q-opers & Quantum/Classical duality

Talk @ UC Davis QMAP seminar 11/9/18

almost complete w/ A Zeitlin, D Sage (LSU)

⓪ Motivation and goals

- (quantum) (q-) geometric Langlands correspondence

$D_\hbar(\mathrm{Bun}_G) \leftrightarrow D_{\frac{1}{m\hbar}}(\mathrm{Bun}_{{}^LG})$ — Bun_G on X - compact Riemann surface

Classical (critical level) $\hbar \to 0$: $D_{\mathrm{crit}}(\mathrm{Bun}_G) \leftrightarrow \mathcal{L}oc_{{}^LG}$ — moduli stack of flat ${}^LG$-bundles

In this talk we'll be talking about difference version of this

q - Geometric Langlands

- Recent progress in enumerative algebraic geometry and integrable systems (I gave my 1st talk last quarter)

Okounkov et al; my recent work w/ Zeitlin

Quantum K-theory of symplectic resolutions, dualities between integrable systems which arise in gauge/string theory

Brief history

- Ed Frenkel pointed out importance of opers to Langlands (07') — learned it with enumerative geometry
- Drinfeld & Sokolov (85) studied them locally as gauge equivalence classes of certain diff operators valued in $\mathfrak{g}$; L-operators of generalized KdV
- Belinson, Drinfeld (05') made a coordinate-independent definition
- Frenkel (07') Miura structure

① SL(2) opers

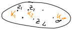
1.1 Def A $GL(2)$-oper on $\mathbb{P}^1$ is a triple $(E, \nabla, \mathcal{L})$, where E - rank-2 vector bundle on $\mathbb{P}^1$, connection $\nabla\colon E \to E\otimes K$, $\mathcal{L}$ - line subbundle such that induced map $\bar\nabla\colon \mathcal{L} \to E/\mathcal{L}\otimes K$ is an iso

If the structure group can be reduced to $SL(2)$ we can get an $SL(2)$ oper.

Pick a trivialization on E. In the vicinity of any point z we have

$$s(z)\wedge\nabla_z s(z) \neq 0, \qquad \nabla_z = \partial_z\cdot\nabla$$

Talk about $SL(2)$-opers w/ regular singularities and trivial monodromies around singular points



An $SL(2)$ q-oper w/ sing at $z_1, z_2, \dots z_L, \infty$ of weights $k_1, k_2, \dots k_\infty$ is the same as above, when $\bar\nabla$ is an iso everywhere except $z_1 \dots z_L, \infty$ where it has order of vanishing k_i

$$s(z)\wedge\nabla_z s(z) \sim (z-z_i)^{k_i} \qquad (1)$$

Trivial monodromy condition ⇒ after an appropriate gauge transformation A is trivial on $\mathbb{P}^1\setminus\infty$ subbundle $\mathcal{L}$ has section

$$s(z) = \begin{pmatrix} q_+(z) \\ q_-(z)\end{pmatrix}, \quad q_+, q_- \text{ have no common roots}$$

then (1) reads

$$q_+\partial_z q_- - \partial_z q_+ q_- = \rho(z), \quad \rho(z) = \prod_{i=1}^{L}(z-z_i)^{k_i} \qquad (2)$$

By a gauge transformation we can make $\deg(q_-) < \deg(q_+)$ and q_- is monic (always possible)

Nondegenerate ⇒ none of z_i's are roots of q_- ⇒ each root of q_- has multiplicity 1

Let $k = \sum_{i=1}^{L} k_i = \deg\rho(z)$ ⇒ $\deg(q_-) + \deg(q_+) = k+1$ ⇒ $\deg(q_-) = \ell_- < \frac{k}{2}$

Rewrite (2) as

$$\partial_z\left(\frac{q_+(z)}{q_-(z)}\right) = -\frac{\rho(z)}{q_-(z)^2}$$

Compute residues at each w_i we get

$$\sum_{m=1}^{L}\frac{k_m}{z_m - w_i} = \sum_{j\neq i}\frac{2}{w_j - w_i}, \quad i = 1,\dots,\ell_-$$

Bethe equations for $sl(2)$ Gaudin model at level $k-2\ell_- > 0$

to study ∞ do analytical transform $z \to \frac{1}{z}$

Theorem 1 There is a 1-1 correspondence between spectrum of the sl_2 Gaudin model and space of non-degenerate sl_2 opers w/ trivial monodromy and regular singularities

1.2 Miura structures

Relax the condition of nonnegative level $K - 2\ell_-$.

