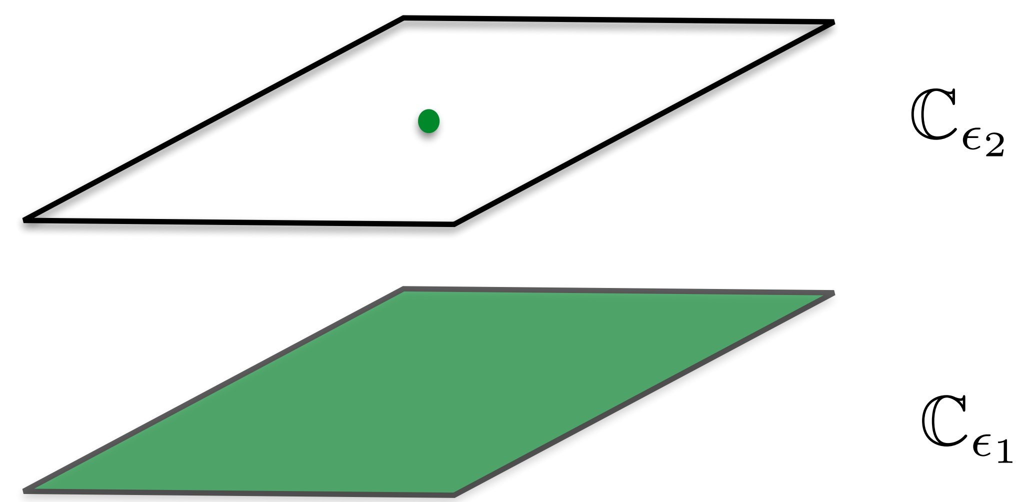
In the presence of Abelian orbifold

$$\Gamma = \mathrm{diag}(1\ \omega\ 1\ \omega^{-1}) \quad \omega^n = 1$$

$$\epsilon_1\,\epsilon_2\,\epsilon_3\,\epsilon_4$$

Produces U(n) $\mathcal{N}$=1* theory on $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2}$ with maximal monodromy defect along $\mathbb{C}_{\epsilon_1}$ and adjoint mass ϵ_3