Miura oper $(E, \nabla, \mathcal{L})$ together with additional subbundle $\hat{\mathcal{L}}$ preserved by ∇

$\mathcal{L}, \hat{\mathcal{L}}$ span E everywhere except maybe finitely many points.

Thus for any oper we have a family of Miura opers parameterized by G/B flag [Frenkel]

Pick $\hat{\mathcal{L}}$ being generated by $\hat{s} = \begin{pmatrix}1\\0\end{pmatrix}$

Theorem 2 There is a 1-1 relation between solutions of Bethe Ansatz equations of the Gaudin model and $SL(2)$ Miura opers w/ reg sing & trivial monodromies with weights at finite points z_1, z_2

1.3 Connections and differential operators

Eigenvalues of Gaudin Hamiltonian directly from oper

Apply $SL(2)$-gauge transformation $g(z) = \begin{pmatrix} q_- & -q_+ \\ 0 & q_-^{-1}\end{pmatrix}$ so that $g(z)s(z) = \begin{pmatrix}0\\1\end{pmatrix}$ to a trivial connection

$$A = -(\partial_z g)g^{-1} = \begin{pmatrix} -\frac{\partial_z q_-}{q_-} & -\rho(z) \\ 0 & \frac{\partial_z q_-}{q_-}\end{pmatrix} \xrightarrow[\text{gauge transform } \begin{pmatrix}\rho^{-1/2} & 0\\ 0 & \rho^{1/2}\end{pmatrix}]{\text{diagonal}} A(z) = \begin{pmatrix} -u(z) & -1 \\ 0 & u(z)\end{pmatrix},$$

where $u(z) = -\frac{\partial_z \rho}{2\rho} + \frac{\partial_z q_-}{q_-} = -\sum_m \frac{k_m/2}{z - z_m} + \sum_i \frac{1}{z - w_i}$

Apply Miura transform $g(z) = \begin{pmatrix}1 & 0\\ u & 1\end{pmatrix}$ so the new connection read $B(z) = \begin{pmatrix}0 & -1\\ -t(z) & 0\end{pmatrix}$,

where $t(z) = \partial_z u(z) + u^2(z) = \sum_m \frac{k_m(k_m+2)/4}{(z - z_m)^2} + \sum \frac{c_m}{z - z_m}$ ← eigenvalues of Gaudin Hamiltonians

$$c_m = k_m \left(\sum_{n \neq m} \frac{k_n}{z_m - z_n} - \sum_{i=1}^{\ell} \frac{1}{z_m - w_i} \right)$$

Horizontal section $f = \begin{pmatrix}f_1\\f_2\end{pmatrix}$ to $B(z)$ is determined by $\left(\partial_z - B(z)\right)\begin{pmatrix}f_1\\f_2\end{pmatrix} = 0$

$(\partial_z^2 - t(z)) f_1(z) = 0$ ← projective connection

1.4 Irregular singularities [FFTL, FFR]

$A \sim d + \begin{pmatrix}a & \\ & -a\end{pmatrix} dz$ on $\mathbb{P}^1 \setminus \infty$ section $s(z) = e^{-\begin{pmatrix}a & \\ & -a\end{pmatrix} z}\begin{pmatrix}q_+\\q_-\end{pmatrix}$

then Bethe equations will take the following form $-2a + \sum_m \frac{k_m}{z_m - w_i} = \sum_{j \neq i} \frac{2}{w_j - w_i}$

Theorem 3 There is a 1-to-1 correspondence between solutions of inhomogeneous Bethe equations and nondegenerate $SL(2)$ opers w/ regular singularities at z_1, z_2 and w/ a double pole w/ 2-residue $-\begin{pmatrix}a & \\ & -a\end{pmatrix}$

2) Q-opers

Let $q \in \mathbb{C}^\times$, consider E^q – pull back of E – vector bundle over $\mathbb{P}^1$ under map $z \mapsto qz$

2.1

Consider map of vector bundles $A \; E \to E^q$ gives linear map of given bases $E_z \longrightarrow E_{qz}$

q-gauge transformations / change of trivialization by $g(z)$: $A(z) \mapsto g(qz) A(z) g^{-1}(z)$

Let $D_q \; E \to E^q$ be $S(z) \mapsto S(qz)$ then $D_q(S) = AS$

Def A meromorphic $(GL(N), q)$ – connection over $\mathbb{P}^1$ is (E, A), where E is a trivializable vect bundle of rank N over $\mathbb{P}^1$ and A is a meromorphic section of the sheaf $Hom_{\mathcal{O}_{\mathbb{P}^1}}(E, E^q)$ ν for which $A(z)$ is invertible. If there exists a trivialization where $\det A(z) = 1$ then we have an $(SL(N), q)$ oper