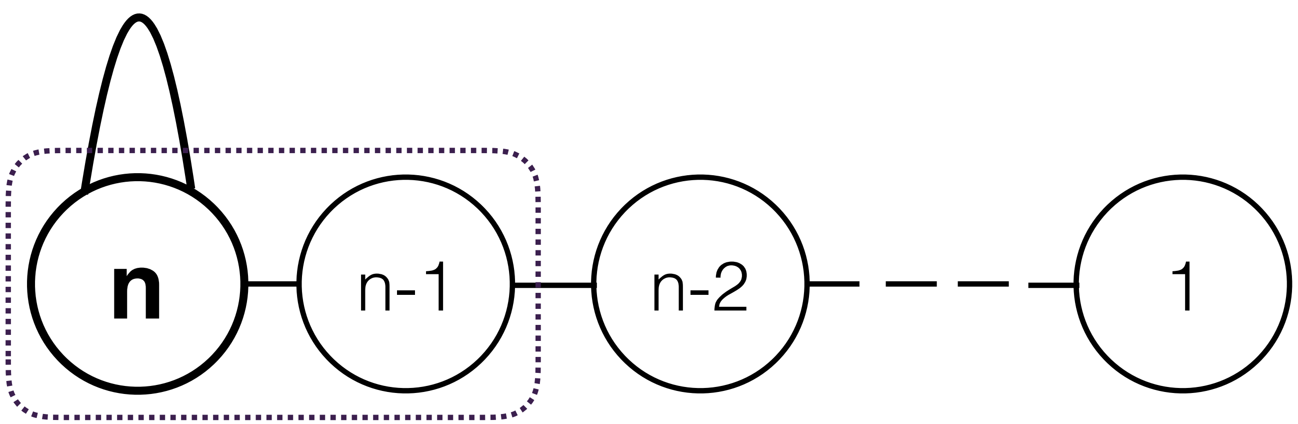
Folded Instantons

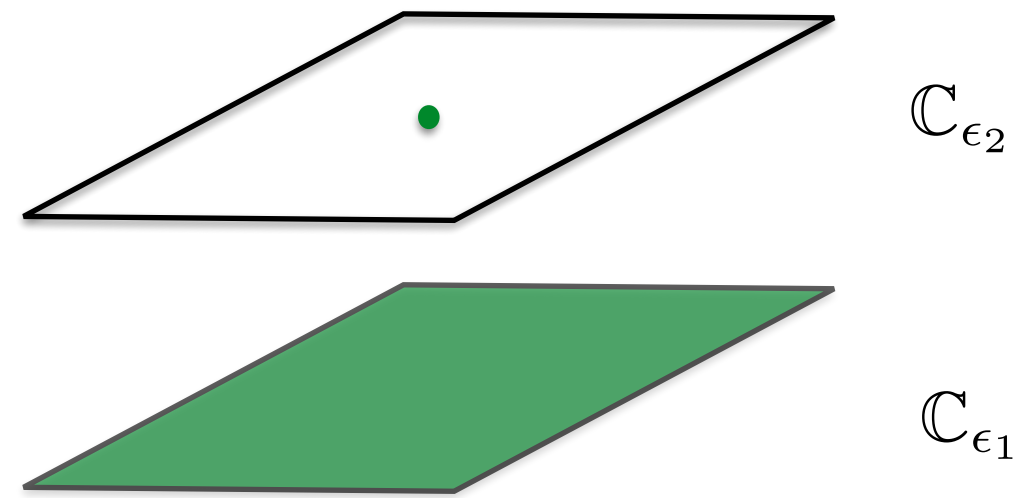
Take n_{12}=n, n_{13} = 2

In the presence of Abelian orbifold

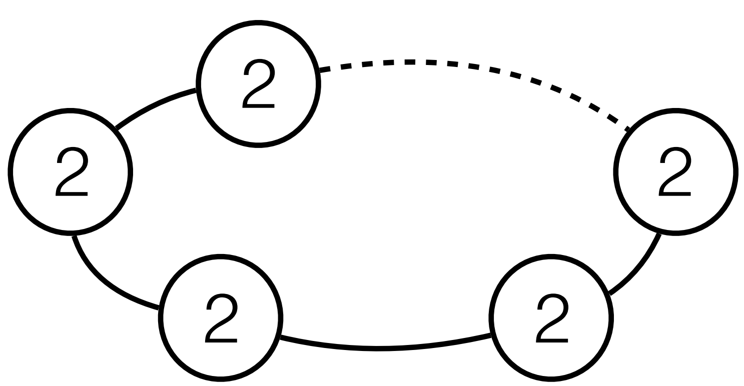
$$\Gamma = \mathrm{diag}(1 \ \omega \ 1 \ \omega^{-1}) \qquad \omega^n = 1$$
$$\epsilon_1 \, \epsilon_2 \, \epsilon_3 \, \epsilon_4$$

Produces U(n) $\mathcal{N}$=1* theory on $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2}$ with maximal monodromy defect along $\mathbb{C}_{\epsilon_1}$ and adjoint mass ϵ_3





Together with necklace quiver with n U(2) gauge groups on $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_3}$

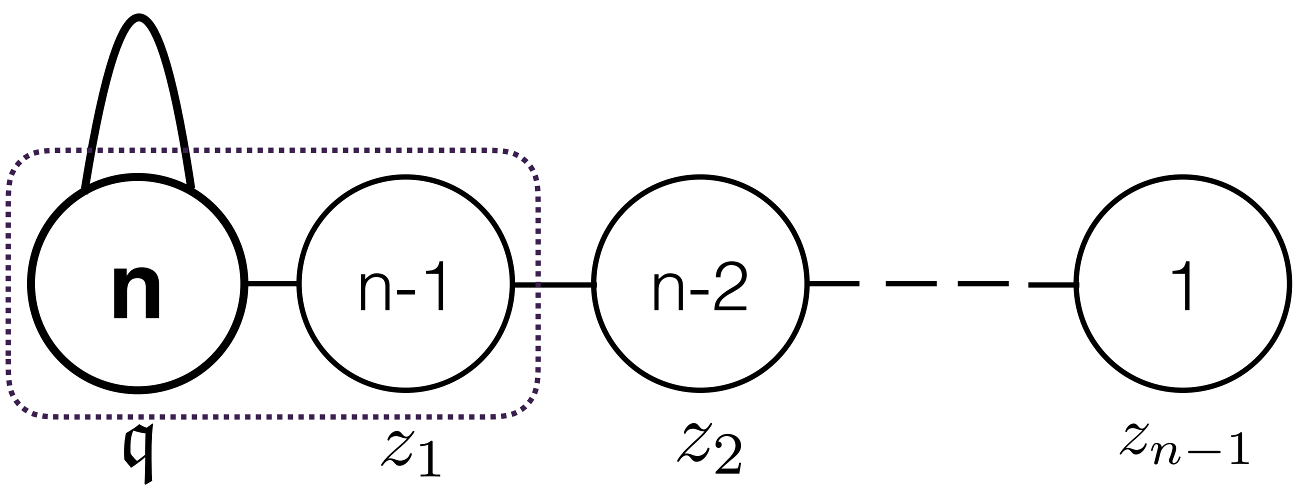


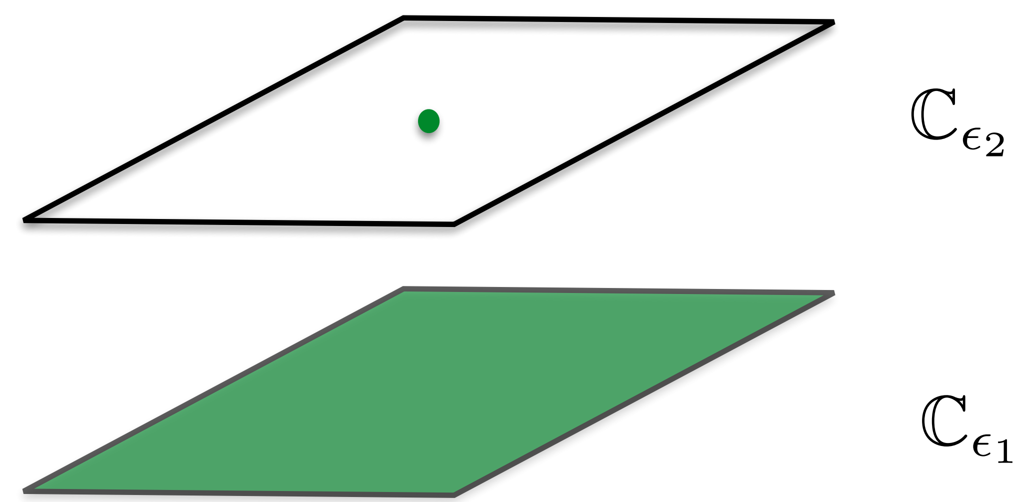
Folded Instantons

Take $n_{12}=n$, $n_{13} = 2$

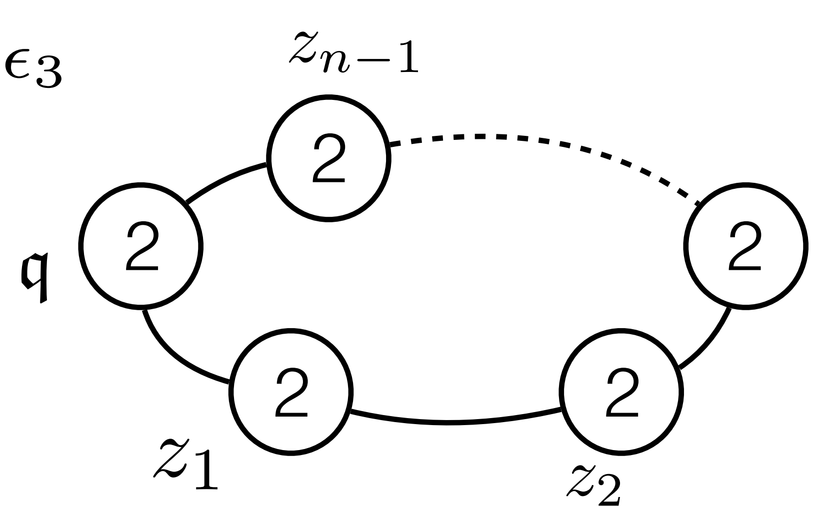
In the presence of Abelian orbifold $\Gamma = \mathrm{diag}(1\ \omega\ 1\ \omega^{-1})$ acting on $\epsilon_1\, \epsilon_2\, \epsilon_3\, \epsilon_4$, with $\omega^n = 1$