Now look at $SL(2)$

Def 1) $A(GL(2), q)$ – oper is a triple $(E, A, \mathcal{L})$, where (E, A) – $GL(2)$ q-connection; $\mathcal{L}$ is a line subbundle such that the induced map $\bar{A} \; \mathcal{L} \to (E/\mathcal{L})^q$ is an iso

2) $(SL(2), q)$ – oper w/ regular singularities $z_1, z_2, \; z_L \neq 0, \infty$ w/ weights $k_1, \; k_L$ is a meromorphic $(SL(2), q)$ – oper for which $\bar{A}$ is an iso everywhere on $\mathbb{P}^1 \setminus \{0, \infty\}$ except the points $z_m, q^{-1} z_m, q^{-2} z_m, \; q^{-k_m+1} z_m, \; m = 1, .. L$ $\quad S(qz) \wedge A(z) S(z) \neq 0$

3) An $(SL(2), q)$ – oper $(E, A, \mathcal{L})$ w/ reg singularities is called a Z-twisted q-oper if A is gauge equivalent to Z^{-1}, where $Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$

4) Miura q-oper $(E, A, \mathcal{L}, \hat{\mathcal{L}})$, where $\hat{\mathcal{L}}$ is preserved by A

2.2 Q-Wronskian and Bethe Ansatz

Theorem 4 There is a 1-to-1 correspondence between the set of non-degenerate solutions of SL_2 XXZ Bethe equations and the set of non-degenerate Z-twisted $(SL(2), q)$-opers w/ regular sing at $z_1, \; z_L \neq 0, \infty$ w/ weights $k_1, \; k_L$ and can be represented by meromorphic q-connection

$$\left(D_q^2 - T(qz) D_q - \frac{\rho(qz)}{\rho(z)} \right) f_1 = 0$$

Proof $S(qz) \wedge A(z) S(z) = \rho(z)$

$$\zeta^{-1} Q_+(z) Q_-(qz) - \zeta Q_+(qz) Q_-(z) = \prod_{m=1}^{L} \prod_{j=0}^{k_m - 1} (z - q^{-j} z_m)$$

Can always make $Q_-(z) = \prod_{i=1} (z - w_i)$

assume that q-lattices of z_a, w_i do not overlap

Then

$$\prod_{m=1}^{L} \frac{w_i - q^{1-k_m} z_m}{w_i - q z_m} = \zeta^{-2} q^{\ell - k} \prod_{j=1}^{\ell} \frac{q w_i - w_j}{w_i - q w_j}, \quad i = 1, \; \ell$$

Analogously get XXZ Hamiltonians (transfer matrix)

$$A(z) = \begin{pmatrix} a(z) & \rho(z) \\ 0 & a^{-1}(z) \end{pmatrix}, \quad a(z) = \zeta^{-1} Q_-(qz) Q_-^{-1}(z) \implies \hat{A}(z) = \begin{pmatrix} 0 & \rho(z) \\ -\rho^{-1}(z) & T(z)\rho^{-1}(qz) \end{pmatrix}$$

$$T(z) = \zeta^{-1} \rho(q^{-1}z) \frac{Q_-(qz)}{Q(z)} + \zeta \rho(z) \frac{Q_-(q^{-1}z)}{Q_-(z)}$$ – eigenvalue of the XXZ transfer matrix

23 Trigonometric Ruijsenaars-Schneider Model w/ 2 degrees of freedom

Take $\deg Q_{\pm}(z) = 1$, $\rho(z) = (z-z_+)(z-z_-)$

$Q_- = z - p_-$, $Q_+ = c(z - p_+)$, $c = q^{-1}(\zeta - \zeta^{-1})$

From quantum Wronskian relation

$$z^2 - \frac{z}{q}\left[\frac{\zeta - q\zeta^{-1}}{\zeta - \zeta^{-1}} p_+ + \frac{q\zeta - \zeta^{-1}}{\zeta - \zeta^{-1}} p_-\right] + \frac{p_+ p_-}{q} = (z - z_+)(z - z_-)$$

from where tRS Hamiltonians

$H_1 = q(z_+ + z_-)$

$H_2 = z_+ z_-$

qWr $\Leftrightarrow$ $\det\left(z\mathbb{1} - L_{tRS}\right) = \rho(z)$

24 SL(N) Generalization

25 Opers hierarchy

26 Enumerative geometry