Produces U(n) $\mathcal{N}$=1* theory on $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2}$ with maximal monodromy defect along $\mathbb{C}_{\epsilon_1}$ and adjoint mass ϵ_3





Together with necklace quiver with n U(2) gauge groups on $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_3}$



Gauge coupling constants

qW-algebra as large-n limit

Origami partition function combines instanton and perturbative data of both theories

$$\mathcal{Z}^{\Gamma} = \mathcal{Z}^{\text{pert}} \cdot \sum_{\lambda} \left[\prod_{\omega \in \Gamma^{\vee}} \mathfrak{q}_{\omega}^{k_{\omega}} \right] \varepsilon \left[-\tilde{T}_{\lambda}^{\Gamma} \right]$$

Taking limits $\mathfrak{q} \to 0\,, \qquad \epsilon_2 \to 0$

we get 3d quiver defect gauge theory T*Fln on $\mathbb{C}_{\epsilon_1} \times S^1$

and finite linear 5d quiver on $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_3} \times S^1$

qW-algebra as large-n limit

Origami partition function combines instanton and perturbative data of both theories

$$\mathcal{Z}^{\Gamma} = \mathcal{Z}^{\text{pert}} \cdot \sum_{\lambda} \left[\prod_{\omega \in \Gamma^{\vee}} \mathfrak{q}_{\omega}^{k_{\omega}} \right] \varepsilon \left[-\tilde{T}_{\lambda}^{\Gamma} \right]$$

Taking limits $\mathfrak{q} \to 0 \,, \qquad \epsilon_2 \to 0$

we get 3d quiver defect gauge theory T*Fln on $\mathbb{C}_{\epsilon_1} \times S^1$

and finite linear 5d quiver on $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_3} \times S^1$

Locus $a_k = q_1^{\lambda_k} q_3^{n-k}$ truncates vortex functions to polynomials and simultaneously Higgses the 5d theory (truncates instanton series)

$$[a_i, a_j] = \frac{1}{j} \delta_{i+j,0} \frac{1-q_1^{|j|}}{1-q_2^{|j|}}$$

Elliptic Deformation

[PK Sciarappa]

If we don't take the limit $\mathfrak{q} \to 0$ trigonometric integrable system is promoted to elliptic RS model

eRS Hamiltonian *eigenvalues coincide with eigenvalues of the* ***quantum multiplication*** *operator in* ***quantum K-theory*** *ring of the instanton moduli space (Hilbert Scheme of points).*

$$\left\langle W_{\Box}^{U(n)} \right\rangle \Big|_{\lambda} \sim \mathcal{E}_1^{(\lambda)} = 1 - (1-q)(1-t^{-1}) \sum_s \sigma_s \Big|_{\lambda}$$

sigmas are determined by Bethe Ansatz equations for ADHM quiver

Elliptic deformation — Quantization

What's next?

Add more equivariant parameters
From 4 to 5 to 6 dimensions
From *cohomology* to *K-theory* to *elliptic* cohomology
What is the maximal number of parameters? 5?

Connection to Higgs branch approach by Beem and Rastelli
The VOA is recovered by passing to cohomology of a BRST-like operators which respects Higgs branch

Higher dimensional CFTs and Higher Spin Theories by Vasiliev [Gopakumar Gaberdiel]
qW-algebra structure was recently found in HS theories